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THE p-PARTS OF BRAUER CHARACTER DEGREES IN p-SOLVABLE GROUPS

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Let G be a finite group. Fix a prime integer p and let e be the largest integer such that p^e divides the degree of some irreducible Brauer character of G with respect to the same prime p. The primary object of this paper is to obtain information about the structure of Sylow p-subgroups of a finite p-solvable group G in knowledge of e.

As applications, we obtain a bound for the derived length of the factor group of a solvable group G relative to its unique maximal normal p-subgroup in terms of the arithmetic structure of its Brauer character degrees and a bound for the derived length of the factor group of G relative to its Fitting subgroup in terms of the maximal integer e when p runs through the prime divisors of the order of G.

All groups considered are finite. Let G be a group and p be a prime. We denote by $\mathrm{IBr}_p(G)$ the set of irreducible Brauer characters of G with respect to the prime p. For the same prime p, let $e_p(G)$ be the largest integer e such that p^e divides $\varphi(1)$ for some $\varphi \in \mathrm{IBr}_p(G)$. Let P be a Sylow p-subgroup of G. Then the Sylow p-invariants of G are defined as follows:

(1) $b_p(G)$, where $p^{b_p(G)}$ is the order of P;

(2) $c_p(G)$, the class of P, that is, the length of the (upper or) lower central series of P;

(3) $dl_p(G)$, the length of the derived series of P;

(4) $ex_p(G)$, where $p^{ex_p(G)}$ is the exponent of P, that is, the greatest order of any element of P.

For a p-solvable group G, we let $l_p(G)$ and $r_p(G)$ denote the p-length and p-rank (respectively) of G, i.e. $r_p(G)$ is the largest integer r such that p^r is the order of a p-chief factor of G.

We give a linear bound for $r_p(G/O_p(G))$ and a logarithmic bound for $l_p(G/O_p(G))$ in terms of $e_p(G)$. Then, using induction on $l_p(G/O_p(G))$, we obtain bounds for $c_p(G/O_p(G))$, $dl_p(G/O_p(G))$ and $ex_p(G/O_p(G))$ in terms of $e_p(G)$.

As one application, we bound the derived length of $G/F(G)$ for a solvable group G in terms of $f(G)$, where

$$f(G) = \max\{e_p(G) \mid p \mid |G|\}$$

and $F(G)$ is the Fitting subgroup of G.

1. p-rank and p-length. For p-solvable G, we bound in this section $r_p(G/O_p(G))$ and $l_p(G/O_p(G))$ in terms of $e_p(G)$.

LEMMA 1.1. *Let G have a nilpotent normal p-complement M and let $O_p(G) = 1$. Then*
(1) $b_p(G) \leq 2e_p(G)$;
(2) *if $|G|$ is odd, then* $b_p(G) \leq e_p(G)$;
(3) *if a Sylow p-subgroup of G is abelian, then* $b_p(G) \leq e_p(G)$.

Proof. Let $\Phi(M)$ denote the Frattini subgroup of M. Since M is nilpotent, $M' \leq \Phi(M)$ and hence $M/\Phi(M)$ is abelian (see Gorenstein [**3**, Chapter 6, Theorem 1.6]). Notice that $\Phi(M) \triangleleft G$. Consider the factor group $G/\Phi(M)$. Let P be a Sylow p-subgroup of G. Then

$$G/\Phi(M) = MP/\Phi(M) = M/\Phi(M) \cdot P\Phi(M)/\Phi(M).$$

Write $G_1 = G/\Phi(M)$, $M_1 = M/\Phi(M)$ and $P_1 = P\Phi(M)/\Phi(M)$. Then M_1 is an abelian normal p-complement of G_1, and $P_1 = P\Phi(M)/\Phi(M) \cong P/P \cap \Phi(M) \cong P$. By Huppert [**5**, Chapter 3, Satz 3.18], P acts faithfully on M_1, and hence P_1 acts faithfully on M_1. Then $O_p(G_1) = 1$ and P_1 acts faithfully on $\mathrm{Irr}(M_1)$ which is an abelian p'-group.

By Corollary 2.4 of Passman [**16**], there exists $\theta \in \mathrm{Irr}(M_1)$ such that $|I_{P_1}(\theta)| \leq |P_1|^{1/2}$. So $|P_1 : I_{P_1}(\theta)| \geq |P_1|^{1/2}$. By Clifford's Theorem, $p^{e_p(G_1)} \geq |P_1|^{1/2}$. Since $|P_1| = |P|$ and $e_p(G) \geq e_p(G_1)$, $2e_p(G) \geq b_p(G)$. This gives (1).

If $|G|$ is odd or P is abelian, then we can apply Lemma 2.2 and Corollary 2.4 of Passman [**16**] to conclude that there exists $\theta \in \mathrm{Irr}(M_1)$ such that $I_{P_1}(\theta) = 1$. By Clifford's Theorem, $|P_1| \leq p^{e_p(G_1)}$, that is, $b_p(G) \leq e_p(G_1)$. So (2) and (3) hold. □

LEMMA 1.2. *Let G have a normal p-complement M and let $O_p(G) = 1$, where p is an odd prime. Let $M = M_1 \times \cdots \times M_n$, where all M_i's are isomorphic nonabelian simple groups. Then*
(1) $b_p(G) \leq 2e_p(G)$;
(2) *if* $e_p(G) = 1$, *then* $b_p(G) = 1$.

