PACIFIC JOURNAL OF MATHEMATICS
Vol. 149, No. 1, 1991

CONJUGATES OF EQUIVARIANT HOLOMORPHIC MAPS OF SYMMETRIC DOMAINS

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In this paper we construct the conjugates of equivariant holomorphic maps of symmetric domains associated to morphisms of arithmetic varieties. We also prove that the conjugate of a Kuga fiber variety is another Kuga fiber variety.

0. Introduction. Let $\mathbf{G}$ be a simply connected semisimple algebraic group over $\mathbf{Q}$ that does not contain direct factors defined over $\mathbf{Q}$ and compact over $\mathbf{R}$, and let K be a maximal compact subgroup of the semisimple Lie group $G = \mathbf{G}(\mathbf{R})$. We assume that the symmetric space $D = G/K$ has a complex structure. Let Γ be a torsion free arithmetic subgroup of $\mathbf{G}$ and let $X = \Gamma \backslash D$ be the corresponding arithmetic variety. For each $\sigma \in \mathrm{Aut}(X)$ it is known (cf. [**5**], [**6**], [**7**], [**10**]) that the conjugate X^σ of X is also an arithmetic variety.

Let $\mathbf{G}'$ be another semisimple algebraic $\mathbf{Q}$-group, and consider the corresponding objects G', K', D', Γ' and X' as in the case of $\mathbf{G}$. Let $\rho: G \to G'$ be a homomorphism of Lie groups and $\tau: D \to D'$ a holomorphic map such that (ρ, τ) is an equivariant pair and $\rho(\Gamma) \subset \Gamma'$. Then τ induces the morphism $\phi: X \to X'$ of arithmetic varieties. Let D^σ and D'^σ be the universal covering spaces of X^σ and X'^σ respectively, and let $\tau^\sigma: D^\sigma \to D'^\sigma$ be the lifting of $\phi^\sigma: X^\sigma \to X'^\sigma$. Let G_0 and G'_0 be the connected components of the identity of $\mathrm{Aut}(D^\sigma)$ and $\mathrm{Aut}(D'^\sigma)$ respectively. If $\Gamma^\sigma \subset G_0$ and $\Gamma'^\sigma \subset G'_0$ are the fundamental groups of X^σ and X'^σ respectively, then we have the following result, Theorem 5.2 of this paper.

THEOREM. *There exist a finite covering G_1^σ of G_0^σ and a homomorphism $\rho_1^\sigma: G_1^\sigma \to G_0'^\sigma$ such that ρ_1^σ and τ^σ are equivariant and $\rho_1^\sigma(\Gamma^\sigma)$ is contained in Γ'^σ.*

As an application of this result we consider the conjugates of Kuga fiber varieties. Let $\mathbf{G}' = \mathrm{Sp}(V, \beta)$ for some $\mathbf{Q}$-vector space V and a nondegenerate alternating bilinear form β, and assume that $X = \Gamma\backslash D$ is compact. Then from the equivariant pair (ρ, τ) we can construct a

Kuga fiber variety $\pi: Y \to X$ which is a fiber bundle such that X and Y are complex projective varieties, π is a morphism of varieties, and the fibers are polarized abelian varieties (see §6 for details). For each $\sigma \in \mathrm{Aut}(\mathbf{C})$ we obtain the conjugate $\pi^\sigma: Y^\sigma \to X^\sigma$ of the Kuga fiber variety $\pi: Y \to X$. Then we have the following theorem, Theorem 6.3 of the text.

THEOREM. *$\pi^\sigma: Y^\sigma \to X^\sigma$ is a Kuga fiber variety.*

The above theorem is known when $\sigma: Y \to X$ is a family of abelian varieties associated to a PEL-type (cf. [**14**]) and it is also proved in [**9**] for Kuga fiber varieties constructed under certain assumptions. The theorem is also an immediate consequence of the main theorems in [**5**] and [**10**] in the case that ϕ arises from a homomorphism $\tilde{\rho}: \mathbf{G} \to \mathbf{G}'$ of algebraic groups defined over $\mathbf{Q}$ and τ maps the CM-points of D to the CM-points of D'. In general, however, τ does not necessarily map the CM-points to CM-points (see [**5**, Proposition 1.11] for a necessary and sufficient condition for τ to map CM-points to CM-points).

This paper contains a part of the results in my Ph.D. thesis. I would like to thank my thesis advisor Professor Michio Kuga. I would also like to thank the referee for various helpful suggestions.

1. Prouniversal covering manifolds. In this section we shall review some of the results in [**6**] and [**7**]. Let $\mathbf{G}$ be a simply connected semisimple algebraic group over $\mathbf{Q}$ that does not contain direct factors defined over $\mathbf{Q}$ and compact over $\mathbf{R}$, and let $G = \mathbf{G}(\mathbf{R})$ be the group of real elements of $\mathbf{G}$. If K is a maximal compact subgroup of G, we assume that the associated symmetric space $D = G/K$ has a G-invariant complex structure.

Let Γ be an arithmetic subgroup of $\mathbf{G}$ that does not contain elements of finite order. Then the quotient space $X = \Gamma\backslash D$ has a natural structure of a complex manifold. Such complex manifolds are called arithmetic varieties. By a theorem of Baily and Borel ([**3**]) X has a structure of an algebraic variety over $\mathbf{C}$, and this structure is unique by a theorem of Borel ([**4**]).

Let $\{\Gamma_k | k = 1, 2, 3, \dots\}$ be an inductive system of subgroups of finite index of Γ such that each Γ_k is an arithmetic subgroup and

$$\Gamma_k \subset \Gamma_j \quad \text{for } j < k.$$

Then, for each k, the quotient space $X_k = \Gamma_k \backslash D$ is an arithmetic variety which is a finite unramified covering manifold of X, and the collection $\{X_k | k \geq 1\}$ is a projective system of finite unramified covering manifolds of X. The projective limit

$$\widehat{D} = \varprojlim X_k$$

has a natural structure of a non-connected complex manifold, which does not depend on the representation of $\widehat{D}$ as a projective limit. If

$$\hat{x} = \varprojlim x_k$$

is an element of $\widehat{D}$ with $x_k \in X_k$ for each k, then the open neighborhoods of $\hat{x}$ in $\widehat{D}$ are determined as follows: If $\{U_k\}$ is a collection of open sets $U_k \subset X_k$ containing x_k for each k, then the open neighborhood of $\hat{x}$ associated to $\{U_k\}$ is given by

$$N(\hat{x}, \{U_k\}) = \{\hat{y} = \varprojlim y_k \in \widehat{D} | y_k \in U_k \text{ for all } k\}.$$

