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SPECTRAL SYMMETRY OF THE DIRAC OPERATOR FOR COMPACT AND NONCOMPACT SYMMETRIC PAIRS

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The aim of this paper is to prove a vanishing of theorem for the Dirac operator on a symmetric pair. In fact, we prove a stronger result: that the Dirac operator has spectral G-symmetry.

THEOREM 1.1. *Let (G, K) be a symmetric pair of rank two or greater, of compact or noncompact type and $\Gamma \subset G$ a co-compact discrete subgroup. Let ρ be a metric on $\Gamma\backslash G$ whose lift to G is G-left and K-right invariant. Then, the Dirac operator has spectral G-symmetry: that is, for each eigenvalue λ the eigenspace V_λ is G-isomorphic to the eigenspace $V_{-\lambda}$.*

COROLLARY 1.2. *The equivariant η-function vanishes identically: $\eta_G(s, g) = 0$.*

The importance of the eta invariant and questions of spectral symmetry has long been recognized, see [**1**]. If $\dim G \neq 4k + 3$, the spectrum is symmetric for algebraic reasons. However, as the example in [**4**] shows, this spectrum need not be symmetric if $\dim G = 3$. For an odd dimensional simply connected Lie group with bi-invariant metric, the map $x \mapsto x^{-1}$ is an orientation reversing isometry and we again get spectral symmetry. However, this map may well not descend to quotients $\Gamma\backslash G$; for example, we know the spectrum for $SO(3) \cong SU(2)/\{\pm 1\}$ is not symmetric. Furthermore, if G is a noncompact rank one group and Γ a co-compact discrete subgroup then, with respect to certain natural metrics on $\Gamma\backslash G$, the spectrum fails to be symmetric, see [**6**]. Thus, the result does not hold in the rank one case.

In §2 we discuss the case of a symmetric pair of compact type. This is done in some detail. Section 3 contains the case of noncompact type. Since this is similar to the compact type, we concentrate on presenting the changes in the new case. We do not consider the case of a symmetric pair of Euclidean type.

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2. Spectral symmetry for a symmetric pair of compact type. Let (G, K) be a symmetric pair of compact type. Then the Lie algebra of G decomposes as $\mathscr{G} = \mathscr{K} \otimes \mathscr{P}$ with bracket relations $[\mathscr{K}, \mathscr{K}] \subset \mathscr{K}$, $[\mathscr{K}, \mathscr{P}] \subset \mathscr{P}$ and $[\mathscr{P}, \mathscr{P}] \subset \mathscr{K}$. With respect to the negative of the Killing form let $E_1, \dots, E_r$ be an orthonormal basis for $\mathscr{K}$ and $E_{r+1}, \dots, E_{r+s}$ one for $\mathscr{P}$ so $r+s = \dim G$ is odd. Throughout this and the following section we shall use the following convention: Latin subscripts run from 1 to r and Greek subscripts from $r+1$ to $r+s$. Let $t > 0$ be a real parameter and set $e_i = E_i/t$ and $e_\alpha = E_\alpha$. Let ρ_t denote the left invariant metric such that $e_1, \dots, e_{r+s}$ is an orthonormal basis of $\mathscr{G}$. Thus for $t \neq 1$ ρ_t is G left-invariant but only K right-invariant. The effect is to scale the metric on the fibers and leave it unchanged on the base of the fibration $K \to G \to G/K$. Further set $w_t = e^q \gamma_1, \dots, e_{r+s}$ where $q = (r+s+1)(r+s+2)/2$ and let ψ^t denote a basic spinor corresponding to the $e_1, \dots, e_{r+s}$ basis. When $t = 1$ the subscript t will be omitted. There is a canonical isomorphism between the Clifford algebra associated to ρ and that associated to ρ_t. Under this isomorphism e_i is the image of E_i, e_α the image of E_α and ψ^t that of ψ. Using this isomorphism, we notice that (with $1 \leq i_j \leq r+s$)

$$e_{e_1} \cdots e_{i_k} \psi^t = E_{i_1} \cdots E_{i_k} \psi \tag{2.1}$$

for any set of basis vectors, where the Clifford product on the left-hand side is relative to the ρ_t but on the right-hand side is relative to $\rho = \rho_1$. This same isomorphism is used implicitly in later expressions.