Proof. Let P be a Sylow p-subgroup of G. Notice that for any $x \in P$ and M_i, $xM_ix^{-1} \in \{M_1, \dots, M_n\}$. We write $N_i = \mathrm{Stab}_P(M_i)$ and $C = \bigcap_{i=1}^n N_i$. Also, we let

$$C_i = C_C(M_i) = \{x \in C \mid xyx^{-1} = y \text{ for all } y \in M_i\}.$$

For each i, $C/C_i \leq \mathrm{Out}(M_i)$. By Lemma 1.3 of Gluck and Wolf [2], C/C_i is a cyclic p-group. Let $\Omega_1(C/C_i) = \langle x \in C/C_i \mid x^p = 1\rangle$, then $\Omega_1(C/C_i)$ is a cyclic group of order p.

On the other hand, since C/C_i acts faithfully on M_i and since $p \nmid |M_i|$, C/C_i acts faithfully on $\mathrm{Irr}(M_i)$ (see Isaacs [9, Theorem 6.32]). Thus

$$\bigcap_{\theta \in \mathrm{Irr}(M_i)} C_{C/C_i}(\theta) = 1.$$

So, there exists some $\theta_i \in \mathrm{Irr}(M_i)$ such that

$$\Omega_1(C/C_i) \cap C_{C/C_i}(\theta_i) = 1.$$

This forces that $C_{C/C_i}(\theta_i) = 1$, that is, $C_C(\theta_i) = C_i$.

Let $\theta = \theta_1 \times \cdots \times \theta_n$. Then $\theta \in \mathrm{Irr}(M) = \mathrm{IBr}_p(M)$ and $I_C(\theta) = \bigcap_{i=1}^n C_C(\theta_i) = \bigcap_{i=1}^n C_i$. Since P acts faithfully on M, $\bigcap_{i=1}^n C_i = 1$, and hence $I_C(\theta) = 1$. Since $C \lhd P$, $MC \lhd G$ and hence $e_P(MC) \leq e_p(G)$. Applying Clifford's Theorem to the group MC, we have $|C| = |C : I_C(\theta)| \leq p^{e_p(MC)}$. Hence

$$\text{(A)} \qquad |C| \leq p^{e_p(G)}.$$

On the other hand, P/C is a permutation group on the set $\{M_1, \dots, M_n\}$ and p is an odd prime. So, by Corollary 1 of Gluck [1], we may assume without loss of generality that $\mathrm{Stab}_{P/C}\{M_1, \dots, M_t\} = 1$, that is $\mathrm{Stab}_P\{M_1, \dots, M_t\} = C$ for some suitable $t \in \{1, \dots, n\}$.

Choose $\theta_j \in \mathrm{Irr}(M_j)$ with $\theta_j \neq 1$, $j = 1, \dots, t$. Let $\theta = \theta_1 \times \cdots \times \theta_t \times 1 \times \cdots \times 1$. Then $\theta \in \mathrm{Irr}(M) = \mathrm{IBr}_p(M)$ and $I_P(\theta) \leq \mathrm{Stab}_p\{M_1, \dots, M_t\} = C$. Applying Clifford's Theorem to the group G, we get $|P : I_P(\theta)| \leq p^{e_p(G)}$. Hence

$$\text{(B)} \qquad |P : C| \leq p^{e_p(G)}.$$

Combining (A) with (B), we obtain $|P| \leq p^{2e_p(G)}$. This gives (1).

Suppose that $e_p(G) = 1$. By (A) and (B), we know that $|C| \leq p$ and $|P : C| \leq p$.

In the following, we want to show that either $C = P$ or $C = 1$. Assume not. Then we have $|C| = p$ and $|P| = p^2$.

Since P acts faithfully on M, C acts faithfully on M. Since $|C| = p$, there exists some M_k such that $C_k = C_C(M_k) = 1$, that is, C acts faithfully on M_k. Thus C acts faithfully on $\mathrm{Irr}(M_k)$. Hence there exists some $\theta_k \in \mathrm{Irr}(M_k)$ such that $\theta_k \neq 1$ and $C_C(\theta_k) = 1$.

Since $\mathrm{Stab}_p\{M_{t+1}, \ldots, M_n\} = \mathrm{Stab}_p\{M_1, \ldots, M_t\}$, we may assume without loss of generality that $k = 1$. Choose $\theta_j \in \mathrm{Irr}(M_j)$ with $\theta_j \neq 1$, $j = 2, \ldots, t$. Let $\theta = \theta_1 \times \theta_2 \times \cdots \times \theta_t \times 1 \times \cdots \times 1$. Then $\theta \in \mathrm{Irr}(M) = \mathrm{IBr}_p(M)$ and $I_P(\theta) \leq \mathrm{Stab}_P\{M_1, \ldots, M_t\} = C$. So $I_P(\theta) = I_C(\theta) = \bigcap_{i=1}^t C_C(\theta_i) = 1$ (because of $C_C(\theta_1) = 1$). Thus there exists $\theta \in \mathrm{Irr}(M) = \mathrm{IBr}_p(M)$ such that $I_P(\theta) = 1$. By Clifford's Theorem, $|P| = |P : I_P(\theta)| \leq p$. This contradicts to $|P| = p^2$. So (2) holds. □

THEOREM 1.3. *Let G be p-solvable. Then*

$$r_p(G/O_p(G)) \leq 2e_p(G).$$

Proof. By induction on $|G|$, we may assume without loss of generality that $O_p(G) = 1$.

If $p = 2$, then G is solvable, and we are done by Manz and Wolf [**13**, Theorem 2.3]. In the following, we assume that p is an odd prime.

Let M be a minimal normal subgroup of G and let $N/M = O_p(G/M)$. By the inductive hypothesis, we may assume that $N/M \neq 1$.

Since G is p-solvable and $O_p(G) = 1$, we have the following two cases:

Case 1. M is an elementary abelian q-group for some prime $q \neq p$;

Case 2. M Is the direct product of isomorphic nonabelian simple p'-group.