Let $\pi: D \to X$ and $\pi_k: X_k \to X$ be the canonical projections. To construct a mapping $\hat{\mu}: D \to \widehat{D}$, we fix a point $x \in X$ and two elements $d_0 \in D$ and $\hat{d}_0 \in \widehat{D}$ such that

$$\pi(d_0) = \hat{\pi}(\hat{d}_0) = x.$$

Then for each k there is a unique map $\mu_k: D \to X_k$ such that $\pi_k \circ \mu_k = \pi$. We define the embedding $\hat{\mu}: D \to \widehat{D}$ by

$$\hat{\mu} = \varprojlim \mu_k.$$

PROPOSITION 1.1. (i) *$\hat{\mu}(D)$ is a connected component of $\widehat{D}$.* (ii) *$\hat{\mu}(D)$ is dense in $\widehat{D}$.*

Proof. (i) Let

$$\hat{d}_0 = \varprojlim d_{0k} \in \hat{\mu}(D) \subset \widehat{D}.$$

If

$$\hat{d}_1 = \varprojlim d_{1k} \in \widehat{D}$$

is in the connected component of $\widehat{D}$ containing $\hat{d}_0$, then there is a continuous map $\hat{c}$: $[0, 1] \to \widehat{D}$ such that $\hat{c}(0) = \hat{d}_0$ and $\hat{c}(1) = \hat{d}_1$. Then $\hat{c}$ can be represented by

$$\varprojlim c_k,$$

where each c_k is a continuous map from $[0, 1]$ to X_k such that

$$\pi_{jk} \circ c_j = c_k \quad \text{for } j < k;$$

here the maps $\pi_{jk}: X_j \to X_k$ are the natural projections. If $c: [0, 1] \to D$ is the common lifting of the maps c_k, then $\hat{\mu}(c(0)) = \hat{d}_0$ and

$$\hat{\mu}(c(1)) = \hat{d}_1 \in \hat{\mu}(D).$$

Thus $\hat{\mu}(D)$ is a connected component of $\widehat{D}$.

(ii) Let

$$\hat{x} = \varprojlim x_k \in \widehat{D},$$

and let $N(\hat{x}, \{U_k\})$ be the neighborhood of $\hat{x}$ associated to a collection $\{U_k\}$ of open sets with $x_k \in U_k \subset X_k$ for each k. If x_0 is an element of X with $\pi_k(x_k) = x_0$ for all k, we choose $y_k \in \pi_k^{-1}(x_0) \cap U_k$ for each k. If $y \in D$ is an element with $\mu_k(y) = y_k$ for all k, then $\hat{\mu}(y) \in N(\hat{x}, \{U_k\})$. Thus $\hat{\mu}(D)$ is dense in $\widehat{D}$.

Let $\Gamma(X) \subset \mathrm{Aut}(D)$ be the fundamental group of X and let

$$G_a(X) = \{g \in \mathrm{Aut}(D) | [\Gamma(X): g\Gamma(X)g^{-1} \cap \Gamma(X)] < \infty \text{ and } [\Gamma(X): g^{-1}\Gamma(X)g \cap \Gamma(X)] < \infty\}.$$

The natural homomorphism $G \to \mathrm{Aut}(D)$ induces the homomorphism

$$\alpha: \mathbf{G}(\mathbf{Q}) \to \mathrm{Aut}(D).$$

Since $\mathbf{G}$ has no factors defined over $\mathbf{Q}$ and compact over $\mathbf{R}$, the kernel of α is the center $Z_{\mathbf{Q}}$ of $\mathbf{G}(\mathbf{Q})$ and therefore it is finite. The image of $\Gamma \subset \mathbf{G}(\mathbf{Q})$ under α coincides with $\Gamma(X)$. If $\hat{\pi}: \widehat{D} \to X$ denotes the natural projection, we set

$$\widehat{G}_a(X) = \mathrm{Aut}(\widehat{D}), \quad \widehat{\Gamma}(X) = \{\hat{g} \in \widehat{G}_a(X) | \hat{\pi} \circ \hat{g} = \hat{\pi}\}.$$

To define a homomorphism $\hat{\chi}: G_a(X) \to \widehat{G}_a(X)$, we take an element $g \in G_a(X)$. For each k, g induces a map $\hat{\chi}_k(g): \widehat{D} \to X_k$ (see **[6**, p. 158]). We set

$$\hat{\chi}(g) = \varprojlim \hat{\chi}_k(g) \in \mathrm{Aut}(\widehat{D}).$$

Then we have

$$\hat{\mu}(gd) = \hat{\chi}(g)\hat{\mu}(d) \quad \text{for } d \in D \text{ and } g \in G_a(X).$$

Since $\hat{\mu}(D)$ is dense in $\widehat{D}$, we have

$$\hat{\chi}(g_1 g_2) = \hat{\chi}(g_1)\hat{\chi}(g_2) \quad \text{for } g_1, g_2 \in G_a(X);$$

hence $\hat{\chi}: G_a(X) \to \widehat{G}_a(X)$ is a homomorphism.

PROPOSITION 1.2. *Let* $\hat{d}_0$ *be an element of* $\hat{\mu}(D)$. *Then* $\hat{\chi}$ *is an isomorphism between* $G_a(X)$ *and the subgroup of* $\widehat{G}_a(X)$ *consisting of all* $\hat{g} \in \widehat{G}_a(X)$ *such that* $\hat{d}_0\hat{g} \in \hat{\mu}(D)$.

Proof. See [**6**, Lemma 4].

The group $\widehat{G}_a(X)$ is a complete locally compact topological group relative to the topology in which a basis of neighborhoods of the identity consists of the subgroups of finite index in $\widehat{\Gamma}(X)$.

PROPOSITION 1.3. (i) $G_a(X)$ *is dense in* $\widehat{G}_a(X)$.

(ii) $\hat{\chi}$ *induces an isomorphism between the double cosets* $\Gamma(X) \setminus G_a(X)/\Gamma(X)$ *and* $\widehat{\Gamma}(X) \setminus \widehat{G}_a(X)/\widehat{\Gamma}(X)$.

Proof. (i) See [**6**, Theorem 1].

(ii) This follows from [**6**, Lemma 3 and Lemma 3′].

