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FOURIER COEFFICIENTS OF NON-HOLOMORPHIC MODULAR FORMS AND SUMS OF KLOOSTERMAN SUMS

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This paper studies Fourier coefficients of non-holomorphic modular forms and sums of Kloosterman sums.

1. Introduction. Put $\Gamma = \mathrm{PSL}(2, Z)$ and $H^+ = \{x+iy | y > 0\}$. Consider the Hilbert space $\mathfrak{L}^2(H^+/\Gamma)$ of function $u(z)$ satisfying:

$$u(\gamma z) = u(z) \qquad (\gamma \in \Gamma)$$

and

$$\langle u, u\rangle = \iint_{H^+/\Gamma} |u(z)|^2 \frac{dx\,dy}{y^2} < +\infty.$$

Consider the Laplacian Δ on $\mathfrak{L}^2(H^+/\Gamma)$:

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

A function $u(z)$ in $\mathfrak{L}^2(H^+/\Gamma)$ is called a cusp form if the constant term in the Fourier expansion of $u(z)$ vanishes. It is known that the Laplacian Δ has a complete discrete spectral decomposition on the subspace of cusp forms. The Maass wave forms $u_j(z)$ defined by

$$(1) \qquad \Delta u_j(z) = \lambda_j u_j(z), \quad \langle u_j, u_j\rangle = 1,$$

where $\lambda_1 \le \lambda_1 \le \lambda_3 \le \cdots$ are the discrete eigenvalues of Δ, constitute an orthonormal basis for the subspace of cusp forms. Note that $\lambda_1 > \frac{3}{2}\pi^2$. From (1) we have the Fourier expansion:

$$(2) \qquad u_j(z) = \sqrt{y}\sum_{n\neq 0} \rho_j(n) K_{ik_j}(2\pi|n) e(nx), \quad e(\theta) = e^{2\pi i\theta}$$

where $\lambda_j = \frac{1}{4} + k_j^2$ and $K_{ik_j}(\cdot)$ is the Whittaker function. We have

$$(3) \qquad \#\{k_j|\,|k_j| \le X\} = \tfrac{1}{12}X^2 + cX\log X + O(X)$$

where c is a constant; cf. Venkov [7].

An important problem in the theory of non-holomorphic modular form is to estimate the Fourier coefficients $\rho_j(n)$. The Ramanujan-Peterson conjecture states that for large $|n|$

$$\rho_j(n) \underset{\varepsilon, j}{\ll} |n|^{\varepsilon} \qquad (\varepsilon > 0).$$

A method to study the Fourier coefficients $\rho_j(n)$ of $u_j(z)$ is the non-holomorphic Poincaré series introduced by Selberg **[5]**:

$$P_m(z, s) = \sum_{\gamma \in \Gamma/\Gamma_\infty} (\operatorname{Im} \gamma z)^s e(m\gamma z) \qquad (\operatorname{Re} s > 1),$$

where $m \geq 1$ is an integer and Γ_∞ is the subgroup of translations. The Poincaré series belongs to $\mathfrak{L}^2(H^+/\Gamma)$, and its inner product against a function $u(z) \in \mathfrak{L}^2(H^+/\Gamma)$ gives the mth Fourier coefficient of $u(z)$. Selberg **[5]** obtained the meromorphic continuation of $P_m(z, s)$ to the entire complex s-plane. By considering the inner product of two Poincaré series, Kuznietsov **[4]** developed summation formulas connecting the Fourier coefficients $\rho_j(n)$ and the Kloosterman sum

$$S(m, n; c) = \sum_{\substack{d=1 \\ ad \equiv 1 \ (\operatorname{mod} c)}}^{c} e\left(\frac{am + dn}{c}\right).$$

One of the summation formulas useful to us is equation (9) below. By using the summation formulas, Kuznietsov **[4]** proved that

$$(5) \qquad \sum_{0 < k_j < X} \frac{|\rho_j(n)|^2}{ch\pi k_j} = \frac{1}{\pi^2} X^2 + O(X \log X + X n^{\varepsilon} + n^{1/2+\varepsilon}),$$

and

$$(6) \qquad \sum_{c < T} \frac{S(m, n; c)}{c} \underset{m, n}{\ll} T^{1/6} \log^{1/3} T.$$

The Weil estimate gives

$$|S(m, n; c)| \leq (m, n, c)^{1/2} d(c) c^{1/2},$$

which yields a trivial bound $O(T^{1/2+\varepsilon})$ for the sum in (6).