Consider the group N. Notice that $M = O_{p'}(N)$ and $O_p(N) = 1$. Applying Lemma 1.1 (1) and Lemma 1.2 (1) to the group N, we get $b_p(N) \leq 2e_p(N)$. Since $N \triangleleft G$, $e_p(N) \leq e_p(G)$ by Clifford's Theorem. Hence $b_p(N) \leq 2e_p(G)$.

By the inductive hypothesis, the p-rank $r_p(G/N)$ of G/N does not exceed $2e_p(G/N) \leq 2e_p(G)$. Since $r_p(G) \leq \max\{r_p(G/N), b_p(N)\}$, $r_p(G) \leq 2e_p(G)$. □

By using Lemma 1.1 (2) instead of Lemma 1.1 (1), the same proof yields the following improvement for groups of odd order.

THEOREM 1.4. *Let G be a group of odd order. Then*

$$r_p(G/O_p(G)) \leq e_p(G).$$

We note that Theorem 1.3 and Theorem 1.4 improve Theorem 2.1 of Manz [12], by the Fong-Swan Theorem.

THEOREM 1.5. *Let G be solvable and $e_p(G) < p$. Then*

$$r_p(G/O_p(G)) \leq e_p(G).$$

Proof. By induction on $|G|$, we may assume without loss of generality that $O_p(G) = 1$.

Let M be a minimal normal subgroup of G and let $N/M = O_p(G/M)$. By the inductive hypothesis, we may assume that $N/M \neq 1$. Since $O_p(G) = 1$, M is an elementary abelian q-group with $q \neq p$.

Consider the group N. Since $N \triangleleft G$ and $O_p(G) = 1$, $O_p(N) = 1$. Notice that N has a normal p-complement M and $e_p(N) \leq e_p(G) < p$.

Let P be a Sylow p-subgroup of N. Since $O_p(N) = 1$, P acts faithfully on M by conjugation. Hence P acts faithfully on $\mathrm{Irr}(M)$. Since M is an elementary abelian q-group, $\mathrm{Irr}(M)$ is an abelian q-group. Let $\Omega_1, \dots, \Omega_n$ be the P-orbits of $\mathrm{Irr}(M)$ and $p^f = \max\{|\Omega_1|, \dots, |\Omega_n|\}$. We may assume without loss of generality that $|\Omega_1| = p^f$. Let $\theta_1 \in \Omega_1$. Applying Clifford's Theorem to the group N, we get $|\Omega_1| = |P : I_P(\theta_1)| \leq p^{e_p(N)}$. So $|\Omega_1| < p^p$. By Corollary 2.4 of Passman [16], there exists $\theta \in \mathrm{Irr}(M)$ such that $I_P(\theta) = 1$. We apply Clifford's Theorem to conclude that $|P| = |P : I_P(\theta)| \leq p^{e_p(N)}$. So $b_p(N) \leq e_p(N)$, and hence $b_p(N) \leq e_p(G)$.

By the inductive hypothesis, the p-rank $r_p(G/N)$ of G/N does not exceed $e_p(G/N) \leq e_p(G)$. Since $r_p(G) \leq \max\{r_p(G/N), b_p(N)\}$, $r_p(G) \leq e_p(G)$. □

Recall that the rank $r(G)$ of G is the maximum dimension of all chief-factors of G and $f(G) = \max\{e_p(G) \mid p \mid |G|\}$.

COROLLARY 1.6. *Let G be solvable. Then*
(1) $r(G/F(G)) \leq 2f(G)$;
(2) *if $|G|$ is odd, then* $r(G/F(G)) \leq f(G)$.

Proof. Let p be a prime number such that $p \mid |G|$. Then, by Theorem 1.3 and Theorem 1.4, $r_p(G/O_p(G)) \leq 2e_p(G) \leq 2f(G)$, and if $|G|$ is odd, $r_p(G/O_p(G)) \leq e_p(G) \leq f(G)$. Since $O_p(G) \leq F(G)$, this yields that $r(G/F(G)) \leq 2f(G)$, and if $|G|$ is odd, $r(G/F(G)) \leq f(G)$. $\square$

Combining Wolf [**17**, Theorem 2.3] with Theorem 1.3, we have

THEOREM 1.7. *Let G be p-solvable. Then*
(1) $l_p(G/O_p(G)) \leq 1+\log_p(2e_p(G))$ *if p is not a Fermat prime*; *and*
(2) $l_p(G/O_p(G)) \leq 2+\log_s(2e_p(G)/(p-1))$ *where* $s = (p^2-p+1)/p$.

2. Sylow p-invariants. In this section, we bound $c_p(G/O_p(G))$, $dl_p(G/O_p(G))$ and $ex_p(G/O_p(G))$ for a p-solvable group G in terms of $e_p(G)$. In particular, we show that if $e_p(G) = 1$, then a Sylow p-subgroup of $G/O_p(G)$ is elementary abelian. We also give bounds for $b_p(G/O_p(G))$.

LEMMA 2.1. *Let G have a normal p-complement and let $O_p(G) = 1$. Then*
(1) $dl_p(G) \leq e_p(G)$;
(2) $ex_p(G) \leq e_p(G)$;
(3) $c_p(G) \leq p^{e_p(G)-1}$.

Proof. Let P be a Sylow p-subgroup of G and let H be the normal p-complement of G. Then P acts faithfully on $\mathrm{Irr}(H)$.