Let $\sigma \in \mathrm{Aut}(\mathbf{C})$. Then we consider the complex variety X^σ obtained from X by the base change. Let D^σ be the universal covering manifold of X^σ, and let

$$\Gamma^\sigma = \Gamma(X^\sigma) \subset \mathrm{Aut}(D^\sigma)$$

be the fundamental group of X^σ. If the varieties X_k^σ are the conjugates of X_k, we set

$$\widehat{D}^\sigma = \varprojlim X_k^\sigma, \qquad \widehat{G}_a(X^\sigma) = \mathrm{Aut}(\widehat{D}^\sigma).$$

Then $\widehat{G}_a(X^\sigma)$ is a complete locally compact topological group in the topology of subgroups of finite index in

$$\widehat{\Gamma}(X^\sigma) = \{\hat{g}^\sigma \in \widehat{G}_a(X^\sigma) | \hat{\pi}^\sigma \circ \hat{g}^\sigma = \hat{\pi}^\sigma\},$$

where $\hat{\Gamma}^\sigma \colon \widehat{D}^\sigma \to X^\sigma$ is the natural projection. As in the case of X, we can construct the homomorphism $\hat{\chi}^\sigma \colon G_a(X^\sigma) \to \widehat{G}_a(X^\sigma)$, where

$$G_a(X^\sigma) = \{g^\sigma \in \mathrm{Aut}(D^\sigma) | [\Gamma^\sigma \colon g^\sigma \Gamma^\sigma (g^\sigma)^{-1} \cap \Gamma^\sigma] < \infty \text{ and } [\Gamma^\sigma \colon (g^\sigma)^{-1} \Gamma^\sigma g^\sigma \cap \Gamma^\sigma] < \infty\}.$$

Let $V \subset G_a(X)$ be a subgroup with $[\Gamma(X) \colon V \cap \Gamma(X)] < \infty$, and let $\widehat{V}$ be the closure of $\hat{\chi}(V)$ in $\widehat{G}_a(X)$. If $\hat{g} \in \widehat{V} \subset \widehat{G}_a(X)$, then there are morphisms $g_k \colon Z_k \to X_k$ such that

$$\varprojlim Z_k = \widehat{D} = \varprojlim X_k, \qquad \hat{g} = \varprojlim g_k.$$

Applying σ to g_k, we obtain morphisms $g_k^\sigma: Z_k^\sigma \to X_k^\sigma$. Then

$$\hat{g}^\sigma = \varprojlim g_k^\sigma$$

is an element of $\widehat{G}(X^\sigma) = \text{Aut}(\widehat{D}^\sigma)$. We set $\widehat{V}^\sigma = \{\hat{g}^\sigma | \hat{g} \in \widehat{V}\}$ and define the subgroup V^σ of $G_a(X^\sigma)$ by

$$V^\sigma = (\hat{\chi}^\sigma)^{-1}(\widehat{V}^\sigma \cap \text{Im}(\hat{\chi}^\sigma)).$$

Now let G_a be a subgroup of finite index in $\alpha(\mathbf{G}(\mathbf{Q}))$ containing $\alpha(\Gamma) = \Gamma(X)$, and let

$$G_a^\sigma = (G_a)^\sigma \subset G_a(X^\sigma).$$

PROPOSITION 1.4. *G_a^σ is dense in the connected component of the identity of* $\text{Aut}(D^\sigma)$ *in the ordinary topology.*

Proof. G_a^σ is contained in the connected component of the identity of $\text{Aut}(D^\sigma)$ by [**10**, Lemma 3.7]. The density follows from [**6**, Theorem 5] (see also [**6**, Theorem A.7]).

THEOREM 1.5. *The group $\Gamma^\sigma = \Gamma(X^\sigma)$ is an arithmetic subgroup of the connected component of the identity of* $\text{Aut}(D^\sigma)$.

Proof. This follows from the main theorems in [**6**] and [**7**].

2. The homomorphism $\hat{\rho}$. Let $\mathbf{G}$, G, K, D, Γ and X be as in §1. We consider another semisimple algebraic $\mathbf{Q}$-group $\mathbf{G}'$ and its arithmetic subgroup $\Gamma' \subset \mathbf{G}'(\mathbf{Q})$ that is torsion free. As in the case of $\mathbf{G}$, we associate G', K', D' and the arithmetic variety $X' = \Gamma' \backslash D'$ to $\mathbf{G}'$. Let $\rho: G \to G'$ be a homomorphism, $\phi: X \to X'$ a morphism of varieties, and $\tau: D \to D'$ a holomorphic lifting of ϕ such that ρ and τ are equivariant, i.e.,

$$\tau(gy) = \rho(g)\tau(y) \quad \text{for all } g \in G \text{ and } y \in D.$$

Let $\{\Gamma_k\}$, $\{X_k\}$ and $\widehat{D}$ be as in §1, and let $\{\Gamma'_k\}$ be an inductive system of arithmetic subgroups of finite index of Γ' such that $\rho(\Gamma_k) \subset \Gamma'_k$ for each $k \geq 1$. The quotient spaces $X'_k = \Gamma'_k \backslash D'$ are arithmetic varieties and they form a projective system $\{X'_k\}$ of covering manifolds of X'. The holomorphic map $\tau: D \to D'$ induces a morphism $\phi\colon X_k \to X'_k$ for each $k \geq 1$. We set

$$\widehat{D}' = \varprojlim X'_k, \qquad \widehat{G}_a(X') = \operatorname{Aut}(\widehat{D}'),$$

and define the holomorphic map $\hat{\tau}\colon \widehat{D} \to \widehat{D}'$ by

$$\hat{\tau} = \varprojlim \phi_k.$$

For each $\hat{g} \in \widehat{G}_a \subset \widehat{G}_a(X) = \operatorname{Aut}(\widehat{D})$, there are elements

$$g_k \in \alpha^{-1}(G_a) \subset \mathbf{G}(\mathbf{Q})$$

(recall that $\operatorname{Ker}(\alpha) = Z_{\mathbf{Q}}$ is finite) such that

$$\hat{g} = \varprojlim p_k(g_k),$$

where the morphisms $p_k(g_k)$ are determined as follows: Let

$$\Gamma_{g,k} = \Gamma_k \cap g_k^{-1}\Gamma_k g_k,$$
$$X_{g,k} = \Gamma_{g,k} \backslash D.$$

Then, for each k, we define

$$p_k(g_k)\colon X_{g,k} \to X_k$$

to be the morphism of arithmetic varieties induced by the left multiplication $g_k\colon D \to D$, $D = G/K$. We have

$$g_k = \gamma_{j,k} g_j \quad \text{with } \gamma_{j,k} \in \Gamma_j \text{ for } j \leq k.$$