The Linnik-Selberg conjecture states that

$$(7) \qquad \sum_{c \leq T} \frac{S(m, n; c)}{c} \underset{\varepsilon}{\ll} T^{\varepsilon} \qquad (T > (m, n)^{1/2}, \ \varepsilon > 0).$$

To deal with the estimate of $\rho_j(n)$, Selberg [5] introduced the above conjecture.

Another method to study the sum of Kloosterman sum in (6) is by the Kloosterman zeta function introduced by Selberg [5]:

$$Z_{m,n}(s) = \sum_{c=1}^{\infty} \frac{S(m, n; c)}{c^{2s}} \qquad \left(\operatorname{Re} s > \frac{3}{4}\right). \tag{8}$$

Selberg [5] obtained the meromorphic continuation of $Z_{m,n}(s)$ to the entire complex plane. A useful characterization of $Z_{m,n}(s)$ may be found in (7.26) of Kuznietsov [4].

Goldfeld and Sarnak [3] have given a very simple proof of the bound $O(T^{1/6+\varepsilon})$ for the sum in (6) by proving a good bound on $Z_{m,n}(s)$ in the critical strip.

Equation (5) means that on the average $|\rho_j(n)|^2/ch\pi k_j$ is bounded with respect to the indices k_j from 0 to X. In this paper, we will show the following:

THEOREM 1. *We have for* $n^{1+\varepsilon} \ll t$ $(\varepsilon > 0)$,

$$\sum_{|k_j - t| < 1} \frac{|\rho_j(n)|^2}{ch\pi k_j} \ll t \qquad (t \to +\infty).$$

Theorem 1 means that on the average $|\rho_j(n)|^2/ch\pi k_j$ is bounded with respect to k_j in short interval.

With Theorem 1, we will show furthermore

THEOREM 2. *For any* $f(t) \to +\infty$ *and* $f(t) = o(t)$ *as* $t \to +\infty$, *and* $n^{1+\varepsilon} \ll t$ $(\varepsilon > 0)$, *we have*

$$\sum_{|k_j - t| < f(t)} \frac{|\rho_j(n)|^2}{ch\pi k_j} \sim \frac{4}{\pi^2} t f(t) \qquad (t \to +\infty)$$

and

THEOREM 3. *For* $Y \geq 10$, *we have*

$$\int_Y^{eY} \left(\sum_{c \leq x} \frac{S(m, n; c)}{d} \right)^2 \frac{dx}{x} \underset{m,n}{\ll} \log Y.$$

It may be interesting to note that we get as a by-product of the proof of Theorem 2 the following:

THEOREM 4. *For any $\sigma \in \mathbb{C}$, we have*

$$\int_{-\infty}^{\infty} \Gamma\left(\sigma - \frac{1}{2} - ir\right) \Gamma\left(\sigma - \frac{1}{2} + ir\right) dr = \pi 2^{2-2\sigma} \Gamma(2\sigma - 1).$$

Theorem 4 would follow immediately from the proof of Theorem 2. In view of (3), it may be interesting to compare Kuznietsov's estimate (5) with Theorems 1 and 2. Theorem 3 means that the sum in (6) is "very small" for almost all x and for most of the time better than the Linnik-Selberg conjecture. More precisely, for $Y \geq 10$ and $f(x) \nearrow \infty$, let $M_Y \subset [Y, eY]$ such that

$$\left| \sum_{c<x} \frac{S(m, n; c)}{c} \right| \geq f(x) \log^{1/2} x \qquad (x \in M_Y).$$

Then Theorem 3 shows that the Lebesgue measure of M_Y is $O(f(Y)^{-2} Y)$.

By putting $\sigma = \frac{3}{4} + 1/\log n$ in Lemma 1, Theorem 1 follows immediately. We prove Theorem 3 by establishing Lemma 2, which is analogous to the explicit formula in the theory of prime number, and by using Gallagher's mean-value inequality for exponential sum which is Lemma 3. The method imitates an idea of Gallagher [**2**].