Write $\Omega = \mathrm{Irr}(H)$. Let $\Omega_1, \ldots, \Omega_n$ be the P-orbits of Ω. Then P acts transitively on each Ω_i. Let $\varphi_i : P \to S(\Omega_i)$ be the homomorphism induced by the action, where $S(\Omega_i)$ is the permutation group on Ω_i. Then we can define a homomorphism from P into the direct product

$$S(\Omega_1) \times \cdots \times S(\Omega_n)$$

as follows:

$$\begin{aligned} &\varphi : P \to S(\Omega_1) \times \cdots \times S(\Omega_n), \\ &\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x)), \qquad x \in P. \end{aligned}$$

Since P acts faithfully on Ω, $\operatorname{Ker}\varphi = 1$, and hence φ is an injection. By the definition of φ, we know that

$$P \cong \varphi(P) \leq \varphi_1(P) \times \cdots \times \varphi_n(P),$$

where each $\varphi_i(P) \leq S(\Omega_i)$ is a p-group.

Since $dl(\varphi_i(P)) \leq dl(P)$, $\max\{dl(\varphi_i(P)) : i = 1, \ldots, n\} \leq dl(P)$. On the other hand, we have that

$$\begin{aligned} dl(P) &= dl(\varphi(P)) \\ &\leq dl(\varphi_1(P) \times \cdots \times \varphi_n(P)) \\ &= \max\{dl(\varphi_i(P)) : i = 1, \ldots, n\}. \end{aligned}$$

Hence, $dl_p(G) = \max\{dl(\varphi_i(P)) : i = 1, \ldots, n\}$. Similarly,

$$ex_p(G) = \max\{ex_p(\varphi_i(P)) : i = 1, \ldots, n\},$$

and

$$c_p(G) = \max\{c_p(\varphi_i(P)) : i = 1, \ldots, n\}.$$

Let $p^f = \max\{|\Omega_i| : i = 1, \ldots, n\}$. We may assume without loss of generality that $|\Omega_1| = p^f$. By Huppert [5, Chapter 3, Satz 15.3], $dl_p(G) \leq f$, $ex_p(G) \leq f$ and $c_p(G) \leq p^{f-1}$. Choose $\theta \in \Omega_1$. By Clifford's Theorem, $|\Omega_1| = |P : I_P(\theta)| \leq p^{e_p(G)}$. So $f \leq e_p(G)$. Thus the conclusions (1)–(3) hold. □

THEOREM 2.2. *Let G be p-solvable. Then*
(1) $dl_p(G/O_p(G)) \leq l_p(G/O_p(G))e_p(G)$;
(2) $ex_p(G/O_p(G)) \leq l_p(G/O_p(G))e_p(G)$;
(3) $c_p(G/O_p(G)) \leq l_p(G/O_p(G))p^{e_p(G)-1}$.

Proof. Since $e_p(G) = e_p(G/O_p(G))$, we may assume without loss of generality that $O_p(G) = 1$. We use induction on $l_p(G)$.

Write $E = O_{p'}(G)$ and $M = O_{p',p}(G)$. Since $O_p(G) = 1$, $O_p(M) = 1$. Clearly, M has a normal p-complement E. Thus, by Lemma 2.1, we have that

$$dl_p(M) \leq e_p(M), \quad ex_p(M) \leq e_p(M), \quad \text{and} \quad c_p(M) \leq p^{e_p(M)-1}.$$

Since $M \lhd G$, $e_p(M) \leq e_p(G)$. Hence,

$$\text{(A)} \quad dl_p(M) \leq e_p(G), \qquad ex_p(M) \leq e_p(G), \qquad c_p(M) \leq p^{e_p(G)-1}.$$

Since $M = O_{p',p}(G)$, $O_p(G/M) = 1$ and $l_p(G/M) = l_p(G) - 1$. Then the induction yields that

$$\begin{aligned} &dl_p(G/M) \leq l_p(G/M)e_p(G/M), \\ &ex_p(G/M) \leq l_p(G/M)e_p(G/M), \quad \text{and} \\ &c_p(G/M) \leq l_p(G/M)p^{e_p(G/M)-1}. \end{aligned}$$

Hence, (B)

$$dl_p(G/M) \le (l_p(G)-1)e_p(G),$$
$$ex_p(G/M) \le (l_p(G)-1)e_p(G),$$
$$c_p(G/M) \le (l_p(G)-1)p^{e_p(G)-1}.$$

By (A) and (B), we have the conclusions. □

Combining Theorem 1.7 with Theorem 2.2, we have

COROLLARY 2.3. *Let G be p-solvable. Then*
(1) *if p is not a Fermat prime, then*

$$dl_p(G/O_p(G)) \le (1+\log_p(2e_p(G)))e_p(G),$$
$$ex_p(G/O_p(G)) \le (1+\log_p(2e_p(G)))e_p(G),$$
$$c_p(G/O_p(G)) \le (1+\log_p(2e_p(G)))p^{e_p(G)-1};$$

(2) *if p is a Fermat prime, then*

$$dl_p(G/O_p(G)) \le (2+\log_s(2e_p(G)/(p-1)))e_p(G),$$
$$ex_p(G/O_p(G)) \le (2+\log_s(2e_p(G)/(p-1)))e_p(G),$$
$$c_p(G/O_p(G)) \le (2+\log_s(2e_p(G)/(p-1)))p^{e_p(G)-1},$$

where $s = (p^2-p+1)/p$.

In the rest of this section, we give some improvements on the bounds we just got for the cases $e_p(G) = 1$, 2 and $e_p(G) < p$.