The Dirac operator is

$$P_t = \sum \omega_t e_i \nabla^t_{e_i} + \sum \omega_t e_\alpha \nabla^t_{e_\alpha} \tag{2.2}$$

where ∇^t is the Levi-Civita connection corresponding to ρ_t. We can identify the space of sections $\Gamma(\underset{\sim}{S})$ with $C_\infty(\Gamma\backslash G) \otimes S$ using left translation. Then for a basic spinor $\psi^t = 1 \otimes s^t$

$$\begin{aligned} P_t(f \otimes s^t) &= \sum \nu(e_i) f \otimes \omega_t e_i s^t + \sum \nu(e_\alpha) f \otimes \omega_t e_\alpha s^t \\ &\quad + f P_t(1 \otimes s^t) \\ &= \frac{1}{t} \sum \nu(E_i) f \otimes \omega E_s s + \sum \nu(E_\alpha) f \otimes \omega E_\alpha s \\ &\quad + f P_t(1 \otimes s^t). \end{aligned} \tag{2.3}$$

If we define $Q_K = \sum \nu(E_i) \otimes \omega E_i$, $Q_P = \sum \nu(E_\alpha) \otimes \omega E_\alpha$ and $Q_t = 1/tQ_K + Q_P$ then we see that

$$P_t(f \otimes s^t) = Q_t(f \otimes s) + fP_t(1 \otimes s^t). \tag{2.4}$$

Thus it remains to calculate $P_t\psi^t$. First we calculate ∇^t.

PROPOSITION 2.1. (i) $\nabla^t_{e_i} e_j = (1/t^2)\nabla_{E_i} E_j$,

(ii) $\nabla^t_{e_i} e_\beta = (2/t - t)\nabla_{E_i} E_\beta$,

(iii) $\nabla^t_{e_\alpha} e_j = t\nabla_{E_\alpha} E_j$,

(iv) $\nabla^t_{e_\alpha} e_\beta = \nabla_{E_\alpha} E_\beta$.

Proof. These follow from the following formulae:

$$\begin{aligned} &\text{(i)} && \langle \nabla^t_{e_i} e_j , e_k \rangle_t = \tfrac{1}{t} \langle \nabla_{E_i} E_j , E_k \rangle , \\ &\text{(ii)} && \langle \nabla^t_{e_i} e_\beta , e_\gamma \rangle_t = (2/t - t) \langle \nabla_{E_i} E_\beta , E_\gamma \rangle , \\ &\text{(iii)} && \langle \nabla^t_{e_\alpha} e_j , e_\gamma \rangle_t = t \langle \nabla_{E_\alpha} E_j , E_\gamma \rangle , \\ &\text{(iv)} && \langle \nabla^t_{e_\alpha} e_\beta , e_k \rangle_t = t \langle \nabla_{E_\alpha} E_\beta , E_k \rangle , \end{aligned} \tag{2.5}$$

and the observation that all similar expressions with an odd number of Greek subscripts are zero. These formulae use the notation $\langle\ ,\ \rangle_t$ for the inner product given by ρ_t. The calculations are similar to those of [3]. In obtaining these formulae, we use the fact that $\operatorname{ad} E_i$ (for $1 \le i \le r+s$) is ρ_t-skew. For orthonormal left invariant vector fields X, Y and Z there is the formula

$$\langle \nabla_X Y , Z \rangle = \frac{1}{2}(\langle Z , [X , Y] \rangle - \langle Y , [X , Z] \rangle - \langle X , [Y , Z] \rangle). \tag{2.6}$$

From this, we see $\nabla_{E_i} E_j = \frac{1}{2}[E_i , E_j]$, which is also useful.

From [2] $\chi(X) = -\frac{1}{4}\sum [X , E_i]E_i - \frac{1}{4}\sum [X , E_\alpha]E_\alpha$. We make the following definitions:

$$\begin{aligned} \chi_K(X) &= -\frac{1}{4}\sum [X , E_i]E_i , \\ \chi_P(X) &= -\frac{1}{4}\sum [X , E_\alpha]E_\alpha , \\ M_K &= \sum \omega E_i \chi_K(E_i) , \\ A &= \sum \omega_i E_i \chi_P(E_i) , \\ M &= \sum \omega E_i \chi(E_i) + \sum \omega E_\alpha \chi(E_\alpha). \end{aligned} \tag{2.7}$$

Clearly $\chi(X) = \chi_K(X) + \chi_P(X)$ and χ_K is the spin representation of $\mathscr{K}$ extended to act on S. If the isotropy representation $K \to \mathrm{SO}(\mathscr{P})$ lifts to spin then this induces $\chi_P|\mathscr{K}$, see Lemma 2.1 of [5].

