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ON THE VALUES OF A ZETA FUNCTION AT NON-POSITIVE INTEGERS

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Let

$$\tilde{\zeta}(s) = \sum_{g_1=1}^{\infty} \sum_{g_2=1}^{\infty} \sum_{g_3=0}^{\infty} [g_1 g_2 + (g_1 + g_2) g_3]^{-s}, \qquad \operatorname{Re} s > \frac{3}{2},$$

be the zeta function associated with the principal Delaunay-Voronoi cone. A general theory asserts that $\tilde{\zeta}(s)$ has an analytic continuation which is holomorphic in the whole complex plane except possible poles at $s = 3/2$, $s = 1$ and $s = 1/2$. In this paper, we shall compute the values of $\tilde{\zeta}(s)$ at non-positive integers. It is not surprising to see that these values are rational numbers and can be expressed explicitly in terms of Bernoulli numbers; i.e.

$$\tilde{\zeta}(-m) = -\frac{1}{2}\left(\frac{B_{m+1}}{m+1}\right)^2 + (-1)^{m+1} \frac{B_{2m+2}(1 + 2^{2m+2})}{2^{2m+1}(2m+2)^2} + \frac{\delta_{0m}}{4}.$$

1. Introduction and the main theorem. Let

$$(1) \qquad \tilde{\zeta}(s) = \sum_{g_1=1}^{\infty} \sum_{g_2=1}^{\infty} \sum_{g_3=0}^{\infty} [g_1 g_2 + (g_1 + g_2) g_3]^{-s}, \qquad \operatorname{Re} s > 3/2,$$

be the zeta function associated with the principal Delaunay-Voronoi cone Ω (see [5]) as defined by

$$(2) \qquad \Omega = \left\{ \begin{bmatrix} \lambda_1 + \lambda_3 & -\lambda_3 \\ -\lambda_3 & \lambda_2 + \lambda_3 \end{bmatrix} | \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\}.$$

By the general theory as in [7], this zeta function is absolutely convergent for $\operatorname{Re} s > 3/2$ and hence it defines a holomorphic function of a complex variable s. Furthermore, $\tilde{\zeta}(s)$ has an analytic continuation which is holomorphic in the whole complex plane except possible poles at $s = 3/2$, $s = 1$ and $s = 1/2$.

In this paper, we shall prove the following

MAIN THEOREM. *For any integer m with $m \geq 0$, $\tilde{\zeta}(-m)$ is a rational number. More precisely,*

$$\tilde{\zeta}(-m) = -\frac{1}{2}\left(\frac{B_{m+1}}{m+1}\right)^2 + (-1)^{m+1} \frac{B_{2m+2}(1 + 2^{2m+2})}{2^{2m+1}(2m+2)^2} + \frac{\delta_{0m}}{4}.$$

Here δ_{0m} is the Kronecker delta function and the Bernoulli numbers B_m $(m \geq 1)$ are defined by

$$\begin{cases} \dfrac{t}{e^t - 1} = 1 + \displaystyle\sum_{m=1}^{\infty} \frac{B_m t^m}{m!}, \quad |t| < 2\pi, \\ B_0 = 1. \end{cases}$$

2. The integral expression of $\tilde{\zeta}(s)$. To obtain an integral expression for $\tilde{\zeta}(s)\Gamma(s)\Gamma(s-\frac{1}{2})\pi^{1/2}$, we need the following lemma.

LEMMA 1. *Let Y be the variable of 2×2 real symmetric matrix, and G be a 2×2 positive definite symmetric matrix. Then we have, for* $\operatorname{Re} s \geq 3/2$,

$$\int_{Y>0} (\det Y)^{s-3/2} e^{-\operatorname{tr}(YG)} dY = (\det G)^{-s} \pi^{1/2} \Gamma(s) \Gamma(s - 1/2).$$

Here $Y > 0$ means that Y is positively definite.

Proof. See p. 226 of **[1]**.