COROLLARY 2.4. *Let G be solvable and $e_p(G) < p$. Then*
(1) *if p is not a Fermat prime, then*

$$dl_p(G/O_p(G)) \le e_p(G),$$
$$ex_p(G/O_p(G)) \le e_p(G),$$
$$c_p(G/O_p(G)) \le p^{e_p(G)-1};$$

(2) *if p is a Fermat prime, then*

$$dl_p(G/O_p(G)) \le 2e_p(G),$$
$$ex_p(G/O_p(G)) \le 2e_p(G),$$
$$c_p(G/O_p(G)) \le 2p^{e_p(G)-1}.$$

Proof. By Theorem 1.5 and Wolf [**17**, Theorem 2.3], $l_p(G/O_p(G)) \le 1+\log_p(e_p(G))$, if p is not a Fermat prime; and $l_p(G/O_p(G)) \le 2+\log_s(e_p(G)/(p-1))$, if p is a Fermat prime, where $s = (p^2-p+1)/p$. Since $e_p(G) < p$, $l_p(G/O_p(G)) \le 1$, if p is not a Fermat prime; and

$l_p(G/O_p(G)) \le 2$, if p is a Fermat prime. Hence the conclusions follow from Theorem 2.2. □

For a group G, Michler [**14**] and Okuyama [**15**] show that if $e_p(G) = 0$, then G has a normal Sylow p-subgroup. The assertion for p-solvable groups is elementary and well-known.

THEOREM 2.5. *Let G be p-solvable and $e_p(G) = 1$. Then*

$$r_p(G/O_p(G)) \le 1 .$$

Proof. The case $p = 2$ is done by Theorem 1.5. In the following, we assume that p is an odd prime. By induction on $|G|$, we may assume without loss of generality that $O_p(G) = 1$.

Let M be a minimal normal subgroup of G and $N/M = O_p(G/M)$. By the inductive hypothesis, we may assume that $N/M \neq 1$. Since G is p-solvable and $O_p(G) = 1$, we have the following two cases:

Case 1. M is an elementary abelian q-group for some prime $q \neq p$.

Case 2. M is the direct product of isomorphic nonabelian simple p'-groups.

Consider the group N. Since $N \triangleleft G$ and $O_p(G) = 1$, $O_p(N) = 1$. Then $1 \le e_p(N) \le e_p(G) = 1$. Thus $e_p(N) = 1$. Notice that N has a normal p-complement M. By Lemma 2.1, N/M is an abelian p-group. Applying Lemma 1.1 (3) and Lemma 1.2 (2) to the group N, we get $b_p(N) = 1$.

By the inductive hypothesis, the p-rank $r_p(G/N) \le 1$. Since $r_p(G) \le \max\{r_p(G/N), b_p(N)\}$, $r_p(G) \le 1$. □

Since $l_p(G) \le r_p(G)$ (see Huppert [**5**, Chapter 6, Hauptsatz 6.6 (c)]), we get the following corollary by combining Theorem 2.2 with Theorem 2.5.

COROLLARY 2.6. *Let G be p-solvable and $e_p(G) = 1$. Then a Sylow p-subgroup of $G/O_p(G)$ is an elementary abelian p-group.*

For $e_p(G) = 2$, there is no general result similar to Corollary 2.6. Let $G = S_3 \operatorname{wr} Z_2$. Then $e_2(G) = 2$ and $O_2(G) = 1$. The Sylow 2-subgroup of G is $Z_2 \operatorname{wr} Z_2$, which is not abelian.

However, for solvable groups, we have the following corollary.

COROLLARY 2.7. *Let G be solvable and $e_p(G) = 2$ with $p \geq 5$. Then*

(1) $dl_p(G/O_p(G)) \leq 2$;
(2) $ex_p(G/O_p(G)) \leq 2$;
(3) $c_p(G/O_p(G)) \leq p$.

Proof. For $p \geq 5$, by Theorem 1.5 and Wolf [**17**, Theorem 2.3], $l_p(G/O_p(G)) \leq 1$ if $e_p(G) = 2$. Hence the conclusions follow from Theorem 2.2. □

In closing this section, we include the following remark, which tells us that logarithmic bounds for the Sylow p-invariants of $G/O_p(G)$ in terms of $e_p(G)$ are probably the best bounds we can expect.

REMARK 2.8. Fix a prime p. Let $G_0 \neq 1$ be a p'-group. We construct groups by iterated wreath products as follows: let $G_1 = G_0 \operatorname{wr} Z_p$ and $G_2 = G_1 \operatorname{wr} Z_p$. Following this way, we have $G_n = G_{n-1} \operatorname{wr} Z_p$ for any natural number n.

By Hall and Higman [**4**, Lemma 3.5.1], $dl_p(G_n) = ex_P(G_n) = n$. Since $O_p(G_1) = 1$ and $|G_1|_p = p$, $e_p(G_1) = 1$. In the following, we use an induction argument on n to show that

$$p^{n-1} \leq e_p(G_n) \leq (p^n - 1)/(p-1).$$

Suppose that $p^{n-2} \leq e_p(G_{n-1}) \leq (p^{n-1} - 1)/(p-1)$. By the definition of G_n, $G_n = (G_{n-1} \times \underset{p\text{-times}}{\cdots} \times G_{n-1}) \rtimes Z_p$. Let $H_n = G_{n-1} \times \underset{p\text{-times}}{\cdots} \times G_{n-1}$. Then $H_n \triangleleft G_n$ and $e_p(H_n) = pe_p(G_{n-1})$. Hence $p^{n-1} \leq e_p(H_n) \leq e_p(G_n)$. In particular, $p^{n-1} \leq e_p(G_n)$.