Recall that $\{\Gamma'_k\}$ is an inductive system of arithmetic subgroups of finite index of Γ' such that $\rho(\Gamma_k) \subset \Gamma'_k$ for each k. We set

$$\Gamma'_{g,k} = \Gamma'_k \cap \rho(g_k)^{-1}\Gamma'_k \rho(g_k),$$
$$X'_{g,k} = \Gamma'_{g,k} \backslash D'$$

for each k. Then we have $\rho(\Gamma_{g,k}) \subset \Gamma'_{g,k}$, and hence the left multiplication maps $\rho(g_k)\colon D' \to D'$ induce the morphisms

$$p'_k(\rho g_k)\colon X'_{g,k} \to X'_k.$$

Since we have

$$\rho(g_k) = \rho(\gamma_{j,k} g_j) = \rho(\gamma_{j,k})\rho(g_j)$$

with

$$\rho(\gamma_{j,k}) \in \rho(\Gamma_j) \subset \Gamma'_j \quad \text{for } j \leq k,$$

the collection $\{\rho(g_k)\}$ of liftings $\rho(g_k)$ of $p'_k(\rho g_k)$ determines the projective limit

$$\varprojlim p'_k(\rho g_k) \in \widehat{G}_a(X') = \operatorname{Aut}(\widehat{D}').$$

Thus we obtain a homomorphism $\hat{\rho}\colon \widehat{G}_a \to \widehat{G}'_a$ defined by

$$\hat{\rho}(\hat{g}) = \varprojlim p'_k(\rho g_k)$$

for all $\hat{g} = \varprojlim g_k \in \widehat{G}_a$.

PROPOSITION 2.1. *If $\hat{\tau}\colon \widehat{D} \to \widehat{D}'$ and $\hat{\rho}\colon \widehat{G}_a \to \widehat{G}_a(X')$ are as described above, then*

$$\hat{\tau}(\hat{g}\hat{y}) = \hat{\rho}(\hat{g})\hat{\tau}(\hat{y}) \quad \text{for all } \hat{g} \in \widehat{G}_a \text{ and } \hat{y} \in \widehat{D}.$$

Proof. If $\mu_k\colon D \to X_k$, $\mu_{g,k}\colon D \to X_{g,k}$, $\mu'_k\colon D' \to X'_k$ and $\mu'_{g,k}\colon D' \to X'_{g,k}$ are the natural covering maps, then we have

$$\mu'_k\tau(g_k y) = \phi_k(g_k\mu_{g,k}(y)),$$
$$\mu'_k\rho(g_k)\tau(y) = (p'(\rho g_k))\phi_{g,k}(\mu_{g,k}(y))$$

for all $y \in D$, where

$$\phi_{g,k}\colon X_{g,k} \to X'_{g,k}$$

is the morphism induced by $\tau\colon D \to D$ for each k with

$$\tau = \varprojlim \phi_{g,k}.$$

Since $\tau(g_k y) = \rho(g_k)\tau(y)$ for all $y \in D$, we have

$$\phi_k(g_k\mu_{g,k}(y)) = (p'_k(\rho g_k))\phi_{g,k}(\mu_{g,k}(y))$$

for all $y \in D$. Since $\mu_{g,k}\colon D \to X_{g,k}$ is surjective and

$$\hat{\rho}(\hat{g}) = \varprojlim p'_k(\rho g_k),$$

it follows that

$$\hat{\tau}(\hat{g}\hat{y}) = \hat{\rho}(\hat{g})\hat{\tau}(\hat{y}) \quad \text{for all } \hat{y} = \varprojlim (\mu_{g,k}(y)) \in \widehat{D}.$$

3. Conjugates of $\hat{\rho}$. Let $\{X_k\}$ and $\{X'_k\}$ be the projective systems of finite unramified covering manifolds of X and X', respectively, considered in §2. We fix an element $\sigma \in \operatorname{Aut}(\mathbf{C})$. The varieties X^σ, X'^σ, X_k^σ and $X_k'^\sigma$ are arithmetic varieties, and the collections

$\{X_k^\sigma\}$ and $\{X_k'^\sigma\}$ are projective systems of finite unramified covering manifolds of X^σ and X'^σ respectively (see [5], [6], [7], [10]). Let $\phi_k^\sigma: X_k^\sigma \to X_k'^\sigma$ be the conjugate morphism of $\phi_k: X_k \to X_k'$ for each k. Let $\widehat{D}$, $\widehat{G}_a(X^\sigma)$ and $\widehat{\Gamma}(X^\sigma)$ be as in §1, and let

$$\widehat{D}'^\sigma = \varprojlim X_k'^\sigma, \qquad \widehat{G}_a(X'^\sigma) = \mathrm{Aut}(\widehat{D}'^\sigma),$$
$$\widehat{\Gamma}(X'^\sigma) = \{\hat{g}'^\sigma \in \widehat{G}_a(X'^\sigma) | \hat{\pi}'^\sigma \circ \hat{g}'^\sigma = \hat{\pi}'^\sigma\},$$

where $\hat{\pi}'^\sigma: \widehat{D}'^\sigma \to X'^\sigma$ is the natural projection. We define the map $\hat{\tau}^\sigma: \widehat{D}^\sigma \to \widehat{D}'^\sigma$ by

$$\hat{\tau}^\sigma = \varprojlim \phi_k^\sigma.$$

If $\hat{h} \in \widehat{G}_a^\sigma$, then there exist a projective system $\{Z_k\}$ of finite unramified covering manifolds of X^σ and morphisms $h_k: Z_k \to X_k^\sigma$ such that

$$\varprojlim Z_k = \widehat{D}^\sigma = \varprojlim X_k^\sigma, \qquad \hat{h} = \varprojlim h_k.$$

The morphisms h_k induce the morphisms

$$h_k^{\sigma^{-1}}: Z_k^{\sigma^{-1}} \to X_k.$$

Let $\tilde{h}_k^{\sigma^{-1}}: D \to D$ be an element of $\alpha^{-1}(G_a(X)) \subset \mathbf{G}(\mathbf{Q})$ that is a lifting of $h_k^{\sigma^{-1}}$ for each k. We define the homomorphism $\hat{\rho}^\sigma: \widehat{G}_a^\sigma \to \widehat{G}_a(X'^\sigma)$ by

$$\hat{\rho}^\sigma(\hat{h}) = \varprojlim (p_k'(\rho(\tilde{h}_k^{\sigma^{-1}})))^\sigma$$

where

$$p_k'(\rho(\tilde{h}_k^{\sigma^{-1}})): W_k' \to X_k'$$

is the morphism induced from $\rho(\tilde{h}_k^{\sigma^{-1}}): D' \to D'$ for each k.

