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A NOTE ON THE STABILITY THEOREM OF J. L. BARBOSA AND M. DO CARMO FOR CLOSED SURFACES OF CONSTANT MEAN CURVATURE

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The theorem of Barbosa and do Carmo asserts that the only stable compact hypersurface of constant mean curvature in R^{n+1} is the round n-sphere. We present an elementary proof of this fact by considering the 2-parameter family $y = s(x + t\xi)$ where x is the constant mean curvature immersion and ξ is the unit normal vector field.

I. Introduction. Let M be a compact oriented n-manifold and $x\colon M \to R^{n+1}$ an immersion of M into R^{n+1}. For such an immersion we compute the n-area $A(x)$

$$A(x) = \int_M dS \tag{1}$$

where dS is the n-area element on M induced by the immersion x. We can also compute the "oriented" volume $V(x)$ enclosed by the immersed surface $x(M)$. It is given by the formula

$$V(x) = \frac{1}{n+1}\int_M (x\cdot\xi)\,dS \tag{2}$$

where ξ is the unit normal vector field determined by the orientation of M and the immersion x.

Let $x_t\colon (-\varepsilon, \varepsilon)\times M \to R^{n+1}$ be a one-parameter family of immersions of M into R^{n+1} with $x_0 = x$. A necessary and sufficient condition that the area functional $A(x_t)$ have a critical value at $t = 0$ for all variations x_t for which $V(x_t)$ is constant is that the immersed surface have constant mean curvature H. Furthermore, such an immersion is said to be stable if for all volume-preserving perturbations the second derivative of $A(x_t)$ at $t = 0$ is non-negative.

In a recent paper [**1**] J. L. Barbosa and M. do Carmo proved the following theorem.

THEOREM [**1**]. *Let M be a compact oriented n-manifold and let $x\colon M \to R^{n+1}$ be an immersion with non-zero constant mean curvature*

H. Then x is stable if and only if $x(M) \subset R^{n+1}$ is a (round) sphere S^n in R^{n+1}.

The stability of the round sphere follows from the isoperimetric inequality. In their proof of the theorem Barbosa and do Carmo consider a particular variation vector field whose first-order change of volume is zero and show that the appropriate second variation is negative unless the surface $x(M)$ is a round sphere. The purpose of this paper is to exhibit a simple one-parameter family of immersions which preserves volume, allows easy calculation of the area and enables us to prove the stability theorem. The family is explicitly described and its variation vector field is precisely the one considered in [**1**].

II. The alternate proof. Let $x\colon M \to R^{n+1}$ be the given compact immersion where we suppose that $x(M)$ has constant mean curvature H. Let $x_t = x + t\xi$ be the one-parameter family of parallel surfaces to x. It is easily seen that x_t has the same unit normal vector field as x. Furthermore the area $A(x_t)$ and volume $V(x_t)$ enclosed by x_t are easily computed.

$$A(x_t) = \int_M \prod_{i=1}^{n} (1 - k_i t)\, dS \tag{3}$$

where k_i are the principal curvatures of $x = x_0$. This is a polynomial of degree n in t and may be expanded in the form

$$\begin{aligned}
A(x_t) &= a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n,\\
a_0 &= \int dS = A(x_0),\\
a_1 &= -\int (k_1 + \cdots + k_n)\, dS = -nHa_0,\\
a_2 &= \int H_2\, dS, \quad H_2 = \prod_{i<j} k_i k_j,\\
a_k &= (-1)^k \int H_k\, dS, \quad H_k = \sum_{i_1 < \cdots < i_k} k_{i_1} k_{i_2} \cdots k_{i_k}.
\end{aligned}$$

This is an essentially well-known formula. For the case $n = 2$ see Stoker [**2**, p. 352].

The other key formula is also well known, see Stoker [**2**, p. 352], namely

$$dV(x_t)/dt = A(x_t) \quad \text{so that} \tag{5}$$

(6) $$V(x_t) = v_0 + v_1 t + v_2 t^2 + \cdots + v_{n+1} t^{n+1}$$

where $v_1 = a_0$, $2v_2 = a_1 = -nHa_0$, etc. We shall give proofs of (3) and (5) in the appendix. The family x_t is not volume preserving. In order to obtain a volume-preserving family we apply the appropriate homothety. Namely, let $y = sx_t$ be a two-parameter family of immersions. Clearly we have

(7) $$A(sx_t) = s^n A(x_t) = s^n(a_0 + a_1 t + \cdots + a_n t^n)$$

(8) $$V(sx_t) = s^{n+1} V(x_t) = s^{n+1}(v_0 + v_1 t + \cdots + v_{n+1} t^{n+1}).$$

We now determine $s = s(t)$ by setting $V(sx_t) = v_0$. By use of formula (8) and the binomial expansion we obtain the series for s^n (needing terms only through t^2) and substitute into (7). Calling $A(t) \equiv A[s(t)x_t]$ we find

(9) $$\begin{aligned} A(t) = a_0 &+ \left[-\left(\frac{n}{n+1}\right)\left(\frac{v_1}{v_0}\right) a_0 + a_1\right] t \\ &+ \left\{\left[\frac{n(2n+1)}{2(n+1)^2}\left(\frac{v_1}{v_0}\right)^2 - \left(\frac{n}{n+1}\right)\left(\frac{v_2}{v_0}\right)\right] a_0 \right. \\ &\qquad \left. + \left(\frac{-n}{n+1}\right)\left(\frac{v_1}{v_0}\right) a_1 + a_2\right\} t^2 + \cdots . \end{aligned}$$