On the other hand, let $\varphi \in \mathrm{IBr}_p(G_n)$ such that $\varphi(1) = p^{e_p(G_n)}m$. Choose $\theta \in \mathrm{IBr}_p(H_n)$ such that $\varphi \in \mathrm{IBr}_p(G_n \mid \theta)$. By Clifford's Theorem, $\varphi(1) = e\theta(1)$ with a positive integer e. Also, by Lemma 3.2 of Isaacs [**8**], φ is an irreducible constituent of θ^{G_n}. Thus $\varphi(1) \leq \theta^{G_n}(1) = |G_n : H_n|\theta(1) = p\theta(1)$. So $0 < e \leq p$. This yields that $e_p(G_n) \leq e_p(H_n) + 1$. Hence $e_p(G_n) \leq p(p^{n-1} - 1)/(p-1) + 1 = (p^n - 1)/(p-1)$.

Now we consider bounding the $b_p(G/O_p(G))$ for a p-solvable group G in terms of $e_p(G)$.

LEMMA 2.9. *Let G have a solvable normal p-complement H and let $O_p(G) = 1$. Then $b_p(G) \leq 2dl(H)e_p(G)$.*

Proof. We use an induction argument on $dl(H)$. Let P be a Sylow p-subgroup of G. Then P acts on H/H' by conjugation. Let $Q = C_p(H/H')$.

Since P/Q acts faithfully on H/H', P/Q acts faithfully on $\mathrm{Irr}(H/H')$ which is an abelian p'-group. By Corollary 2.4 of Passman **[16]**, there exists $\theta \in \mathrm{Irr}(H/H')$ such that $|C_{P/Q}(\theta)| \leq |P/Q|^{1/2}$. So $|P/Q : C_{P/Q}(\theta)| \geq |P/Q|^{1/2}$. Consider $\theta \in \mathrm{Irr}(H)$ with $H' \leq \mathrm{Ker}\,\theta$. By Clifford's Theorem, $|P : I_p(\theta)| \leq p^{e_p(G)}$. Since $|P : I_P(\theta)| = |P/Q : C_{P/Q}(\theta)|$, $|P/Q|^{1/2} \leq p^{e_p(G)}$. Hence $\log_p |P/Q| \leq 2e_p(G)$. If $Q = 1$, then we are done.

Next, we assume that $Q \neq 1$. We claim that Q acts faithfully on H'. Assume not. We may assume without loss of generality that Q acts trivially on H'. Since $Q = C_P(H/H')$, Q acts trivially on H/H'. Since $(|Q|, |H|) = 1$, Q acts trivially on H (see Huppert [**5**, Chapter 3, Hilfssatz 13.3 (b)]). But since P acts faithfully on H, we must have $Q = 1$. This contradicts to $Q \neq 1$.

Write $G_1 = H'Q$. Then G_1 has a normal p-complement H' and $O_p(G_1) = C_Q(H') = 1$. Furthermore, we claim that $G_1 \lhd G$. Since $H' \lhd G$ and $Q \lhd P$, we only need to show that $hQh^{-1} \subseteq H'Q$ for all $h \in H$. Let $q \in Q = C_p(H/H')$. Then $q^{-1}hqh^{-1} \in H'$. Hence $hqh^{-1} \in qH' \subseteq QH' = H'Q$. Thus $G_1 = H'Q \lhd G$. By Clifford's Theorem, $e_p(G_1) \leq e_p(G)$.

Since $dl(H') = dl(H) - 1 < dl(H)$, by induction, $\log_p |Q| \leq 2dl(H')e_p(G_1) = 2(dl(H) - 1)e_p(G_1)$. Hence,

$$\begin{aligned}
b_p(G) &= \log_p |P| \\
&= \log_p(|Q|\,|P/Q|) \\
&= \log_p |Q| + \log_p |P/Q| \\
&\leq 2(dl(H) - 1)e_p(G_1) + 2e_p(G) \\
&\leq 2(dl(H) - 1)e_p(G) + 2e_p(G) \\
&= 2dl(H)e_p(G),
\end{aligned}$$

which is the claim. □

The following Lemma is a corollary of Lemma 1.1.

LEMMA 2.10. *Let G be solvable, $O_p(G) = 1$ and P a Sylow p-subgroup of G. Let G have a normal p-complement. Then*

$$b_p(G) \leq 2e_p(F(G)P).$$

Proof. Since $O_p(G) = 1$, the Fitting subgroup $F(G)$ is a p'-group. By Huppert [**5**, Chapter 3, Satz 4.2], $C_G(F(G)) \leq F(G)$ and hence $C_P(F(G)) = 1$. Let $G_1 = F(G)P$. Then G_1 has a nilpotent normal p-complement $F(G)$ and $O_p(G_1) = C_P(F(G)) = 1$. By Lemma 1.1, $b_p(G_1) \leq 2e_p(G_1)$. Since $b_p(G_1) = b_p(G)$, $b_p(G) \leq 2e_p(F(G)P)$. □

To handle p-solvable groups with arbitrary p-length, we introduce the following definition.

DEFINITION 2.11. For a prime p and a positive integer n, we define $\lambda_p(n)$ and $\beta_p(n)$ by

$$\lambda_p(n) = \sum_{i=1}^{\infty} [n/p^i]$$

and

$$\beta_p(n) = \sum_{i=0}^{\infty} [n/(p-1)p^i].$$

PROPOSITION 2.12. *If p is a prime and n is a positive integer, then*

$$\lambda_p(n) \leq n-1 \quad \textit{and} \quad \beta_p(n) \leq 2n-1.$$

Proof. Since $\lambda_p(n) \leq (n-1)/(p-1)$, $\lambda_p(n) \leq n-1$. Since $p \geq 2$, $2(p-1)^2 - p \geq (p-1)^2 - (p-1)$, and hence $2n(p-1)^2 - np \geq (p-1)^2 - (p-1)$. So $(2n-1) \geq (np/(p-1)^2) - (1/(p-1))$. Since $\beta_p(n) \leq (np/(p-1)^2) - (1/(p-1))$, $\beta_p(n) \leq 2n-1$. □