Proposition 3.1. *$\hat{\tau}^\sigma(\hat{g}^\sigma\hat{y}^\sigma) = \hat{\rho}^\sigma(\hat{g}^\sigma)\hat{\tau}^\sigma(\hat{y}^\sigma)$ for all $\hat{g}^\sigma \in \widehat{G}_a^\sigma$ and $\hat{y}^\sigma \in \widehat{D}^\sigma$.*

Proof. For each k the relation $\hat{\tau}(\hat{g}\hat{y}) = \hat{\rho}(\hat{g})\hat{\tau}(\hat{y})$ induces the following commutative diagram:

$$\begin{array}{ccc} X_{g,k} & \xrightarrow{\phi_{g,k}} & X'_{g,k} \\ {\scriptstyle p_k(g_k)}\downarrow & & \downarrow{\scriptstyle p_k'(\rho g_k)} \\ X_k & \xrightarrow{\phi_k} & X_k' \end{array}$$

Applying σ to this diagram, we obtain

$$\begin{array}{ccc} X^{\sigma}_{g,k} & \xrightarrow{\phi^{\sigma}_{g,k}} & X'^{\sigma}_{g,k} \\ {\scriptstyle (p_k(g_k))^{\sigma}}\downarrow & & \downarrow{\scriptstyle (p'_k(\rho g_K))^{\sigma}} \\ X^{\sigma}_k & \xrightarrow{\phi^{\sigma}_k} & X'^{\sigma}_k \end{array}$$

so that

$$\phi^{\sigma}_k((p_k(g_k))^{\sigma}x^{\sigma}_k) = (p'_k(\rho g_k))^{\sigma}\phi^{\sigma}_{g,k}(x^{\sigma}_k)$$

for all $x^{\sigma}_k \in X^{\sigma}_{g,k}$. By taking projective limits we obtain

$$\hat{\tau}^{\sigma}(\hat{g}^{\sigma}\hat{y}^{\sigma}) = \hat{\rho}^{\sigma}(\hat{g}^{\sigma})\hat{\tau}^{\sigma}(\hat{y}^{\sigma})$$

for all $\hat{y}^{\sigma} \in \widehat{D}^{\sigma}$.

PROPOSITION 3.2. *$\hat{\rho}(\widehat{\Gamma}(X^{\sigma}))$ is contained in $\widehat{\Gamma}(X'^{\sigma})$.*

Proof. If $\hat{h} \in \widehat{\Gamma}(X^{\sigma}) \subset \widehat{G}_a(X^{\sigma})$, there are morphisms $h_k \colon Z_k \to X^{\sigma}_k$ such that

$$\varprojlim Z_k = \widehat{D}^{\sigma} = \varprojlim X^{\sigma}_k, \quad \hat{h} = \varprojlim h_k \quad \text{and} \quad \pi^{\sigma}_k \circ h_k = \pi^0_k,$$

where $\pi^0_k \colon Z_k \to X^{\sigma}$ is the natural projection. Applying σ^{-1}, we obtain the morphisms $h^{\sigma^{-1}}_k \colon Z^{\sigma^{-1}}_k \to X_k$ such that

$$\pi_k \circ h^{\sigma^{-1}}_k = (\pi^0_k)^{\sigma^{-1}} \quad \text{for all } k,$$

where the maps $(\pi^0_k)^{\sigma^{-1}} \colon Z_k \to X$ are the natural projections. If the maps $\tilde{h}^{\sigma^{-1}}_k \colon D \to D$ with $\tilde{h}^{\sigma^{-1}}_k \in \mathbf{G}(\mathbf{Q})$ are as before, we have $\tilde{h}^{\sigma^{-1}}_k(\Gamma) \subset \Gamma$, and therefore

$$\rho(\tilde{h}^{\sigma^{-1}}_k)(\Gamma') \subset \Gamma' \quad \text{for all } k.$$

Thus the morphisms $p'_k(\rho(\tilde{h}^{\sigma^{-1}}_k)) \colon W'_k \to X'_k$ satisfy

$$\pi'^0_k \circ p'_k(\rho(\tilde{h}^{\sigma^{-1}}_k)) = \pi'_k \quad \text{for all } k,$$

where the maps $\pi'^0_k \colon W'_k \to X'$ are the natural projections. Hence we have

$$(\pi'^0_k)^{\sigma} \circ (p'_k(\rho(\tilde{h}^{\sigma^{-1}}_k)))^{\sigma} = (\pi'^0_k)^{\sigma}$$

for all k. Thus it follows that $\hat{\rho}^{\sigma}(\hat{h}) \in \widehat{\Gamma}(X'^{\sigma})$.

4. The homomorphism ρ^σ. Let $X^\sigma, D^\sigma, \Gamma^\sigma, G_a^\sigma, G_a(X^\sigma), \widehat{D}^\sigma, \widehat{\Gamma}(X^\sigma), \widehat{G}_a(X^\sigma)$ be as in §1, and let $\widehat{D}'^\sigma$, $\widehat{\Gamma}_a(X'^\sigma)$, $\widehat{G}(X'^\sigma)$ be as in §3. Let D'^σ be the universal covering space of X'^σ, and let

$$\Gamma'^\sigma = \Gamma(X'^\sigma) \subset \mathrm{Aut}(D'^\sigma)$$

be the fundamental group of X'^σ. We set

$$G_a(X'^\sigma) = \{g'^\sigma \in \mathrm{Aut}(D'^\sigma) | [\Gamma'^\sigma : g'^\sigma \Gamma'^\sigma (g'^\sigma)^{-1} \cap \Gamma'^\sigma] < \infty \text{ and } [\Gamma'^\sigma : (g'^\sigma)^{-1} \Gamma'^\sigma g'^\sigma \cap \Gamma'^\sigma] < \infty\}.$$

Since the map $\hat{\mu}: D \to \widehat{D}$ and the homomorphism $\hat{\chi}: G_a(X) \to \widehat{G}_a(X)$ described in §1 are injective, we shall identify each of $D^\sigma, D'^\sigma, G_a(X^\sigma), G_a(X'^\sigma)$ with its embedded image in $\widehat{D}^\sigma$, $\widehat{D}'^\sigma$, $\widehat{G}_a(X^\sigma)$, $\widehat{G}_a(X'^\sigma)$ respectively. By Proposition 1.1(i) and the above identification, D^σ and D'^σ are connected components of $\widehat{D}^\sigma$ and $\widehat{D}'^\sigma$ respectively. We shall assume that D'^σ is the connected component of $\widehat{D}'^\sigma$ containing the image of D^σ under the map τ^σ.