PROPOSITION 1. *For* $\operatorname{Re} s > 3/2$, *we have*

$$\begin{aligned} &\tilde{\zeta}(s)\Gamma(s)\Gamma(s-1/2)\pi^{1/2} \\ &\quad = 2\int_0^{\infty} u^{2s-4}\, du \int_0^1 (1-r^2)^{s-3/2} r\, dr \\ &\qquad \times \int_0^{2\pi} u^3[(e^{u(1+r\sin\theta)}-1) \\ &\qquad\qquad \times (e^{u(1-r\sin\theta)}-1)(1-e^{-2u(1-r\cos\theta)})]^{-1} d\theta. \end{aligned}$$

Proof. For $\operatorname{Re} s > 3/2$, by Lemma 1, we have

$$\begin{aligned} &\tilde{\zeta}(s)\Gamma(s)\Gamma(s-1/2)\pi^{1/2} \\ &\quad = \sum_{g_1=1}^{\infty}\sum_{g_2=1}^{\infty}\sum_{g_3=0}^{\infty}\int_{Y>0} (\det Y)^{s-3/2} \exp\{-y_1 g_1 - y_2 g_2 \\ &\qquad\qquad - (y_1 + y_2 - 2y_{12})g_3\}\, dy_1\, dy_2\, dy_{12} \\ &\quad = \int_{Y>0} (\det Y)^{s-3/2}[(e^{y_1}-1)(e^{y_2}-1) \\ &\qquad\qquad \times (1-e^{-y_1-y_2+2y_{12}})]^{-1}\, dy_1\, dy_2\, dy_{12}. \end{aligned}$$

By changing of variables: $u = (y_1 + y_2)/2$, $v = (y_1 - y_2)/2$, $w = y_{12}$; the integral is transformed into

$$\begin{aligned} 2\int_{u^2-v^2-w^2>0,\ u>0} &(u^2 - v^2 - w^2)^{s-3/2} \\ &\times [(e^{u+v}-1)(e^{u-v}-1)(1-e^{-2u+2w})]^{-1}\, du\, dv\, dw. \end{aligned}$$

Let $v = up$, $w = uq$ and let $p = r\sin\theta$, $q = r\cos\theta$. Then

$$\begin{aligned}\tilde{\zeta}(s)&\Gamma(s)\Gamma(s-1/2)\pi^{1/2}\\ &= 2\int_0^\infty u^{2s-4}\,du\int_0^1(1-r^2)^{s-3/2}r\,dr\\ &\quad\times\int_0^{2\pi}u^3[(e^{u(1+r\sin\theta)}-1)(e^{u(1-r\sin\theta)}-1)\\ &\qquad\times(1-e^{-2u(1-r\cos\theta)})]^{-1}\,d\theta,\end{aligned}$$

as asserted.

Let

$$\begin{aligned}F(u,r)=\int_0^{2\pi}&u^3[(e^{u(1+r\sin\theta)}-1)(e^{u(1-r\sin\theta)}-1)\\ &\times(1-e^{-2u(1-r\cos\theta)})]^{-1}\,d\theta\end{aligned}$$

and

$$I(s,u,F)=\frac{1}{\Gamma(s-1/2)\pi^{1/2}}\int_0^1(1-r^2)^{s-3/2}rF(u,r)\,dr.$$

$I(s,u,F)$ as a function of s, is holomorphic for all $s\neq 1$ since it is a quotient of two generalized functions in s which has simple poles at negative half integers [7].

Denote by $L(\varepsilon)$ the contour in the complex plane consisting of the interval $[\varepsilon,+\infty)$ twice, in both directions (in and out) and the circle $|z|=\varepsilon$ in counterclockwise direction. Then the integral expression of $\tilde{\zeta}(s)\Gamma(s)$ can be transformed into a contour integral of the form

$$2(e^{4\pi is}-1)^{-1}\int_{L(\varepsilon)}u^{2s-4}I(s,u,F)\,du.$$

With the functional equation of gamma function $\Gamma(s)\Gamma(1-s) = 2\pi ie^{\pi is}(e^{2\pi is}-1)^{-1}$, we get

$$\tilde{\zeta}(s)=2\Gamma(1-s)e^{-\pi is}(e^{4\pi is}-1)^{-1}\frac{1}{2\pi i}\int_{L(\varepsilon)}u^{2s-4}I(s,u,F)\,du.$$

This gives the analytic continuation of $\tilde{\zeta}(s)$.