THEOREM 2.13. *Let G be p-solvable and $O_{p'}(G)$ be solvable. Suppose that $O_p(G) = 1$. Then*
(1) $b_p(G) \leq 6dl(O_{p'}(G))e_p(O_{p',p}(G)) - 1$; *and*
(2) $b_p(G) \leq 4dl(O_{p'}(G))e_p(O_{p',p}(G)) - 1$ *unless p is a Fermat prime.*

Proof. Write $E = O_{p'}(G)$ and $M = O_{p',p}(G)$. Since $O_p(G) = 1$, $O_p(M) = 1$. Clearly, M has a solvable normal p-complement E. Thus $b_p(M) \leq 2dl(O_{p'}(G))e_p(O_{p',p}(G))$ by Lemma 2.9.

Let $b_p(M) = m$, hence $|M/E| = p^m$. By Wolf [**17**, Corollary 2.1], we have that
(1) $b_p(G) \leq m + \beta_p(m)$; and
(2) $b_p(G) \leq m + \lambda_p(m)$ unless p is a Fermat prime.
Applying Proposition 2.12, we obtain
(1) $b_p(G) \leq 3m - 1$; and
(2) $b_p(G) \leq 2m - 1$ unless p is a Fermat prime.

Since $m = b_p(M)$, we get

(1) $b_p(G) \leq 6dl(O_{p'}(G))e_p(O_{p',p}(G)) - 1$; and

(2) $b_p(G) \leq 4dl(O_{p'}(G))e_p(O_{p',p}(G)) - 1$ unless p is a Fermat prime. $\square$

Similarly, applying Lemma 2.10, we obtain the following Theorem.

THEOREM 2.14. *Let G be solvable, $O_p(G) = 1$ and P a Sylow p-subgroup of $O_{p',p}(G)$. Then*

(1) $b_p(G) \leq 6e_p(F(G)P) - 1$; *and*

(2) $b_p(G) \leq 4e_p(F(G)P) - 1$ *unless p is a Fermat prime.*

3. The derived length of solvable groups. Let $n = \prod_{i=1}^{k} p_i^{a_i}$ be the prime number decomposition of a natural number n $(a_i \neq 0)$. We define

$$\omega(n) = \sum_{i=1}^{k} a_i .$$

For a group G, we let

$$\omega(G) = \max\{\omega(\chi(1)) \mid \chi \in \mathrm{Irr}(G)\}$$

and

$$\omega_p(G) = \max\{\omega(\varphi(1)) \mid \varphi \in \mathrm{IBr}_p(G)\} .$$

Recall that $f(G) = \max\{e_p(G) \mid p \mid |G|\}$.

For a solvable group G, we obtain a bound for the derived length of $G/O_p(G)$ in terms of $\omega_p(G)$ and a quadratic bound for the derived length of $G/F(G)$ in terms of $f(G)$.

LEMMA 3.1. *Let G be solvable with $O_p(G) = 1$ and $l_p(G) = 1$. Then $dl(G) \leq 5\omega_p(G)$.*

Proof. Since $O_p(G) = 1$ and $l_p(G) = 1$, $e_p(G) \geq 1$. Thus $\omega_p(G) \geq e_p(G) \geq 1$. If $\omega_p(G) = 1$, then $dl(G) \leq 4$ by Huppert [7, Theorem 1]. So, $dl(G) \leq 5\omega_p(G)$, and we are done in this case.

In the following, we assume that $\omega_p(G) \geq 2$. We have two cases to consider.

Case 1. $O_{p',p}(G) = G$.

By Lemma 2.1, $dl(G/O_{p'}(G)) \leq e_p(G)$, and hence $dl(G/O_{p'}(G)) \leq \omega_p(G)$. Since $O_{p'}(G)$ is a p'-group, $\omega_p(O_{p'}(G)) = \omega(O_{p'}(G))$.

If $\omega(O_{p'}(G)) \geq 2$, then, by Huppert [**6**, Theorem 3], $dl(O_{p'}(G)) \leq 2\omega(O_{p'}(G))$, and hence $dl(O_{p'}(G)) \leq 2\omega_p(O_{p'}(G))$. Since $O_{p'}(G) \lhd G$, $\omega_p(O_{p'}(G)) \leq \omega_p(G)$ by Clifford's Theorem. So $dl(O_{p'}(G)) \leq 2\omega_p(G)$. Thus

$$\begin{aligned} dl(G) &\leq dl(O_{p'}(G)) + dl(G/O_{p'}(G)) \\ &\leq 2\omega_p(G) + \omega_p(G) = 3\omega_p(G). \end{aligned}$$

If $\omega(O_{p'}(G)) \leq 1$, then, by Isaacs and Passman [**10**, Theorem 6.1], $dl(O_{p'}(G)) \leq 3$, and hence

$$dl(G) \leq dl(O_{p'}(G)) + dl(G/O_{p'}(G)) \leq 3 + \omega_p(G).$$

Since $\omega_p(G) \geq 2$, $dl(G) \leq 2\omega_p(G) + \omega_p(G) = 3\omega_p(G)$.

Case 2. $O_{p',p,p'}(G) = G$.

Write $M = O_{p',p}(G)$. By what we have just proved in the above, we have that

(1) if $\omega_p(M) = 1$, then $dl(M) \leq 4$;

(2) if $\omega_p(M) \geq 2$, then $dl(M) \leq 3\omega_p(M)$.

Since $M \lhd G$, $\omega_p(M) \leq \omega_p(G)$. Furthermore, since $\omega_p(G) \geq 2$, $3\omega_p(G) \geq 6$. Thus $dl(M) \leq 3\omega_p(G)$.