PROPOSITION 4.1. *If D'^σ is the connected component of $\widehat{D}'^\sigma$ chosen as above, then $\hat{\rho}^\sigma(G_a^\sigma)$ is contained in $G_a(X'^\sigma)$.*

Proof. Let $d_0^\sigma \in D^\sigma \subset \widehat{D}^\sigma$. Then we have

$$\hat{\tau}^\sigma(d_0^\sigma) \in D'^\sigma \subset \widehat{D}'^\sigma.$$

By Proposition 1.2 we have

$$G_a^\sigma \subset G_a(X^\sigma) = \{\hat{g}^\sigma \in \widehat{G}_a(X^\sigma) | \hat{g}^\sigma d_0^\sigma \in D^\sigma\},$$
$$G_a(X'^\sigma) = \{\hat{g}'^\sigma \in \widehat{G}_a(X'^\sigma) | \hat{g}'^\sigma \hat{\tau}^\sigma(d_0^\sigma) \in D'^\sigma\}.$$

If $g^\sigma \in G_a^\sigma$, then $g^\sigma d_0^\sigma \in D^\sigma$; hence from the continuity of $\hat{\tau}^\sigma$ it follows that

$$\hat{\tau}^\sigma(g^\sigma d_0^\sigma) \in \hat{\tau}^\sigma(D^\sigma) \subset D'^\sigma.$$

Thus we get

$$\hat{\rho}^\sigma(g^\sigma)\hat{\tau}^\sigma(d_0^\sigma) = \hat{\tau}^\sigma(g^\sigma d_0^\sigma) \in D'^\sigma.$$

Therefore it follows that

$$\hat{\rho}^\sigma(g^\sigma) \in G_a(X'^\sigma) \quad \text{for all } g^\sigma \in G_a^\sigma.$$

Now we define the homomorphism $\rho^\sigma: G_a^\sigma \to G_a(X'^\sigma)$ and $\tau^\sigma: D^\sigma \to D'^\sigma$ by

$$\rho^\sigma = \hat{\rho}^\sigma|_{G_a^\sigma}, \qquad \tau^\sigma = \hat{\tau}^\sigma|_{D^\sigma}.$$

Then ρ^σ and τ^σ satisfy the relation

$$\tau^\sigma(g^\sigma d^\sigma) = \rho^\sigma(g^\sigma)\tau^\sigma(d^\sigma)$$

for all $g^\sigma \in G_a^\sigma$ and $d^\sigma \in D^\sigma$.

5. Conjugates of equivariant pairs. Let $\rho: G \to G'$ and $\tau: D \to D'$ be equivariant as before so that we have

$$\tau(gz) = \rho(g)\tau(z)$$

for all $g \in G$ and $z \in D$. Also as before let $\Gamma \subset G$ and $\Gamma' \subset G'$ be torsion free arithmetic subgroups with $\rho(\Gamma) \subset \Gamma'$, and let $\phi: X \to X'$ be the morphism induced by τ where $X = \Gamma \setminus D$ and $X' = \Gamma' \setminus D'$. If $\rho^\sigma: G_a^\sigma \to G_a'^\sigma$ and $\tau^\sigma: D^\sigma \to D'^\sigma$ are as in §4, we denote by G_0^σ, $G_0'^\sigma$ the connected components of the identity of $\mathrm{Aut}(D^\sigma)$, $\mathrm{Aut}(D'^\sigma)$ respectively. By Proposition 1.4 G_a^σ is dense in G_0^σ; hence by Proposition 3.1 and Proposition 4.1, we have

$$\tau^\sigma(hy) = \rho^\sigma(h)\tau^\sigma(y)$$

for all $h \in G_a^0$ and $y \in D^\sigma$. We set

$$H = \{(g, g') \in G_0^\sigma \times G_0'^\sigma | \tau^\sigma(gy) = g'\tau^\sigma(y) \text{ for all } y \in D^\sigma\}.$$

Since the set $\{(g, \rho^\sigma(g)) | g \in G_a^\sigma\}$ is contained in H and G_a^σ is dense in G_0^σ, the projection map $\mathrm{pr}_1: H \to G_0^\sigma$ is surjective.

PROPOSITION 5.1. *H is a reductive Lie group.*

Proof. Let K be the kernel of the projection map $\mathrm{pr}_1: H \to G_0^\sigma$. Then we have

$$\begin{aligned} K &= \{(1, g') \in G_0^\sigma \times G_0'^\sigma | \tau^\sigma(y) = g'\tau^\sigma(y) \text{ for all } y \in D^\sigma\} \\ &\cong \{g' \in G_0'^\sigma | \tau^\sigma(y) = g'\tau^\sigma(y) \text{ for all } y \in D^\sigma\} = \bigcap_{y \in D^\sigma} \mathrm{Iso}(\tau^\sigma(y)), \end{aligned}$$

where $\mathrm{Iso}(\tau^\sigma(y))$ is the isotropy subgroup of $\tau^\sigma(y)$ in $G_0'^\sigma$. Hence K is a compact Lie group, and therefore a reductive Lie group. Since there is an exact sequence

$$1 \to K \to H \xrightarrow{\mathrm{pr}_1} G_0^\sigma \to 1$$

of Lie groups with K and G_0^σ reductive, it follows that H is reductive.

Theorem 5.2. *There exist a finite covering G_1^σ of G_0^σ and a homomorphism $\rho_1^\sigma : G_1^\sigma \to G_0'^\sigma$ of Lie groups such that ρ_1^σ and τ^σ are equivariant and $\rho_1^\sigma(\Gamma^\sigma)$ is contained in Γ'^σ.*

Proof. Decompose the reductive Lie group H into an almost direct product of simple Lie groups and simple tori, and let G_1^σ be the product of the simple factors of H which map nontrivially to G_0^σ (see [**2**, III.2.2] for a similar argument). Then the kernel of the map from G_1^σ to G_0^σ is finite, and hence G_1^σ is a finite covering Lie group of G_0^σ. We define the action of G_1^σ on D^σ and the homomorphism $\rho_1^\sigma : G_1^\sigma \to G_0'^\sigma$ by

$$(g, g') \cdot y = gy, \qquad \rho_1^\sigma(g, g') = g'$$

for all $y \in D^\sigma$ and $(g, g') \in G_1^\sigma \subset H \subset G_0^\sigma \times G_0'^\sigma$. Then we have

$$\tau^\sigma((g, g') \cdot y) = \tau^\sigma(gy) = g'\tau^\sigma(y) = \rho_1^\sigma(g, g')\tau^\sigma(y)$$

for all $(g, g') \in G_1^\sigma$ and $y \in D^\sigma$. Hence we obtain a homomorphism of Lie groups ρ_1^σ from a finite covering G_1^σ of G_0^σ to $G_0'^\sigma$ such that