PROPOSITION 2. *For all $s\in C-\{\pm 1/2,\pm 3/2,\dots\}$, we have*

$$\tilde{\zeta}(s)=2\Gamma(1-s)e^{-\pi is}(e^{2\pi is}+1)^{-1}\frac{1}{2\pi i}\int_{L(\varepsilon)}u^{2s-4}I(s,u,F)\,du$$

where

$$I(s,u,F)=\frac{1}{\Gamma(s-1/2)\pi^{1/2}}\int_0^1(1-r^2)^{s-3/2}rF(u,r)\,dr$$

with

$$F(u, r) = \int_0^{2\pi} u^3[(e^{u(1+r\sin\theta)} - 1)(e^{u(1-r\sin\theta)} - 1) \times (1 - e^{-2u(1-r\cos\theta)})]^{-1}\, d\theta.$$

REMARK. From the analytic continuation of $\tilde{\zeta}(s)$ given in Proposition 2, it seems that it might also have possible poles at negative half integers. However these poles can be eliminated by the gamma function appearing in the expression of $I(s, u, F)$. In fact, $\tilde{\zeta}(s)$ is analytic at negative half integers; but it is hard to see from our formula.

3. The values of $\tilde{\zeta}(s)$ at non-positive integers. The analytic continuation of $I(s, u, F)$ was given in a more general context in [3]. When s is a non-positive integer, the value $\tilde{\zeta}(s)$ depends only on the continuation of $I(s, u, F)$ when $|u| < \varepsilon$, due to the fact that the contour integral along $[\varepsilon, +\infty]$ twice in opposite direction cancel each other. Furthermore, these values can be obtained by the theorem of residue.

PROPOSITION 3. *The values of* $\tilde{\zeta}(s)$ *at* $s = 0, -1, -2, \dots, -m, \dots$, *are rational numbers and*

$$\tilde{\zeta}(-m) = -\frac{\Gamma(2m+2)}{2^{2m+1}\pi} N(-m)$$

where

$$N(s) = \int_0^1 (1-r^2)^{s-3/2} r\, dr \int_0^{2\pi} Q_{2m+3}(r, \theta)\, d\theta, \qquad \mathrm{Re}\, s > 1,$$

with

$$\begin{aligned} &Q_{2l+1}(r, \theta) \\ &= \frac{1}{2}\sum_{k=0}^{l} \frac{B_{2(l-k)}B_{2k}}{(2l-2k)!(2k)!}[(1+r\sin\theta)^{2l-2k-1}(1-r\sin\theta)^{2k-1} \\ &\qquad - (1+r\sin\theta)^{2l-2k-1}(2-2r\cos\theta)^{2k-1} \\ &\qquad - (1-r\sin\theta)^{2l-2k-1}(2-2r\cos\theta)^{2k-1}] \\ &\quad + \frac{1}{8}\delta_{1l} \quad (\delta_{ij} \text{ is the Kronecker's delta function}). \end{aligned}$$

Proof. We have

$$\tilde{\zeta}(-m) = 2\Gamma(1+m)(-1)^{m-1}\frac{1}{2\pi i}\int_{L(\varepsilon)} u^{-2m-4} I(-m, u, F)\, du.$$

In order to compute $\tilde{\zeta}(-m)$, we must find the coefficient of u^{2m+3} in the power expansion of $I(-m, u, F)$. For $|u| < \pi/2$, we have

$$\frac{u}{e^{u(1+r\sin\theta)}-1} = \frac{1}{1+r\sin\theta} - \frac{u}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}(1+r\sin\theta)^{2n-1}u^{2n}}{(2n)!},$$
$$\frac{u}{e^{u(1-r\sin\theta)}-1} = \frac{1}{1-r\sin\theta} - \frac{u}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}(1-r\sin\theta)^{2n-1}u^{2n}}{(2n)!},$$
$$\frac{u}{1-e^{-2u(1-r\cos\theta)}} = \frac{1}{1-r\cos\theta} - \frac{u}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}(2+2r\cos\theta)^{2n-1}u^{2n}}{(2n)!}.$$