LEMMA 2.2. $M = M_K + 3A$.

Proof. Observe that

$$\sum_\gamma E_\gamma[E_\gamma, E_\alpha] = \sum_i E_i[E_i, E_\alpha], \tag{2.8}$$

since

$$\begin{aligned}\sum E_\gamma[E_\gamma, E_\alpha] &= \sum E_\gamma\langle[E_\gamma, E_\alpha], E_i\rangle E_i = \sum -E_\gamma\langle[E_i, E_\alpha], E_\gamma\rangle E \\ &= \sum -[E_i, E_\alpha]E_i = \sum E_i[E_i, E_\alpha].\end{aligned}$$

The result now follows.

PROPOSITION 2.3. $P_t\psi^t = \frac{1}{2t}M_K\psi + \frac{1}{2}(\frac{2}{t} + t)A\psi$.

Proof. We calculate:

$$\begin{aligned}P_t\psi^t = &-\frac{1}{4}\sum \frac{1}{t}\omega E_i(\nabla_{E_i}E_j)E_j\psi \\ &-\frac{1}{4}\sum\left(\frac{2}{t} - t\right)\omega E_i(\nabla_{E_i}E_\beta)E_\beta\psi \\ &-\frac{1}{4}\sum t\omega E_\alpha(\nabla_{E_\alpha}E_\beta)E_\beta\psi - \frac{1}{4}\sum t\omega E_\alpha(\nabla_{E_\alpha}E_\beta)E_\beta\psi \\ = &-\frac{1}{8}\sum\frac{1}{t}\omega E_i[E_i, E_j]E_j\psi \\ &-\frac{1}{8}\sum\left(\frac{2}{t} - t\right)\omega E_i[E_i, E_\beta]E_\beta\psi \\ &-\frac{1}{8}\sum t\omega E_\alpha[E_\alpha, E_\beta]E_\beta\psi - \frac{1}{8}\sum t\omega E_\alpha[E_\alpha, E_\beta]E_\beta\psi \\ = &\ \frac{1}{2t}M_K\psi + \frac{1}{2}\left(\frac{2}{t} + t\right)A\psi,\end{aligned} \tag{2.9}$$

which is the result of the proposition.

COROLLARY 2.4.

$$P_t = 1/tQ_K + Q_P + 1/2t(1 \otimes M_K) + (1/t + t/2)(1 \otimes A).$$

LEMMA 2.5. *The operators Q_K, Q_P, $1 \otimes M_K$, $1 \otimes A$ and hence P_t all commute with the action of $\mathscr{K}$ via the representation $\nu \otimes \chi$.*

Proof. This is another direct calculation. For example in the case of Q_K:

$$
\begin{aligned}
(2.10)\ [Q_K, (\nu\otimes 1 + 1\otimes\chi)E_i]
&= \sum \nu([E_j, E_i])\otimes\omega E_j \\
&\quad + \sum \nu(E_j)\otimes\omega(E_j\chi(E_i) - \chi(E_i)E_j) \\
&= \sum \nu([E_j, E_i])\otimes\omega E_j + \sum \nu(E_j)\otimes\omega[E_j, E_i] = 0.
\end{aligned}
$$

PROPOSITION 2.6. *The operator* P_t *preserves the decomposition* $\Gamma(\underset{\sim}{S}) = L^2(\Gamma\backslash G)\otimes S = \widehat{\bigoplus} V_\lambda \otimes S$ *into isotypic components under the right regular representation* $\nu\otimes 1$ *of* G.

Proof. This is immediate since

$$P_t = Q_t + (1/2t)1\otimes M_K + (1/t + t/2)1\otimes A$$

and Q_t is a linear combination of the operators $\nu(E)$.