Since G/M is a p'-group, $\omega_p(G/M) = \omega(G/M)$. If $\omega(G/M) \leq 1$, $dl(G/M) \leq 3$ by Isaacs and Passman [**10**, Theorem 6.1]. If $\omega(G/M) \geq 2$, $dl(G/M) \leq 2\omega(G/M)$ by Huppert [**6**, Theorem 3], and hence $dl(G/M) \leq 2\omega_p(G/M)$. Since $\omega_p(G/M) \leq \omega_p(G)$ and $2\omega_p(G) \geq 4$, $dl(G/M) \leq 2\omega_p(G)$. Therefore,

$$\begin{aligned} dl(G) &\leq dl(M) + dl(G/M) \\ &\leq 3\omega_p(G) + 2\omega_p(G) = 5\omega_p(G). \end{aligned}$$

This completes the proof of the lemma. □

THEOREM 3.2. *Let G be solvable and $l_p(G/O_p(G)) \geq 1$. Then*

$$dl(G/O_p(G)) \leq 5l_p(G/O_p(G))\Omega_p(G).$$

Proof. We may assume without loss of generality that $O_p(G) = 1$. We use induction on $l_p(G)$. By Lemma 3.1, we can assume that $l_p(G) \geq 2$.

Write $M = O_{p',p}(G)$. Since $O_p(G) = 1$, $O_p(M) = 1$. Clearly, $l_p(M) = 1$. Thus $dl(M) \leq 5\omega_p(M)$ by Lemma 3.1. Since $M \lhd G$, $\omega_p(M) \leq \omega_p(G)$ by Clifford's Theorem. Hence $dl(M) \leq 5\omega_p(G)$.

Since $M = O_{p',p}(G)$, $O_p(G/M) = 1$ and $l_p(G/M) = l_p(G) - 1$. Notice that $1 \le l_p(G/M) < l_p(G)$. Thus, by induction, $dl(G/M) \le 5l_p(G/M)\omega_p(G/M)$. Since $l_p(G/M) = l_p(G) - 1$ and $\omega_p(G/M) \le \omega_p(G)$, $dl(G/M) \le 5(l_p(G) - 1)\omega_p(G)$. Hence,

$$\begin{aligned} dl(G) &\le dl(M) + dl(G/M) \\ &\le 5\omega_p(G) + 5(l_p(G) - 1)\omega_p(G) = 5l_p(G)\omega_p(G), \end{aligned}$$

and the assertion holds. □

Combining Theorem 1.7 with Theorem 3.2, we get

COROLLARY 3.3. *Let G be solvable and $l_p(G/O_p(G)) \ge 1$. Then*
(1) *if p is not a Fermat prime, then*

$$dl(G/O_p(G)) \le 5\omega_p(G)(1 + \log_p(2\omega_p(G)));$$

(2) *if p is a Fermat prime, then*

$$dl(G/O_p(G)) \le 5\omega_p(G)[2 + \log_s(2\omega_p(G)/(p-1))],$$

where $s = (p^2 - p + 1)/p$.

As usual, we denote by $F(G)$ the Fitting subgroup of G.

LEMMA 3.4. *Let G be solvable and $G/F(G) = F(G/F(G))$. Then*

$$dl(G/F(G)) \le 2f(G)^2.$$

Proof. Let p be a prime number such that $p \mid |G|$. By Theorem 2.2, $dl_p(G/O_p(G)) \le l_p(G/O_p(G))e_p(G)$. Combining $l_p(G/O_p(G)) \le r_p(G/O_p(G))$ with Theorem 1.3, we have

$$dl_p(G/O_p(G)) \le 2e_p(G)^2 \le 2f(G)^2.$$

Since $dl_p(G/F(G)) = dl_p(G/O_p(G))$, $dl_p(G/F(G)) \le 2f(G)^2$. Since $G/F(G) = F(G/F(G))$,

$$dl(G/F(G)) = \max\{dl_p(G/F(G)) \mid p \mid |G/F(G)|\}.$$

Thus $dl(G/F(G)) \le 2f(G)^2$. □

THEOREM 3.5. *Let G be solvable. Then*

$$dl(G/F(G)) \le 2(f(G)^2 + f(G) + 1).$$

Proof. Let $F_2/F(G) = F(G/F(G))$. By Corollary 1.6, $r(G/F(G)) \leq 2f(G)$. We use Leisering and Manz [**11**, Lemma 2.3] to embed G/F_2 in the direct product of some $\mathrm{GL}(2f(G), p)$, where p runs through the prime divisors of $|F_2/F(G)|$. Consequently, Theorem 2.5 of Leisering and Manz [**11**] yields that $dl(G/F_2) \leq 2f(G) + 2$.

Applying Lemma 3.4 to the group F_2, we have $dl(F_2/F(F_2)) \leq 2f(F_2)^2$. Hence $dl(F_2/F(G)) \leq 2f(F_2)^2 \leq 2f(G)^2$. Finally,

$$\begin{aligned} dl(G/F(G)) &\leq dl(G/F_2) + dl(F_2/F(G)) \\ &\leq 2f(G) + 2 + 2f(G)^2 \\ &= 2(f(G)^2 + f(G) + 1). \end{aligned} \qquad \square$$

Some remarks are appropriate for this theorem.

(1) If $f(G) = 1$, then $dl(G/F(G)) \leq 2$.

(2) If G has odd order, then $dl(G/F(G)) \leq f(G)^2 + f(G) + 2$.

(3) Let $n(G)$ be the nilpotent length of G. Then $n(G) \leq 2(f(G) + 2)$.

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