The coefficient of u^{2m+3} in the power expansion of

$$u^3[(e^{u(1+r\sin\theta)}-1)(e^{u(1-r\sin\theta)}-1)(1-e^{-2u(1-r\cos\theta)})]^{-1}$$

is $Q_{2m+3}(r, \theta)$, which is a Q-linear combination of functions of following types:

$$F_1(r, \theta) = (1+r\sin\theta)^{m_{11}-1}(1-r\sin\theta)^{m_{12}}(1-r\cos\theta)^{m_{13}},$$
$$F_2(r, \theta) = (1+r\sin\theta)^{m_{21}}(1-r\sin\theta)^{m_{22}-1}(1-r\cos\theta)^{m_{23}},$$
$$F_3(r, \theta) = (1+r\sin\theta)^{m_{31}}(1-r\sin\theta)^{m_{32}}(1-r\cos\theta)^{m_{33}-1},$$

where m_{ij} $(i, j = 1, 2, 3)$ are positive integers or zero. Integrating with respect to θ from 0 to 2π, we get

$$\int_0^{2\pi} F_j(r, \theta)\,d\theta = 2\pi P_j(r^2) \quad \text{if } m_{jj} \geq 1,$$
$$= \frac{2\pi Q_j(r^2)}{\sqrt{1-r^2}} + 2\pi R_j(r^2) \quad \text{if } m_{jj} = 0,$$

where $P_j(X)$, $Q_j(X)$ and $R_j(x)$ are polynomials. Thus the coefficient of u^{2m+3} in $I(-m, u, F)$ is a Q-linear combination of integrals of the forms

$$\frac{2\pi}{\Gamma(-m-1/2)\pi^{1/2}} \int_0^1 (1-r^2)^{-m-3/2} r P_j(r^2)\,dr,$$
$$\frac{2\pi}{\Gamma(-m-1/2)\pi^{1/2}} \int_0^1 (1-r^2)^{-m-2} r Q_j(r^2)\,dr.$$

The values of continuations of these integrals are rational numbers. Consequently, the value $\tilde{\zeta}(-m)$ is a rational number.

4. On the explicit values of $N(-m)$ and the proof of the main theorem. In this section, we shall prove that almost all terms in $Q_{2m+3}(r, \theta)$ have zero contribution to $N(-m)$. The contributions from remaining terms will be computed one by one. Finally, we obtain the explicit expression of $\tilde{\zeta}(-m)$ as shown in Main Theorem.

PROPOSITION 4. *For non-negative integers p and q, not both 0, define*

$$F_{p,q}(s) = \int_0^1 (1-r^2)^{s-3/2} r\,dr \int_0^{2\pi} (1+r\sin\theta)^{2p-1}(1-r\sin\theta)^{2q-1} d\theta \qquad (\operatorname{Re} s > 1).$$

Then $F_{p,q}(s)$ is a rational function with simple poles at $s = \frac{1}{2}, -\frac{1}{2}, \ldots, \frac{3}{2} - p - q$ (and also at $s = 1$ if p or q is zero) and vanishes at $s = 1 - p - q$ unless $p = q$, in which case

$$F_{p,p}(1-2p) = -2\pi(4p-2)(4p-4)\cdots 2/(4p-1)(4p-3)\cdots 3.$$

Proof. Changing from polar coordinate to linear coordinate, $r\cos\theta = x$, $r\sin\theta = y$; we find

$$F_{p,q}(s) = \iint_{x^2+y^2<1} (1+y)^{2p-1}(1-y)^{2q-1}(1-x^2-y^2)^{s-3/2}\,dx\,dy.$$

With $t = x/\sqrt{1-y^2}$ as a new variable in place of x, then

$$\begin{aligned}
F_{p,q}(s) &= \int_{-1}^1 (1+y)^{s+2p-2}(1-y)^{s+2q-2}\,dy \int_{-1}^1 (1-t^2)^{s-3/2}\,dt \\
&= 2^{2s+2p+2q-3} \int_0^1 u^{s+2p-2}(1-u)^{s+2q-2}\,du \\
&\quad \times \int_0^1 v^{-1/2}(1-v)^{s-3/2}\,dv \qquad \left(u = \frac{y+1}{2},\, v = t^2\right) \\
&= 2^{2s+2p+2q-3} \frac{\Gamma(s+2p-1)\Gamma(s+2q-1)\Gamma(s-1/2)\Gamma(1/2)}{\Gamma(2s+2p+2q-2)\Gamma(s)}.
\end{aligned}$$