2. Lemmas. The proof of Lemma 1 is based on the following equation (9) which follows by putting $s_1 = \sigma + it$ and $s_2 = \sigma - it$ in the lemmas in §4.1 and §4.4 of Kuznietsov [**4**].

PROPOSITION. *For $s = \sigma + it$, $\frac{3}{4} < \sigma < \frac{5}{4}$, and any integer $n \geq 1$, we have*

$$\begin{aligned} \text{(9)} \qquad & \pi \left\{ \sum_{j=1}^{\infty} |\rho_j(n)|^2 \Lambda(s; k_j) \right. \\ & \qquad \left. + \frac{1}{\pi} \int_{-\infty}^{\infty} |\sigma_{2ir}(n)|^2 \Lambda(s; r) \frac{ch\pi r}{|\zeta(1+2ir)|^2} dr \right\} \\ & = \Gamma(2\sigma - 1) + (4\pi n)^{2\sigma-1} \\ & \qquad \times \left\{ \frac{2^{3-2\sigma}}{ish2\pi t} \sum_{c=1}^{\infty} \frac{S(n, n; c)}{c^{2\sigma}} \Phi\left(s, \frac{4\pi n}{c}\right) \right\}, \end{aligned}$$

where

$$\Lambda(s\,;\,r) = \frac{|\Gamma(s-\frac{1}{2}+ir)\Gamma(s-\frac{1}{2}-ir)|^2}{|\Gamma(s)|^2}, \quad \sigma_{2ir}(n) = \sum_{d|n} d^{2ir}$$

and for $x > 0$

$$(10) \qquad \Phi(s\,,\,x) = -\pi \int_1^\infty \left(u - \frac{1}{u}\right)^{2\sigma-2} \{(\sin \pi s) J_{2it}(xu) + (\sin \pi \bar{s}) J_{-2it}(xu)\} \frac{du}{u},$$

and $J_{2it}(u)$ *is the Bessel function.*

We need the following estimate for the Bessel function:

$$(11) \qquad J_{it}(u) \ll e^{\pi t/2}(t^2+u^2)^{-1/4} \qquad (t \in \mathbb{R})$$

uniformly in $u > 0$ for $|t| \to +\infty$.

LEMMA 1. *We have for* $\frac{3}{4} < \sigma < \frac{5}{6}$

$$(12) \qquad \sum_{|t-k_j|<1} \frac{|\rho_j(n)|^2}{ch\pi k_j} \ll t + \sqrt{t} n^{2\sigma-1} \left(\sigma - \frac{3}{4}\right)^{-2} \qquad (t \to +\infty).$$

Proof. We take $\frac{3}{4} < \sigma < \frac{5}{6}$ in the Proposition. With the bound in (11), we see from (10) that

$$(13) \quad \Phi(s\,,\,x) \ll e^{2\pi t} \int_1^\infty \left(u - \frac{1}{u}\right)^{2\sigma-2} (t^2 + x^2u^2)^{-1/4} \frac{du}{u} \qquad \left(x = \frac{4\pi n}{c}\right)$$

$$\ll t^{-1/2} e^{2\pi t} \int_1^\infty \left(u - \frac{1}{u}\right)^{2\sigma-2} \left(1 + \left(\frac{x}{t}\right)^2 u^2\right)^{-1/4} \frac{du}{u}$$

$$\ll t^{-1/2} e^{2\pi t}, \quad \text{since } \frac{3}{4} < \sigma < \frac{5}{6}.$$

On considering Weil's bound for $S(m\,,\,n\,;\,c)$ and (13), the second term on the right-hand side of (9) is then

$$(14) \qquad \ll t^{-1/2} n^{2\sigma-1} \left(\sigma - \frac{3}{4}\right)^{-2}.$$

On the other hand, the integral in (9) is non-negative, and the series in (9) is

$$\begin{aligned} &\geq \sum_{|k_j-t|<1} |\rho_j(n)|^2 \frac{|\Gamma(s-\frac{1}{2}+ik_j)\Gamma(s-\frac{1}{2}-ik_j)|^2}{|\Gamma(s)|^2} \\ &\gg \frac{1}{t} \sum_{|k_j-t|<1} \frac{|\rho_j(n)|^2}{ch\pi k_j}, \end{aligned}$$

since $\Gamma(s) = \sqrt{2\pi} e^{-(\pi/2)|t|} t^{\sigma-1/2}(1+O(|t|^{-1}))$, and $|\Gamma(s-\frac{1}{2}-ik_j)| \gg 1$ for $|t-k_j| < 1$.