Let $\Omega_G = -\sum E_i^2 - \sum E_\alpha^2$ and $\Omega_K = -\sum E_i^2$ be the Casimir elements. Set $\Omega_P = \Omega_G - \Omega_K$ and let ρ_K denote half the sum of the positive roots of K. Then define the following operators:

$$
(2.11)\qquad
\begin{aligned}
&\text{(i)} & R_K &= \sum \nu(E_i)\otimes\chi_K(E_i), \\
&\text{(ii)} & R_P &= \sum \nu(E_\alpha)\otimes\chi_P(E_\alpha), \\
&\text{(iii)} & R_M &= \sum \nu(E_i)\otimes\chi_P(E_i), \\
&\text{(iv)} & R_S &= \sum \chi_K(E_i)\chi_P(E_i),
\end{aligned}
$$

where χ_K and χ_P are given in (2.7). Notice that R_S is an operator on S while the other three operate on $C^\infty(G)\otimes S$. Direct calculation now establishes the following result.

PROPOSITION 2.7. *Using the notation* $\{U, V\} = UV + VU$:

(i) $\{Q_K, Q_P\} = 4R_P$,
(ii) $\{Q_K, 1\otimes M_K\} = -6R_K$,
(iii) $\{Q_K, 1\otimes A\} = -2R_M$,
(iv) $\{Q_P, 1\otimes M_K\} = 0$,
(v) $\{Q_P, 1\otimes A\} = -4R_P$,
(vi) $\{M_K, A\} = -6R_S$,
(vii) $Q_K^2 = \nu(\Omega_K)\otimes 1 + 2R_K$,
(viii) $Q_P^2 = \nu(\Omega_P)\otimes 1 + 2R_M$,

(ix) $A^2 = \chi_P(\Omega_K) + 2R_S$,
(x) $M_K^2 = 9\|\rho_K\|^2$.

Proof. To illustrate the proof, we verify part (x):

$$(2.12)\quad \begin{aligned} M_K^2 &= \sum \omega E_i \chi_K(E_i)\omega E_j \chi_K(E_j) = \sum E_i \chi_K(E_i) E_j \chi_K(E_j) \\ &= \tfrac{1}{2}\sum (E_i E_j \chi_K(E_i)\chi_K(E_j) + E_j E_i \chi_K(E_j)\chi_K(E_i) \\ &\qquad + E_i[E_i, E_j]\chi_K(E_j)) \\ &= \frac{1}{2}\Big\{ \sum E_i E_j (\chi_K(E_i)\chi_K(E_j) - \chi_K(E_j)\chi_K(E_i)) \\ &\qquad - 2\sum \chi_K(E_i)^2 \Big\} \\ &\quad - 4\sum \chi_K(E_j)^2 \\ &= \frac{1}{2}\sum E_i E_j \chi_K([E_i, E_j]) - 5\sum \chi_K(E_i)^2. \end{aligned}$$

Now

$$(2.13)\quad \begin{aligned} \sum E_i E_j \chi_K([E_i, E_j]) &= -\sum [E_s, E_j] E_j \chi_K(E_s) \\ &= 4\sum \chi_K(E_s)^2 = -4\chi_K(\Omega_K). \end{aligned}$$

Thus $M_K^2 = 3\chi_K(\Omega_K) = 9\|\rho_K\|^2$, since χ_K is the sum of irreducible representations, each taking the same value, $3\|\rho_K\|^2$, on Ω_K.

The space of sections $\Gamma(\underset{\sim}{S})$ has been decomposed into a completed sum of terms of the form $V_\lambda \otimes S$, $\lambda \in \widehat{G}$, under the action of the group G. Each V_λ is finite-dimensional and we may decompose $V_\lambda \otimes S$ under the $\nu \otimes \chi$ action of $\mathscr{K}$ into isotypic (rather than irreducible) components:

$$(2.14)\quad V_\lambda \otimes S = \bigoplus S_\theta.$$

Now Lemma 2.5 and Proposition 2.6 tell us that P_t leaves S_θ invariant. The next step is to show P_t^2 is constant on S_θ and then that $\operatorname{tr} P_t|S_\theta = 0$. To show $P_t^2|S_\theta$ is constant we show that each of the ten operators of Proposition 2.7 is constant on S_θ. This is clearly the same as showing R_K, R_P, R_M and R_S are constant on S_θ.