This explicit evaluation holds for any complex number p and q (and Re s sufficiently). The statements of the proposition follow easily when p and q are non-negative integers: it is clear that the poles are as stated and that $F_{p,q}(1-p-q) = 0$ for $p \neq q$, since in that case both $\Gamma(s)$ and $\Gamma(2p+2q+2s-2)$; but only one of $\Gamma(2p+s-1)$ and $\Gamma(2q+s-1)$, have simple poles at $s = 1-p-q$, while for $p = q$ all four of these gamma functions have poles and we obtain the formula given in the proposition by comparing residues.

PROPOSITION 5. *For non-negative integers p and q, not both 0, define*

$$G_{p,q}(s) = \int_0^1 (1-r^2)^{s-3/2} r\,dr \int_0^{2\pi} (1-r\cos\theta)^{2p-1}(1-r\sin\theta)^{2q-1}\,d\theta \qquad (\mathrm{Re}\, s > 1).$$

Then $G_{p,q}(s)$ is a rational function with simple poles at $s = \frac{1}{2}, -\frac{1}{2}, \dots, \frac{5}{2}-p-q$ (and also at $s = 1, 0, \dots, 2-p-q$ if p or q is zero) and vanishes at $s = 1-p-q$ unless $p = q$ or p or q is zero, in which case

$$G_{p,q}(1-p-q) = \begin{cases} -2^p(p-1)!\pi/(4p-1)(4p-3)\cdots(2p+1), & \text{if } p = q, \\ (-1)^{p+q}\pi/(p+q), & \text{if } p = 0 \text{ or } q = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We have

$$G_{p,q}(s) = \iint_{x^2+y^2<1} (1-x)^{2p-1}(1-y)^{2q-1}(1-x^2-y^2)^{s-3/2}\,dx\,dy.$$

Suppose that $p \geq 1$ and expand $(1-x)^{2p-1}$ by the binomial theorem. Then $G_{p,q}(s)$ is a sum of double integrals with an inner integral of the form

$$\int_{-a}^{a} x^m (a^2-x^2)^{s-3/2}\,dx \qquad \left(a = \sqrt{1-y^2}\right),$$

which is 0 for odd m. For $m = 2j$, we have

$$\int_{-a}^{a} x^{2j}(a^2-x^2)^{s-3/2}\,dx = \frac{\Gamma(j+1/2)\Gamma(s-1/2)a^{2j+2s-2}}{\Gamma(s+j)}.$$

Hence

$$\begin{aligned} G_{p,q}(s) &= \sum_{j=0}^{p-1} \binom{2p-1}{2j} \frac{\Gamma(s-1/2)\Gamma(j-1/2}{\Gamma(s+j)} \\ &\quad \cdot \int_{-1}^{1} (1-y)^{s+2q-j-2}(1+y)^{s+j-1}\,dy \\ &= \sum_{j=0}^{p-1} \frac{(2p-1)!\pi}{2^{2j} j!(2p-2j-1)!} \cdot \frac{\Gamma(s-1/2)}{\Gamma(s+q+j-1/2)} \\ &\quad \cdot \frac{\Gamma(s+2q+j-1)}{\Gamma(s+q+j)}. \end{aligned}$$

The second factor in the jth summand has poles at half integers between $\frac{1}{2}$ and $\frac{3}{2}-q-j \geq \frac{5}{2}-p-q$. The third factor in the jth summand is a polynomial (unless $q=0$, when it equals $1/(s+j-1)$) which vanishes at $s=1-p-q$ for all j if $q>p$ and for all j except $j=0$ if $q=p$. This leads to the assertions of the proposition. When $p>q=0$, the value of G at $s=1-p-q$ is given by

$$\begin{aligned} G_{p,0}(1-p) &= \sum_{j=0}^{p-1} \frac{(2p-1)!\pi}{2^{2j} j!(2p-2j-1)!} \\ &\qquad \cdot (-1)^j \frac{\Gamma(p-j+1/2)}{\Gamma(p+1/2)} \cdot \frac{-1}{p+j} \\ &= \pi \sum_{j=0}^{p-1} \frac{(-1)^{j+1}(p-1)!}{j!(p-j)!} = \frac{(-1)^p \pi}{p}. \end{aligned}$$