This proves Lemma 1.

LEMMA 2. *We have for* $T < \frac{1}{2}x$

$$\sum_{c \leq x} \frac{S(m, n; c)}{c} = \sum_{|k_j|<T} \frac{\rho_j(n)\overline{\rho_j(m)}}{ch\pi k_j} \frac{\Gamma(2ik_j)}{2ik_j} x^{2ik_j} + O\left(\frac{x^{1/2}\log^2 x}{T}\right),$$

the implicit constant here depends on m, n.

Before proceeding with the proof of Lemma 2, we need several analytic properties of $Z_{m,n}(s)$. On the half plane Re $s > 0$, the poles of $Z_{m,n}(s)$ are located at $s = \frac{1}{2} + ik_j$, and as $t \to \infty$

$$Z_{m,n}(s) \underset{m,n}{\ll} \frac{|s|^{1/2}}{|\sigma - \frac{1}{2}|} \qquad (s = \sigma + it, \sigma \neq \tfrac{1}{2}). \tag{15}$$

Estimate (15) is obvious by using the result and the same method as in the proof of Theorem 1 of Goldfeld and Sarnak **[3]**. On the other hand, by the Lemma of §7.3 of Kuznietsov **[4]**, we have the representation for $Z_{m,n}(s)$ $(s \in \mathbb{C})$:

$$\begin{aligned} &(2\pi\sqrt{mn})^{2s-1} Z_{m,n}(s) \\ &\quad = \sum_{j=1}^{\infty} \frac{\rho_j(n)\overline{\rho_j(m)}}{ch\pi k_j} h(k_j, s) - \frac{\delta_{m,n}}{2\pi} \frac{\Gamma(s)}{\Gamma(1-s)} \\ &\quad + \sum_{l=0}^{\infty} p_{m,n}(l) \frac{\Gamma(s+l)}{\Gamma(2-s+l)} + L_{m,n}(s) \end{aligned} \tag{16}$$

where $L_{m,n}(s)$ denotes the analytic continuation of the function which is defined in the half plane $\mathrm{Re}\ s > \frac{1}{2}$ by the integral

$$L_{m,n}(s) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{n}{m}\right)^{it}\sigma_{2ir}(n)\sigma_{-2ir}(m)\frac{h(r,s)}{\zeta(1+2ir)\zeta(1-2ir)}\,dr,$$

and

$$h(r,s) = \frac{1}{2}\sin(\pi s)\Gamma\left(s-\frac{1}{2}+ir\right)\Gamma\left(s-\frac{1}{2}-ir\right),$$

and

$$p_{m,n}(l) = (2l+1)\sum_{c=1}^{\infty}\frac{S(m,n;c)}{c}J_{2l+1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

By (16), we see that

$$\operatorname*{Res}_{s=1/2+ik_j} Z_{m,n}(s) = \frac{(2\pi\sqrt{mn})^{-2ik_j}}{2}\Gamma(2ik_j)\rho_j(n)\overline{\rho_j(m)}. \tag{17}$$

Consider $s = \sigma + it$ with

$$\left|\sigma - \frac{1}{2}\right| \le \frac{\delta}{\log(|t|+2)} \tag{18}$$

for a small $\delta > 0$. Deforming suitably the integral path in the integral of $L_{m,n}(s)$, we have for s satisfying (18)

$$L_{m,n}(s) = O(\log^2|t|) \tag{19}$$

since $\zeta(x+iy) \neq 0$ and $\zeta(x+iy) \ll \log(|y|+2)$ in the region $x > 1-(c/\log(|y|+2))$ $(c>0)$.