LEMMA 2.8. *The operators R_K, R_P, R_M and R_S are constants on S_θ.*

Proof. First notice that while χ_K and χ_P may not be irreducible the Casimir takes the same value in each irreducible summand, see

[5, Lemma 2.2]. The result, for all except R_S, now follows from the following formulae:

$$\begin{aligned}(2.15)\qquad R_K &= \tfrac{1}{2}(-\nu\otimes\chi_K(\Omega_K)+\nu(\Omega_K)\otimes 1+1\otimes\chi_K(\Omega_K)),\\ R_P &= \tfrac{1}{2}(-\nu\otimes\chi_P(\Omega_P)+\nu(\Omega_P)\otimes 1+1\otimes\chi_P(\Omega_P)),\\ R_M &= \tfrac{1}{2}(-\nu\otimes\chi_P(\Omega_K)+\nu(\Omega_K)\otimes 1+1\otimes\chi_P(\Omega_K)).\end{aligned}$$

For R_S consider the decomposition $\mathscr{G}=\mathscr{K}\otimes\mathscr{P}$. It gives rise to an isomorphism $\mathrm{Cliff}(\mathscr{G})\cong\mathrm{Cliff}(\mathscr{K})\otimes\mathrm{Cliff}(\mathscr{P})$ and thence to one of modules:

$$(2.16)\qquad S\cong S_K\otimes S_P.$$

With respect to this decomposition $\chi_K=\widehat{\chi}_K\otimes 1$ and $\chi_P=1\otimes\widehat{\chi}_P$ so that

$$(2.17)\qquad R_S=\tfrac{1}{2}(-\widehat{\chi}_K\otimes\widehat{\chi}_P(\Omega_K)+\widehat{\chi}_K\otimes 1(\Omega_K)+1\otimes\widehat{\chi}_P(\Omega_K)).$$

COROLLARY 2.9. *The operator* $P_t^2|S_\theta$ *is constant.*

This constant depends on t and θ. In principle it has been calculated but is omitted as the expression is unenlightening.

PROPOSITION 2.10. *If* $\operatorname{rank} G>1$, $\operatorname{tr} P_t|S_\theta=0$.

Proof. Let U_p be the subspace of $\mathrm{Cliff}(\mathscr{G})$ spanned as a vector space by $E_{i_1}E_{i_2}\cdots E_{i_p}$, $i_1<i_2<\cdots<i_p$ (this time without using the convention of Latin and Greek indices). Then for $X\in U_p$, we have

$$(2.18)\qquad \operatorname{tr}X=0\quad\text{for } p\neq 0.$$

Since $M_K=\sum\omega E_i\chi_K(E_i)=\frac{1}{4}\sum\omega E_i[E_i,E_j]E_j$ and $\operatorname{rank}G>1$ (so $\dim\mathscr{G}>3$) it is clear that $M_K\in U_{r+s-3}$. Thus by equation (2.18), since $r+s>3$, $\operatorname{tr}M_K|S=0$. Split S into eigenspaces of $M_K\colon S=(S_K^+\oplus S_K^-)\otimes S_P=(S_K^+\otimes S_P)\oplus(S_K^-\otimes S_P)$. Since $M_K^2=\alpha^2$, $\alpha=3\|\rho_K\|$, there are only two eigenspaces and $\operatorname{tr}M_K=0$ gives $\dim S_K^+=\dim S_K^-$. By considering weights $S_K\cong 2^nV_{\rho_K}$, $n=\frac{1}{2}(l-1)$, so that $S_K^+\cong S_K^-\cong 2^{n-1}V_{\rho_K}$ and $S_\theta=S_\theta^+\oplus S_\theta^-$ with $\dim S_\theta^+=\dim S_\theta^-$. Thus $\operatorname{tr}M_K|S_\theta=0$ and with respect to the decomposition M_K has matrix $\left(\begin{smallmatrix}\alpha&0\\0&-\alpha\end{smallmatrix}\right)$. If B is any operator with matrix $\left(\begin{smallmatrix}u&v\\x&y\end{smallmatrix}\right)$ then $\{M_K,B\}=\left(\begin{smallmatrix}\alpha u&0\\0&-2\alpha y\end{smallmatrix}\right)$. Thus if $\{M_K,B\}$ is constant on S_θ then $u=-y$ and $\operatorname{tr}B|S_\theta=0$. Taking $B=Q_K$, Q_P and A we see $\operatorname{tr}Q_K|S_\theta=\operatorname{tr}Q_P|S_\theta=\operatorname{tr}A|S_\theta=0$. Hence $\operatorname{tr}P_t|S_\theta=0$.

THEOREM 2.11. *P_t has spectral symmetry for all $t > 0$ if* $\operatorname{rank} G > 1$.