The proof of the main theorem. Note that the function $N(s)$ in Proposition 3 is a linear combination of $F_{p,q}(s)$ and $G_{p,q}(s)$ with $p+q=m+1$. More precisely, we have

$$\begin{aligned} N(s) = \frac{1}{2} \sum_{p+q=m+1\,,\,0\leq p\leq m+1} & \frac{B_{2p}B_{2q}}{(2p)!(2q)!}[F_{p,q}(s) - 2^{2p}G_{p,q}(s)] \\ &+ \frac{\delta_{0m}}{8(s-1/2)}. \end{aligned}$$

If m is a positive even integer, then $p \neq q$. Since

$$\begin{aligned} F_{p,q}(-m) &= F_{p,q}(1-p-q) = 0, \quad \text{for } p\neq q, \\ G_{p,q}(-m) &= G_{p,q}(1-p-q) = 0, \quad \text{for } p\neq q,\, p\neq 0,\, q\neq 0, \end{aligned}$$

it follows

$$\begin{aligned} N(-m) &= \frac{B_{2m+2}}{2(2m+2)!}[-2^{2m+2}G_{m+1,0}(-m) - G_{0,m+1}(-m)] \\ &= \frac{(-1)^m B_{2m+2}(2^{2m+2}+1)\pi}{(2m+2)(2m+2)!}. \end{aligned}$$

If m is a positive odd integer, then it is possible that $p = q = (m+1)/2$. Hence

$$\begin{aligned} N(-m) &= \frac{B_{2m+2}}{2(2m+2)!}[-2^{2m+2}G_{m+1,0}(-m) - G_{0,m+1}(-m)] \\ &\quad + \frac{1}{2}\left(\frac{B_{2m+1}}{(m+1)}\right)^2 [F_{(m+1)/2,(m+1)/2}(-m) \\ &\qquad - 2^{m+1}G_{(m+1)/2,(m+1)/2}(-m)] \\ &= \frac{(-1)^m B_{2m+2}(2^{2m+2}+1)\pi}{(2m+2)(2m+2)!} \\ &\quad + \frac{1}{2}\left(\frac{B_{2m+1}}{(m+1)!}\right)^2 \left\{\frac{-2\pi(2^m m!)^2}{(2m+1)!} + \frac{4\pi(2^m m!)^2}{(2m+1)!}\right\} \\ &= \frac{(-1)^m B_{2m+2}(2^{2m+2}+1)\pi}{(2m+2)(2m+2)!} + \frac{1}{2}\left(\frac{B_{2m+1}}{m+1}\right)^2 \cdot \frac{2^{2m+1}\pi}{(2m+1)!}. \end{aligned}$$

But $B_l = 0$ if l is a positive odd integer greater than 1. Consequently, we have for any positive integer m,

$$N(-m) = \frac{(-1)^m B_{2m+2}(2^{2m+2}+1)\pi}{(2m+2)(2m+2)!} + \frac{1}{2}\left(\frac{B_{2m+1}}{m+1}\right)^2 \cdot \frac{2^{2m+1}\pi}{(2m+1)!}.$$

Hence

$$\begin{aligned} \tilde{\zeta}(-m) &= -\frac{(2m+1)!}{2^{2m+1}\pi}N(-m) \\ &= -\frac{1}{2}\left(\frac{B_{2m+1}}{m+1}\right)^2 + \frac{(-1)^{m+1}B_{2m+2}(2^{2m+2}+1)}{2^{2m+1}(2m+2)^2}. \end{aligned}$$

For the case $m = 0$, a direct computation from Proposition 3 yields

$$\tilde{\zeta}(0) = \frac{1}{48}.$$

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