The fact that $A'(0) = 0$ in (9) leads to

(10) $$a_0 + (n+1)Hv_0 = 0.$$

Substituting the identities in (6) and (10) into the coefficient of t^2 in (9) leads to

(11) $$\begin{aligned} A''(0)/2 &= -\int_M \left[\left(\frac{n^2-n}{2}\right) H^2 - H_2\right] dS \\ &= -\frac{1}{2n}\int_M \left(\sum_{i<j}(k_i - k_j)^2\right) dS. \end{aligned}$$

The second equation in (11) is seen as follows [**1**, p. 348]

$$\begin{aligned} (n-1)n^2H^2 - 2nH_2 &= (n-1)\left(\sum_i k_i\right)^2 - 2n\left(\sum_{i<j} k_i k_j\right) \\ &= (n-1)\left(\sum_i k_i^2\right) + 2(n-1)\sum_{i<j} k_i k_j - 2n\sum_{i<j} k_i k_j \\ &= (n-1)\left(\sum_i k_i^2\right) - 2\sum_{i<j} k_i k_j = \sum_{i<j}(k_i - k_j)^2. \end{aligned}$$

From (11) we see that if x is not all umbilic then $A''(0)$ is negative and the immersion is unstable.

Finally, to find the variational vector field for the family $s(t)x_t = s(t)[x + t\xi]$ we differentiate to get

$$z = d(sx_t)/dt|_{t=0} = \dot{s}(0)x + s(0)\xi.$$

But $s(0) = 1$ and an easy calculation using (8) gives $\dot{s}(0) = H$ so that $z = Hx + \xi$ and the normal component is $g = z \cdot \xi = H(x \cdot \xi) + 1$ which is the variation used by Barbosa and do Carmo.

Appendix.

Proof of (3). Introduce local coordinates $(u_1, \dots, u_n)$ with corresponding maps $x(u_1, \dots, u_n)$ and normal vector $\xi(u_1, \dots, u_n)$. Denote by $x_i = \partial x/\partial u_i$ and $\xi_i = \partial \xi/\partial u_i$. The metric on M induced by the map x is given by the matrix $g = (g_{ij})$ where $g_{ij} = (x_i \cdot x_j)$ and the element of area for the immersion is $dS = \sqrt{|g|}du_1\, du_2 \cdots du_n$ where $|g| = \det(g)$. For the immersion $x + t\xi$ the corresponding metric tensor is

$$\tilde{g}_{ij} = (x + t\xi)_i \cdot (x + t\xi)_j = g_{ij} - 2th_{ij} + t^2\gamma_{ij}$$

where $h_{ij} = -(x_i \cdot \xi_j)$ are the components of the second fundamental form and $\gamma_{ij} = (\xi_i \cdot \xi_j)$ determine the third fundamental form. We set $g = (g_{ij})$, $h = (h_{ij})$, $\gamma = (\gamma_{ij})$ and compute

$$\det(\tilde{g}) = \det(g) \cdot \det[I - g^{-1}(2th - t^2\gamma)].$$

But the eigenvalues of $g^{-1}(2th - t^2\gamma)$ are just $2tk_i - t^2k_i^2$ where k_i are the principal curvatures of x. Thus

$$\det(\tilde{g}) = \det(g) \cdot \prod_i (1 - k_i t)^2$$

giving us

$$dS_t = \sqrt{|\tilde{g}|}du_1 du_2 \cdots du_n = \prod_i (1 - k_i t)\, dS.$$

Proof of (5). It is sufficient to prove (5) when $t = 0$.

$$V(x_t) = \frac{1}{n+1} \int_M ((x + t\xi) \cdot \xi) \prod_i (1 - k_i t)\, dS.$$

Therefore

$$\frac{d}{dt} V(x_t)|_{t=0} = \frac{1}{n+1} \int_M [1 - nH(x \cdot \xi)]\, dS.$$

Thus it suffices to show

$$\int_M [1 + H(x \cdot \xi)]\, dS = 0.$$

A proof of this identity in the case $n = 2$ may be found in Stoker [**2**, p. 303]. It is equivalent to (10). For the general case consider the $(n-1)$-form on M given by

$$\omega = \{dx, \ldots (n-1) \text{ times } \ldots, dx, x, \xi\}$$

where we have described each column of an $(n+1) \times (n+1)$ matrix and ω is the determinant of the matrix.

$$d\omega = \{dx, \ldots, dx, dx, \xi\} + \{dx, \ldots, dx, x, d\xi\}.$$

A straightforward calculation gives

$$\{dx, \ldots, dx, dx, \xi\} = n!\, dS,$$
$$\{dx, \ldots, dx, x, d\xi\} = n! H(x \cdot \xi)\, dS.$$

Thus $d\omega = n![1 + H(x \cdot \xi)]\, dS$ and the assertion follows.

References

[1] J. L. Barbosa and M. do Carmo, *Stability of hypersurfaces of constant mean curvature*, Math. Zeitschrift, **185** (1984), 339–353.

[2] J. L. Stoker, *Differential Geometry*, Wiley–Interscience, (1969).

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