Also for s satisfying (18), we have

$$\frac{\Gamma(s)}{\Gamma(1-s)} \ll 1 \tag{20}$$

and

$$\sum_{l=0}^{\infty} p_{m,n}(l)\,\frac{\Gamma(s+l)}{\Gamma(2-s+l)} \ll mn. \tag{21}$$

By using the estimate on Bessel function

$$|J_k(y)| \le \min\left(1, \frac{(y/2)^k}{(k-1)!}\right),$$

we prove (21) as follows: note first that $(2l+1)\ \Gamma(s+l)/\Gamma(2-s+l) \ll 1$. Thus

$$
\begin{aligned}
\sum_{l=0}^{\infty} & p_{m,n}(l)\frac{\Gamma(s+l)}{\Gamma(2-s+l)} \\
&\ll \sum_{l=0}^{\infty}\sum_{1\le c\le 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c}\left|J_{2l+1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right| \\
&\quad + \sum_{l=0}^{\infty}\sum_{c>20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c}\left|J_{2l+1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right| \\
&\ll \sum_{0\le l\le 20\pi\sqrt{mn}}\ \sum_{1\le c\le 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c}\left|J_{2l+1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right| \\
&\quad + \sum_{1\le c\le 20\pi\sqrt{mn}}\ \sum_{l>20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c}\left|J_{2l+1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right| \\
&\quad + \sum_{l>20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c}\sum_{l=0}^{\infty}\left|J_{2l+1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right| \\
&\ll mn + \sum_{1\le c\le 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c}\sum_{l>20\pi\sqrt{mn}}\left(\frac{2\pi\sqrt{mn}}{c}\right)^{2l+1}\frac{1}{(2l)!} \\
&\quad + \sum_{c>20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c}\sum_{l=0}^{\infty}\left(\frac{2\pi\sqrt{mn}}{c}\right)^{2l+1}\frac{1}{(2l!)} \\
&\ll mn + \sum_{l\le c\le 20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c} \\
&\quad + \sum_{c>20\pi\sqrt{mn}} \frac{|S(m,n;c)|}{c}\times\frac{2\pi\sqrt{mn}}{c} \\
&\ll mn.
\end{aligned}
$$

This proves (21). Estimate (21) is obviously not the best, but we are satisfied with this presently.

Also by using Theorem 1 and (5), we have that

$$
\sum_{|k_j-t|>1}\frac{\rho_j(n)\overline{\rho_j(m)}}{ch\pi k_j}h(k_j,x)=O(|t|) \tag{22}
$$

for s satisfying (18) and $\max\{m^{1+\varepsilon}, n^{1+\varepsilon}\} \ll |t|$. Thus by (19), (20),

(21) and (22), equation (16) becomes

$$(23)\quad Z_{m,n}(s) = (2\pi\sqrt{mn})^{1-2s} \sum_{|k_j - t|<1} \frac{\rho_j(n)\overline{\rho_j(m)}}{ch\pi k_j} h(k_j, s) + O(|t|)$$

for s satisfying (18) and $mn \ll |t|$ and $\max\{m^{1+\varepsilon}, n^{1+\varepsilon}\} \ll |t|$.

We are now in a position to prove Lemma 2.

Proof of Lemma 2. Choose $0 < \varepsilon \le \delta/\log(|t|+2)$ for small $\delta > 0$. By (15) and the Lindelöf-Phragmen principle it follows that

$$(24)\qquad |Z_{m,n}(s)| \underset{m,n}{\ll} \frac{|t|^{3/2-2\sigma+2\varepsilon}}{\varepsilon^2}$$

for $\frac{1}{2}+\varepsilon \le \sigma \le \frac{3}{4}+\varepsilon$, since $Z_{m,n}(\frac{3}{4}+\varepsilon) \underset{m,n}{\ll} \varepsilon^{-2}$ by (8); and obviously

$$(25)\qquad |Z_{m,n}(s)| \underset{m,n}{\ll} \frac{|t|^{1/2}}{\frac{1}{2}-\sigma}$$

for $\frac{1}{10} \le \sigma \le \frac{1}{2} - \varepsilon$.