$$\tau^\sigma(h_1 y) = \rho_1^\sigma(h_1)\tau^\sigma(y)$$

for all $h_1 \in G_1^\sigma$ and $y \in D^\sigma$. Thus it follows that ρ_1^σ and τ^σ are equivariant. Now it remains to show that $\rho_1^\sigma(\Gamma^\sigma)$ is contained in Γ'^σ. By Proposition 3.2, $\hat{\rho}(\widehat{\Gamma}(X^\sigma))$ is contained in $\widehat{\Gamma}(X'^\sigma)$; hence $\hat{\rho}^\sigma : \widehat{G}_a^\sigma \to \widehat{G}_a(X'^\sigma)$ induces the mapping

$$\hat{\rho}^\sigma : \widehat{\Gamma}(X^\sigma) \setminus \widehat{G}_a^\sigma / \widehat{\Gamma}(X^\sigma) \to \widehat{\Gamma}(X'^\sigma) \setminus \widehat{G}_a(X'^\sigma)/\widehat{\Gamma}(X'^\sigma)$$

of double cosets. By Proposition 1.3(ii) $\hat{\rho}^\sigma$ induces the mapping

$$\hat{\rho}^\sigma : \Gamma(X^\sigma) \setminus G_a^\sigma / \Gamma(X^\sigma) \to \Gamma(X'^\sigma) \setminus G_a(X'^\sigma)/\Gamma(X'^\sigma).$$

Thus we have

$$\hat{\rho}(\Gamma(X^\sigma)) = \rho^\sigma(\Gamma^\sigma) \subset \Gamma(X'^\sigma) = \Gamma'^\sigma.$$

Since $\rho_1^\sigma(\Gamma^\sigma) = \rho^\sigma(\Gamma^\sigma)$, it follows that $\rho_1^\sigma(\Gamma^\sigma) \subset \Gamma'^\sigma$.

6. Conjugates of Kuga fiber varieties. First, we shall describe the construction of Kuga fiber varieties (see [**1**], [**8**], [**11**, Chapter 4] for details). Let V be a $\mathbf{Q}$-vector space of dimension $2n$, and let L be a lattice in V. Let β be a nondegenerate alternating bilinear form on V such that $\beta(L, L) \subset \mathbf{Z}$. Let

$$\mathrm{Sp}(V, \beta) = \{g \in \mathrm{GL}(V) | \beta(gx, gy) = \beta(x, y) \text{ for all } x, y \in V\}$$

be the symplectic group of the pair (V, β), and let $\mathscr{H}$ denote the Siegel space

$$\mathscr{H} = \{J \in \mathrm{GL}(V_{\mathbf{R}}) | J^2 = -1, \ \beta(x, Jy) \text{ is a positive definite symmetric bilinear form in } x, y \in V_{\mathbf{R}}\}.$$

Then each element $J \in \mathscr{H}$ defines a complex structure on $V_{\mathbf{R}}$ and there is a unique complex analytic structure on $\mathscr{H} \times V_{\mathbf{R}}$ such that the projection $P: \mathscr{H} \times V_{\mathbf{R}} \to \mathscr{H}$ is a complex vector bundle over $\mathscr{H}$. For each J if we denote the complex vector space $(V_{\mathbf{R}}, J)$ by V_J, then the complex torus $A_J = V_J/L$ is an abelian variety with the polarization β. We set

$$A_{\mathscr{H}} = L \backslash \mathscr{H} \times V_{\mathbf{R}},$$

where the action of L on $\mathscr{H} \times V_{\mathbf{R}}$ is given by

$$l \cdot (J, v) = (J, v + l) \quad \text{for } j \in \mathscr{H} \text{ and } l \in L.$$

Then the vector bundle $P: \mathscr{H} \times V_{\mathbf{R}} \to \mathscr{H}$ induces the fiber bundle $\pi_{\mathscr{H}}: A_{\mathscr{H}} \to \mathscr{H}$ whose fibers are abelian varieties polarized by β. We set

$$\mathrm{Sp}(L, \beta) = \{g \in \mathrm{Sp}(V, \beta) | gL = L\},$$

and take a subgroup Γ_S of $\mathrm{Sp}(L, \beta)$ of finite index that contains no elements of finite order. Then, as discussed in §1, the quotient $X = \Gamma_S \backslash \mathscr{H}$ is an arithmetic variety that can be considered as a Zariski open subset of a complex projective variety. Now the fiber bundle $\pi_{\mathscr{H}}: A_{\mathscr{H}} \to \mathscr{H}$ induces the standard family of abelian varieties $\pi_S: Y_S \to X_S$ over X_S.

Let $\mathbf{G}$, G and $D = G/K$ be as in §1. Let $\rho: G \to \mathrm{Sp}(V, \beta)(\mathbf{R})$ be a symplectic representation of G, and let Γ be a torsion free arithmetic subgroup of $\mathbf{G}$ with $\rho(\Gamma) \subset \Gamma_S$. We shall assume that the quotient $X = \Gamma \backslash D$ is compact. Let $\tau: D \to \mathscr{H}$ be a holomorphic map such that ρ and τ are equivariant, and let $\phi: X \to X_S$ be the morphism of varieties induced by τ. By pulling back the fiber bundle $\pi_S: Y_S \to X_X$ via the morphism $\phi: X \to X_S$, we obtain the fiber bundle over the arithmetic variety X whose fibers are abelian varieties polarized by β. It is known (cf. **[8]**, **[11**, Theorem 8.6]) that the fiber space Y of the family of abelian varieties $\pi: y \to X$ has a structure of a complex projective variety. The fiber space Y (or the fiber bundle $\pi: Y \to X$ itself) is called a Kuga fiber variety.

To consider conjugates of Kuga fiber varieties, we shall first state some of the known results of G. Shimura about the families of abelian varieties associated to PEL-types (see e.g. **[12]**, **[15]** for details).

THEOREM 6.1. *Let* Ω *be a normal admissible* PEL-*type* (*see* [**14**] *for its definition*). *Then there exists a family of abelian varieties* $f: W \to U$ *denoted by* $\mathscr{F}(\Omega)$ *with the following properties*:

(i) *For each* $u \in U$, *the fiber* $f^{-1}(u)$ *of* $\mathscr{F}(\Omega)$ *has a* PEL-*structure* Q_u *of type* Ω.

(ii) *For each* PEL-*structure* Q *of type* Ω, *there exists one and only one point* $u \in U$ *such that* Q *is isomorphic to* Q_u.

(iii) U *and* W *are Zariski open subset of projective varieties and* f *is a morphism of varieties.*

(iv) *There exists a number field* k_Ω *such that* U, W *and* f *are defined over* k_Ω.

(v) *For each* $\sigma \in \mathrm{Aut}(\mathbf{C})$ *there exists a* PEL-*type* Ω^σ *and an isomorphism of* $\mathscr{F}(\Omega)^\sigma$ *to* $\mathscr{F}(\Omega^\sigma)$ *defined over* k_{Ω^σ}.