THEOREM 2.12. *The equivariant eta function of the operator P_t on $\Gamma \backslash G$, for rank $G > 1$ at $t > 0$ and any discrete co-compact subgroup Γ vanishes as a K-character: $\eta_K(s, g) = 0$ where $\eta_K(s, g) = \sum_\lambda \operatorname{sign}(\lambda)|\lambda|^{-s} \operatorname{tr}(g|V_\lambda)$ for $g \in K$.*

3. Spectral symmetry for a symmetric pair of noncompact type. Let (G, K) be a symmetric pair of noncompact type. This case is similar to that of the previous section. However, the details are different and we shall be concerned, mainly, with pointing out the differences. Decompose $\mathscr{G} = \mathscr{K} \oplus \mathscr{P}$ and define the metric ρ to be the negative of the Killing form on $\mathscr{K}$, the Killing form on $\mathscr{P}$ and under ρ let $\mathscr{K}$ be orthogonal to $\mathscr{P}$. As before let $E_1, \dots, E_r$ be an orthonormal basis for $\mathscr{K}$; $E_{r+1}, \dots, E_{r+s}$ be one for $\mathscr{P}$ and we shall use the convention that Latin subscripts run from 1 to r and Greek from $r+1$ to $r+s$. Set $e_i = E_i/t$, $e_\alpha = E_\alpha$ and let ρ_t be the metric with $e_1, \dots, e_{r+s}$ as orthonormal basis. Let χ_K, χ_P, Q_K, Q_P, M_K and A be defined by the formulae of the previous section.

Formally we can use the compact dual $\mathscr{G}^*$ of $\mathscr{G}$ to obtain the present results from the previous section. Let $\mathscr{G}_{\mathbf{C}}$ be the complexification of $\mathscr{G}$. Then there is the compact dual $\mathscr{G}^* \subset \mathscr{G}_{\mathbf{C}}$ of $\mathscr{G}$ and a correspondence

$$X \to X \quad \text{for } X \in \mathscr{K}, \qquad X \to iX \quad \text{for } X \in \mathscr{P} \qquad (i = \sqrt{-1}) \tag{3.1}$$

between $\mathscr{G}$ and $\mathscr{G}^*$. Denote by X^* the element of $\mathscr{G}^*$ corresponding to $X \in \mathscr{G}$ so $e_j^* = e_j$ and $e_\alpha^* = ie_\alpha$. There is a metric ρ_t^* on $\mathscr{G}^*$ with orthonormal basis $e_1^*, \dots, e_{r+s}^*$. Formally

$$\rho_t(x, y) = \rho_{it}^*(ix^*, iy^*) \tag{3.2}$$

and so as elements of the Lie algebra one is led to expect

$$P_t\psi^t = iP_{it}^*\psi^{t*}. \tag{3.3}$$

In fact this is true as a direct, rather than formal, calculation shows.

PROPOSITION 3.1. $P_t\psi^t = \frac{1}{2t}M_K\psi + \frac{1}{2}(\frac{2}{t} - t)A\psi$.

Proof. This is essentially the same as the proof of Proposition 2.3. The main changes are as follows. Firstly the invariance of the metric is now given by

$$\begin{aligned}(3.4)\qquad \langle E_\beta\,, [E_i\,, E_\gamma]\rangle &= -\langle E_\gamma\,, [E_i\,, E_\beta]\rangle\,,\\ \langle E_i\,, [E_\beta\,, E_\gamma]\rangle &= +\langle E_\gamma\,, [E_\beta\,, E_i]\rangle\end{aligned}$$

instead of always a negative sign. Thus

$$\sum E_\gamma[E_\gamma\,, E_\alpha] = -\sum E_i[E_i\,, E_\alpha]$$

and so

$$(3.5)\qquad A = -\frac{1}{4}\sum \omega E_i[E_i\,, E_\alpha]E_\alpha = \frac{1}{4}\sum \omega E_\alpha[E_\alpha\,, E_\beta]E_\beta.$$