Consider the integral

$$(26)\qquad I(T) = \frac{1}{2\pi i}\int_{\eta - iT}^{\eta + iT} Z_{m,n}(s)\frac{x^{2s-1}}{2s-1}\,ds \qquad (\eta = \tfrac{3}{4}+\varepsilon)$$

with $T > 0$ not an ordinate of a pole of $Z_{m,n}(s)$. Now by Lemma 3.12 of Titchmarsh **[6]**, we get

$$(27)\qquad \sum_{c\le x} \frac{S(m,n;c)}{c} = \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT} Z_{m,n}(s)\frac{x^{2s-1}}{2s-1}\,ds + O_{m,n}\left(\frac{x^\eta}{T\varepsilon^2}\right).$$

Computations of residues yield

$$(28)\qquad I(T) = \sum_{|k_j|<T} \xi_j \frac{x^{2ik_j}}{2ik_j} + \frac{1}{2\pi i}\int_{1/10-iT}^{1/10+iT} Z_{m,n}(s)\frac{x^{2s-1}}{2s-1}\,ds + \frac{1}{2\pi i}\int_{1/10\pm iT}^{\eta\pm iT} Z_{m,n}\frac{x^{2s-1}}{2s-1}\,ds$$

where ξ_j is the residue of $Z_{m,n}(s)$ at $s = \frac{1}{2} + ik_j$. Using (17), we see that

$$\xi_j \ll \frac{|\rho_j(n)\rho_j(m)|}{ch\pi k_j}|k_j|^{-1/2}. \tag{29}$$

Now we estimate the integrals in (28). By (25), we have first

$$\int_{1/10-iT}^{1/10+iT} Z_{m,n}(s)\frac{x^{2s-1}}{2s-1}\,ds \underset{m,n}{\ll} x^{-4/5}T^{1/2} \tag{30}$$

and

$$\int_{1/10\pm iT}^{1/2-\varepsilon\pm iT} Z_{m,n}(s)\frac{x^{2s-1}}{2s-1}\,ds \underset{m,n}{\ll} \frac{x^{-2\varepsilon}T^{-1/2}}{\varepsilon}. \tag{31}$$

By (24), we have

$$\int_{1/2+\varepsilon\pm iT}^{\eta\pm iT} Z_{m,n}(s)\frac{x^{2s-1}}{2s-1}\,ds \underset{m,n}{\ll} \frac{x^{1/2+\varepsilon}}{\varepsilon^2|t|}\frac{1}{|\log\frac{T}{x}|} \tag{32}$$

for $|\log\frac{T}{x}| \gg 1$.

Finally, by (23) we have

$$\begin{aligned}(33)\quad &\int_{1/2-\varepsilon\pm iT}^{1/2+\varepsilon\pm iT} Z_{m,n}(s)\frac{x^{2s-1}}{2s-1}\,ds \\ &= \sum_{|k_j\mp T|<1}\frac{\rho_j(n)\overline{\rho_j(m)}}{ch\pi k_j} \\ &\quad\cdot\int_{1/2-\varepsilon\pm iT}^{1/2+\varepsilon\pm iT}(2\pi\sqrt{mn})^{1-2s}h(k_j\,,s)\frac{x^{2s-1}}{2s-1}\,ds + O_{m,n}(\varepsilon).\end{aligned}$$

Noting that $|\Gamma(s)| \gg |s|^{-1}$ for $\varepsilon \ll |s| \ll 1$ and by suitably deforming the integral path on the right-hand side of (33) to an upper or lower semi-circle according as $\frac{1}{2}+ik_j$ stays below or above the integral path, we get

$$\int_{1/2-\varepsilon\pm iT}^{1/2+\varepsilon\pm iT}(2\pi\sqrt{mn})^{1-2s}h(j_k\,,s)\frac{x^{2s-1}}{2s-1}\,ds \underset{m,n}{\ll} T^{-3/2}$$

since $|k_j \mp T| < 1$, so the right-hand side of (33) is

$$\begin{aligned}(34)\quad &\underset{m,n}{\ll} \sum_{|k_j\mp T|<1}\frac{|\rho_j(n)\rho_j(m)|}{ch\pi k_j}T^{-3/2}+\varepsilon \\ &\underset{m,n}{\ll} T^{-1/2}+\varepsilon,\end{aligned}$$

by Theorem 1.

Putting (30), (31), (32), and (34) together, equation (28) becomes, for $\log|\frac{T}{x}| \gg 1$ and $\varepsilon = \delta \log^{-1} T$,

$$I(T) = \sum_{|k_j|<T} \xi_j \frac{x^{2ik_j}}{2ik_j} + O_{m,n}\left(\frac{x^{1/2}\log^2 xT}{T}\right) \tag{35}$$

for $T \leq \frac{1}{2}x$, which combined with (27) yield

$$\sum_{c\leq x} \frac{S(m, n; c)}{c} = \sum_{|k_j|<T} \xi_j \frac{x^{2ik_j}}{2ik_j} + O_{m,n}\left(\frac{x^{1/2}\log^2 xT}{T}\right). \tag{36}$$

This completes the proof of Lemma 2.