Proof. See [**12**, Theorem 5.3] and [**14**, Proposition 3.1]; see also [**5**, §8].

THEOREM 6.2. *Let* Ω *be as in Theorem* 6.1, *and let* $f: W \to U$ *be a family of abelian varieties associated to* Ω. *Let* $f': W' \to U'$ *be another family of abelian varieties such that each fiber* $(f')^{-1}(u)$ *for* $u \in U$ *has a* PEL-*structure* Q'_u. *Suppose that at least one* Q'_u *is of type* Ω *and that the family* $\{Q'_u | u \in U\}$ *of* PEL-*structures satisfies the property* (ii) *in Theorem* 6.1. *Then there is a biregular isomorphism between* $f: W \to U$ *and* $f': W' \to U'$.

Proof. See [**12**, Theorem 5.4 and Theorem 5.5].

THEOREM 6.3. *If* $\pi: Y \to X$ *is a Kuga fiber variety and* $\sigma \in \mathrm{Aut}(\mathbf{C})$, *then the conjugate* $\pi^\sigma: Y^\sigma \to X^\sigma$ *is also a Kuga fiber variety.*

Proof. Let $G_S = \mathrm{Sp}(V, \beta)(\mathbf{R})$. We shall apply Theorem 5.2 to the equivariant pair

$$\rho: G \to G_S, \qquad \tau: D \to \mathscr{H}$$

and the morphism $\phi: X \to X_S$ of arithmetic varieties. For $\sigma \in \mathrm{Aut}(\mathbf{C})$, we consider the conjugate morphism

$$\phi^\sigma: X^\sigma \to X_S^\sigma$$

and its lifting $\tau^\sigma: D^\sigma \to D_S^\sigma$. Since Γ_S and H can be obtained from an admissible and normal PEL-type (see [**14**, §4–§6]), X_S^σ is isomorphic to $\Gamma_S^{(\sigma)} \backslash \mathscr{H}'$ where $\mathscr{H}' = \mathscr{H}$ and $\Gamma_S^{(\sigma)}$ is an arithmetic subgroup of $(G_S)_0^\sigma$, the connected component of the identity of $\mathrm{Aut}(D_S^\sigma)$ (see e.g. [**14**, (3.2)]). Let

$$\pi_S^{(\sigma)}: Y_S^{(\sigma)} \to X_S^\sigma$$

be the standard family of abelian varieties obtained from $\Gamma_S^{(\sigma)}$, $\mathscr{H}$ and $\mathrm{Sp}(V, \beta)$. By Theorem 5.2 there exist a Lie group G_1^σ and a homomorphism $\rho_1^\sigma: G_1^\sigma \to (G_S)_0^\sigma$ such that ρ_1^σ and τ^σ are equivariant and $\rho_1^\sigma(\Gamma^\sigma)$ is contained in $\Gamma_S^{(\sigma)}$. Hence we obtain a Kuga fiber variety $\pi^{(\sigma)}: Y^{(\sigma)} \to X^\sigma$ by pulling back the standard family

$$\pi_S^{(\sigma)}: Y_S^{(\sigma)} \to X_S^\sigma$$

via the morphism $\phi^\sigma: S^\sigma \to X_S^\sigma$:

$$\begin{array}{ccc} Y^{(\sigma)} & \xrightarrow{\phi_Y^{(\sigma)}} & Y_S^{(\sigma)} \\ {\scriptstyle \pi^{(\sigma)}}\downarrow & & \downarrow{\scriptstyle \pi_S^{(\sigma)}} \\ X^\sigma & \xrightarrow{\phi^\sigma} & X_S^\sigma \end{array}$$

On the other hand, by applying σ to the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\phi_Y} & Y_S \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_S} \\ X & \xrightarrow{\phi} & X_S \end{array}$$

we obtain the following commutative diagram:

$$\begin{array}{ccc} Y^\sigma & \xrightarrow{\phi_Y^\sigma} & Y_S^\sigma \\ {\scriptstyle \pi^\sigma}\downarrow & & \downarrow{\scriptstyle \pi_S^\sigma} \\ X^\sigma & \xrightarrow{\phi^\sigma} & X_S^\sigma \end{array}$$

If $\pi_S: Y_S \to X_S$ is a family of abelian varieties $\mathscr{F}(\Omega)$ associated to a PEL-type Ω, then by Theorem 6.1 the conjugate $\pi_S^\sigma: Y_S^\sigma \to X_S^\sigma$ is a family $\mathscr{F}(\Omega^\sigma)$ associated to the PEL-type Ω^σ. Note that the Lie group $(G_S)_0^\sigma$, the symmetric domain $\mathscr{H}$ and the arithmetic subgroup

$\Gamma_S^{(\sigma)}$ coincide with the corresponding objects associated to the standard family $\pi_S^{(\sigma)}: Y_S^{(\sigma)} \to X_S^\sigma$. If the construction of the fiber variety associated to a PEL-type described in [**12**, (3.14)] and [**13**, (5.1)] is used, then Y_S^σ can be constructed in the same way as $Y_S^{(\sigma)}$. In particular, at least one fiber of $\pi_S^{(\sigma)}: Y_S^{(\sigma)} \to X_S^\sigma$ has a PEL-structure $(Q^{(\sigma)})_u$ $(u \in X_S^\sigma)$ that is of type Ω^σ. The standard family $\pi_S^{(\sigma)}$ certainly satisfies the property (ii) in Theorem 6.1. To show that the conjugate family $\pi_S^\sigma: Y_S^\sigma \to X_S^\sigma$ satisfies the same property, let $(Q^\sigma)_u$ and $(Q^\sigma)_v$ be the PEL-structures of the fibers over $u, v \in X_S^\sigma$. Then u is equal to v by [**13**, Proposition 4.4] (see also [**15**, Theorem 4]); hence the conjugate family π_S^σ also satisfies the property (ii) in Theorem 6.1. Thus by Theorem 6.2 there is a biregular isomorphism between the families $\pi_S^\sigma: Y_S^\sigma \to X_S^\sigma$ and $\pi_S^{(\sigma)}: Y_S^{(\sigma)} \to X_S^\sigma$. Therefore there is an isomorphism between the diagram for $\pi^{(\sigma)}: Y^{(\sigma)} \to X^\sigma$ and the one for $\pi^\sigma: Y^\sigma \to X^\sigma$ given above; hence it follows that $\pi^\sigma: Y^\sigma \to X^\sigma$ is a Kuga fiber variety.

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Received August 31, 1989 and in revised form March 15, 1990.

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