The formula $\nabla_X Y = 1/2[X\,, Y]$ no longer holds for all X and Y. Instead we have

$$\begin{aligned}(3.6)\qquad \nabla_{E_i}E_j &= \tfrac{1}{2}[E_i\,, E_j]\,, & \nabla_{E_i}E_\beta &= \tfrac{3}{2}[E_i\,, E_\beta]\,,\\ \nabla_{E_\alpha}E_j &= -\tfrac{1}{2}[E_\alpha\,, E_j]\,, & \nabla_{E_\alpha}E_\beta &= \tfrac{1}{2}[E_\alpha\,, E_\beta].\end{aligned}$$

Then equations (2.5) in the noncompact case become

$$\begin{aligned}(3.7)\qquad &\text{(i)} & \langle\nabla^t_{e_i}e_j\,, e_k\rangle_t &= \tfrac{1}{t}\langle\nabla_{E_i}E_j\,, E_k\rangle\,,\\ &\text{(ii)} & \langle\nabla^t_{e_i}e_\beta\,, e_\gamma\rangle_t &= \tfrac{1}{3}\left(\tfrac{2}{t}+t\right)\langle\nabla_{E_i}E_\beta\,, E_\gamma\rangle\,,\\ &\text{(iii)} & \langle\nabla^t_{e_\alpha}e_j\,, e_\gamma\rangle_t &= t\langle\nabla_{E_\alpha}E_j\,, E_\gamma\rangle\,,\\ &\text{(iv)} & \langle\nabla^t_{e_\alpha}e_\beta\,, e_k\rangle_t &= t\langle\nabla_{E_\alpha}E_\beta\,, E_k\rangle.\end{aligned}$$

As before the other expressions analogous to these with an odd number of Greek indices are zero. The result of Proposition 2.1 is now:

$$\begin{aligned}(3.8)\qquad &\text{(i)} & \nabla^t_{e_i}e_j &= (1/t^2)\nabla_{E_i}E_j\,,\\ &\text{(ii)} & \nabla^t_{e_i}e_\beta &= \tfrac{1}{3}\left(\tfrac{2}{t}+t\right)\nabla_{E_i}E_\beta\,,\\ &\text{(iii)} & \nabla_{e_\alpha}e_j &= t\nabla_{E_\alpha}E_j\,,\\ &\text{(iv)} & \nabla^t_{e_\alpha}e_\beta &= \nabla_{E_\alpha}E_\beta.\end{aligned}$$

The proof is completed by a calculation similar to that used to prove Proposition 2.3.

The list of relations in Proposition 2.7 takes the following form where the operators R_K, R_P, R_M and R_S are defined by the formulae (2.11).

PROPOSITION 3.2.

(i) $\{Q_K, Q_P\} = -4R_P$,
(ii) $\{Q_K, 1 \otimes M_K\} = -6R_K$,
(iii) $\{Q_K, 1 \otimes A\} = -2R_M$,
(iv) $\{Q_P, 1 \otimes M_K\} = 0$,
(v) $\{Q_P, 1 \otimes A\} = 4R_P$,
(vi) $\{M_K, A\} = -6R_S$,
(vii) $Q_K^2 = \nu(\Omega_K) \otimes 1 + 2R_K$,
(viii) $Q_P^2 = \nu(\Omega_P) \otimes 1 - 2R_M$,
(ix) $A^2 = \chi_P(\Omega_K) \otimes 1 + 2R_S$,
(x) $M_K^2 = 9||\rho_K||^2$.

Now let Γ be any co-compact discrete subgroup of G. Then the space of L^2-sections of the spin bundle $\underset{\sim}{S}$ over $\Gamma\backslash G$ decomposes into a completed sum of unitary representations of G. For $\lambda \in \widehat{G}$ let V_λ^Γ be the isotypic summand of type λ so that

$$L^2(\underset{\sim}{S}) = \widehat{\bigoplus} V_\lambda^\Gamma \otimes S. \tag{3.9}$$

The representations λ with $V_\lambda^\Gamma \neq 0$ occurring in this sum are, in general, not explicitly known. Each term in this sum decomposes further into $\mathscr{K}$-types under the action $\nu \otimes \chi$:

$$V_\lambda^\Gamma \otimes S = \bigoplus S_\theta. \tag{3.10}$$

The arguments of §2 go through word for word. So there is spectral symmetry for P_t on each S_θ providing rank $G > 1$. Consequently we have the following theorem.

THEOREM 3.3. *The equivariant eta function for the operator P_t on $\Gamma\backslash G$ vanishes as a K-character for G a real semi-simple Lie group of rank > 1 and Γ a co-compact discrete subgroup.*

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