By putting $T = x^{1/3}\log^{4/3} x$ in (36), we get $O(x^{1/6}\log^{2/3} x)$ on the right-hand side of (36) which is slightly inferior to Kuznietsov's bound (6).

LEMMA 3. *Let $A(u) = \sum_v c(v)e^{ivu}$ be an absolutely convergent series with complex coefficients $c(v)$ and real indices v. Then for $T > 0$*

$$\int_{-T}^{T} \left|\sum_v c(v)e^{ivu}\right|^2 du \ll \int_{-\infty}^{\infty} \left|T \sum_{t<v<t+T^{-1}} c(v)\right|^2 dt.$$

Proof. This is Lemma 1 of Gallagher [**1**].

3. Proofs of Theorems. We prove first Theorem 2. Take $\sigma = \frac{3}{4} + 1/\log n$. Then the Proposition of §2 gives

$$\sum_{|k_j-t|<1} \frac{|\rho_j(n)|^2}{ch\pi k_j} \ll t \qquad (n^{1+\varepsilon} \ll t), \tag{37}$$

which is the assertion of Theorem 1.

In view of $|\zeta(1+ir)|^{-1} \ll \log|r|$ $(|r| \to +\infty)$, a rough estimate gives

$$\int_{-\infty}^{\infty} |\sigma_{2ir}(n)|^2 \Lambda(s; r) \frac{ch\pi r}{|\zeta(1+2ir)|^2} dr \ll t^{-1}\log t\, d^2(n). \tag{38}$$

We have, for $k_j \geq t + \sqrt{f(t)}$,

$$\begin{aligned} \Lambda(s, k_j) &= \frac{|\Gamma(s-1/2+ik_j)\Gamma(s-1/2-ik_j)|^2}{|\Gamma(s)|^2} \\ &\ll e^{\pi t} t^{1-2\sigma} e^{-2\pi k_j} |t+k_j|^{2\sigma-2} |k_j - t|^{2\sigma-2}, \end{aligned} \tag{39}$$

and for $k_j \leq t - \sqrt{f(t)}$

$$\Lambda(s\,;\,k_j) \ll e^{-\pi t} t^{1-2\sigma} |t+k_j|^{2\sigma-2} |t-k_j|^{2\sigma-2}. \tag{40}$$

On considering (37) and (5), inequalities (39) and (40) give rise to

$$\sum_{|t-k_j|\geq\sqrt{f(t)}} |\rho_j(n)|^2 \Lambda(s\,;\,k_j) = o(1) \qquad (t \to +\infty). \tag{41}$$

Now (14) together with (38) and (41) yield, by virtue of (9),

$$\sum_{|k_j-t|<\sqrt{f(t)}} |\rho_j(n)|^2 \Lambda(s\,;\,k_j) = \frac{1}{\pi}\Gamma(2\sigma-1) + o(1), \tag{42}$$

since $n^{1+\varepsilon} \ll t$. And also for $|k_j - t| < \sqrt{f(t)}$

$$\Lambda(s\,;\,k_j) = 2^{2\sigma-2} t^{-1} e^{-\pi k_j} \left|\Gamma\left(s - \frac{1}{2} - ik_j\right)\right|^2 (1 + o(1)).$$

Substituting this into (42), we obtain

$$\begin{aligned} \sum_{|k_j-t|<\sqrt{f(t)}} |\rho_j(n)|^2 e^{-\pi k_j} \left|\Gamma\left(s - \frac{1}{2} - ik_j\right)\right|^2 \\ = \frac{2^{2-2\sigma}}{\pi}\Gamma(2\sigma-1)t + o(t), \end{aligned} \tag{43}$$

since $\sqrt{f(t)} = o(t)$.

Taking integrals on both sides of (43) yields

$$\begin{aligned} \int_{t-f(t)}^{t+f(t)} \sum_{|k_j-r|<\sqrt{f(r)}} |\rho_j(n)|^2 e^{-\pi k_j} \left|\Gamma\left(\sigma - \frac{1}{2} + i(r-k_j)\right)\right|^2 dr \\ = \frac{2^{3-2\sigma}}{\pi}\Gamma(2\sigma-1)tf(t) + o(tf(t)). \end{aligned} \tag{44}$$

Interchanging the order of summation and integral in (44), the left-hand side of (44) becomes

$$\begin{aligned} \sum_{|k_j-t|<f(t)} |\rho_j(n)|^2 e^{-\pi k_j} \int_{k_j-\sqrt{f(t)}}^{k_j+\sqrt{f(t)}} \left|\Gamma\left(\sigma - \frac{1}{2} + i(r-k_j)\right)\right|^2 dr \\ + o(tf(t)), \end{aligned} \tag{45}$$

by using (37). Note further that

$$\int_{k_j-\sqrt{f(t)}}^{k_j+\sqrt{f(t)}} \left|\Gamma\left(\sigma - \frac{1}{2} + i(r-k_j)\right)\right|^2 dr$$
$$= \int_{-\infty}^{\infty} \left|\Gamma(\sigma - \frac{1}{2} + ir)\right|^2 dr + O(e^{-\pi\sqrt{f(t)}}).$$

From this and (44) and (45) it follows that

$$(46)\quad \sum_{|k_j - t| < f(t)} |\rho_j(n)|^2 e^{-\pi k_j}$$
$$\sim \frac{2^{3-2\sigma}}{\pi}\left(\int_{-\infty}^{\infty}\left|\Gamma\left(\sigma - \frac{1}{2} + ir\right)\right|^2 dr\right)^{-1} \Gamma(2\sigma - 1) t f(t)$$

for $\sigma = \frac{3}{4} + 1/\log n$ and $n^{1+\varepsilon} \ll t$.

Now if we fix n, then we see from the proof that (46) holds good uniformly for σ in an interval $I \subset (\frac{3}{4}, \infty)$. By analytic continuation, there is a constant ξ for which

$$(47)\quad \xi \int_{-\infty}^{\infty} \Gamma\left(\sigma - \frac{1}{2} + ir\right)\Gamma\left(\sigma - \frac{1}{2} - ir\right) dr = 2^{2-2\sigma}\Gamma(2\sigma - 1) \qquad (\sigma \in \mathbb{C}).$$

Indeed $\xi = \frac{1}{\pi}$, since

$$\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2} + ir\right)\right|^2 dr = \int_{-\infty}^{\infty} \frac{\pi}{ch\pi r} dr = \pi.$$

This completes the proof of Theorem 1, and equation (47) gives the proof of Theorem 4.

Finally we prove Theorem 3. For $Y \geq 10$, $Y \leq x \leq eY$, and $Y^{2/3} \leq T \leq \frac{1}{2}Y$, Lemma 2 gives

$$(48)\qquad \sum_{c \leq x} \frac{S(m, n; c)}{c} = \sum_{|k_j| < T} \xi_j \frac{x^{2ik_j}}{2ik_j} + o(1).$$

On applying Lemma 3 to (48), we get

$$
\begin{aligned}
&\int_Y^{eY}\left(\sum_{c\le x}\frac{S(m,n;c)}{c}\right)^2\frac{dx}{x}\\
&\ll \int_Y^{eY}\left|\sum_{|k_j|<T}\frac{\xi_j}{2k_j}x^{2ik_j}\right|^2\frac{dx}{x}+o(1)\\
&=\int_{\log Y}^{1+\log Y}\left|\sum_{|k_j|<T}\frac{\xi_j}{2k_j}e^{2ik_ju}\right|^2 du+o(1)\\
&\ll \int_{-T-1}^{T+1}\left|\sum_{t<k_j<t+1}\frac{\xi_j}{2k_j}\right|^2 dt+o(1)\\
&\ll \int_1^{T+1}\left(\sum_{|k_j-t|<1}\frac{|\rho_j(n)\rho_j(m)|}{ch\pi k_j}k_j^{-3/2}\right)^2 dt+o(1), \quad \text{by (29)},\\
&\ll_{m,n} \int_1^{T+1} t^{-1}\,dt+o(1) \quad \text{by Theorem 1},\\
&\ll_{m,n} \log T\\
&\ll_{m,n} \log Y.
\end{aligned}
$$

This completes the proof of Theorem 3.

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