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ON THE ROMANOV KERNEL AND KURANISHI'S L^2-ESTIMATE FOR $\overline{\partial}_b$ OVER A BALL IN THE STRONGLY PSEUDO CONVEX BOUNDARY

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As is proved by Kerzman-Stein, over a compact strongly pseudo convex boundary in C^n, Szegö projection S is the operator defined by Henkin-Ramirez *modulo compact operators*. While, over *a special ball*, U_ε, in the strongly pseudo convex boundary, in order to obtain a local embedding theorem of CR-structures, Kuranishi constructed the Neumann type operator N_b for $\overline{\partial}_b$ and so we have *a local Szegö operator* by

$$S_{U_\varepsilon} = \mathrm{id} - \overline{\partial}_b^* N_b \overline{\partial}_b \quad \text{on } U_\varepsilon,$$

where $\overline{\partial}_b^*$ means the adjoint operator of $\overline{\partial}_b$. There might be a relation between S_{U_ε} and the Romanov kernel like the case of the Szegö operator and the Henkin-Ramirez kernel. We study this problem and show some estimates for the Romanov kernel.

0. Introduction. Let $(M, {}^\circ T'')$ be an abstract strongly pseudo convex CR-manifold. Then as is well known, if $\dim_R M = 2n - 1 \geq 7$, $(M, {}^\circ T'')$ is locally embeddable in a complex euclidean space $C^n((Ak3), (K))$. In the proof of this local embedding theorem, it is shown that: over *a special ball* in the strongly pseudo convex boundary, an L^2-estimate for $\overline{\partial}_b$, which is stronger than the standard L^2-estimate, is established and so the L^2-solution operator for $\overline{\partial}_b$ is obtained. This operator plays an essential role in our local embedding theorem. Therefore it must be important to study this solution operator for $\overline{\partial}_b$ precisely.

In order to get a solution operator, there exists another method. By using an integral formula, a local solution operator for $\overline{\partial}_b$ is constructed explicitly by Henkin and Harvey-Polking. Obviously, these solution operators are different. And it seems quite interesting to study the relation between the L^2-solution for $\overline{\partial}_b$ and the explicit solution, obtained by using an integral formula. We recall the $\overline{\partial}$-case over a strongly pseudo convex domain in C^n. In this case, the explicit solution, constructed by Lieb and Range, is a certain kind of the essential part of the Kohn's L^2-solution. Therefore we could hope for a similar result in the $\overline{\partial}_b$ case over *a special ball* in the strongly pseudo convex boundary. As mentioned already, our L^2-a priori estimate is different from the standard L^2-estimate. Therefore in the above sense, it seems to be natural to consider that the explicit solution operator would satisfy the similar L^2-estimate. In this paper, we discuss this point over rigid hypersurfaces in C^n (for the definition, see §3 in this paper). And we prove our a priori estimate (Main Theorem in §5 in this paper) for the explicit solution operator.

1. CR-structure and $\overline{\partial}_b$-operator. Let M be a real hypersurface in C^n. Let p be a reference point of M. We assume that p is a smooth point, namely let ρ be a defining function of M in a neighborhood of p in C^n, i.e., there is a neighborhood $V(p)$ of p satisfying:

$$M \cap V(p) = \{q : q \in V(p),\ \rho(q) = 0\}$$

and

$$d\rho \neq 0 \quad \text{over } M \cap V(p).$$

Then over $M \cap V(p)$, we can introduce an CR-structure induced from C^n. Namely, let

$$^{\circ}T'' = T''C^n \cap C \otimes TM \quad \text{over } M \cap V(p).$$

Then this $^{\circ}T''$ satisfies

(1-1) $\quad ^{\circ}T'' \cap {}^{\circ}\overline{T''} = 0, \quad f\text{-}\dim_C(C \otimes TM/(^{\circ}T'' + {}^{\circ}\overline{T''})) = 1,$

(1-2) $[\Gamma(M \cap V(p), {}^{\circ}T''), \Gamma(M \cap V(p), {}^{\circ}T''] \subset \Gamma(M \cap V(p), {}^{\circ}T'').$

This pair $(M \cap V(p), {}^{\circ}T'')$ is called a CR-structure, or a CR-manifold.

Let $(M \cap V(p), {}^{\circ}T'')$ be a CR-manifold. We introduce a C^∞ vector bundle decomposition

$$C \otimes TM = {}^{\circ}T'' + {}^{\circ}\overline{T''} + C\zeta, \tag{1-3}$$

where

(1-3-1) $\quad \zeta$ is a real vector field,

(1-3-2) $\quad \zeta_q \notin {}^{\circ}T''_q + {}^{\circ}\overline{T''_q}$ for q in $M \cap V(p)$.

By using this decomposition, we have a Levi form

$$L(X, Y) = \sqrt{-1}[X, \overline{Y}]_\zeta \quad \text{for } X, Y \text{ in } \Gamma(M \cap V(p), {}^{\circ}T''),$$

where $[X, \overline{Y}]_\zeta$ means the ζ-part of $[X, \overline{Y}]$ according to (1-3). As is well known, this map L makes sense for elements X, Y in $^{\circ}T''$. And if this Levi form is positive or negative definite, $(M \cap V(p), {}^{\circ}T'')$

is called a strongly pseudo convex real hypersurface. Next we briefly explain $\overline{\partial}_b$-complex. For u in $\Gamma(M \cap V(p), C)$, we set

$$\overline{\partial}_b u(x) = Xu \quad \text{for } X \text{ in } {}^\circ T'',$$

where $\Gamma(M \cap V(p), c)$ means the spacing consisting of C^∞ functions over $M \cap V(p)$. Namely we have a first order differential operator

$$\overline{\partial}_b : \Gamma(M \cap V(p), c) \to \Gamma(M \cap V(p), ({}^\circ T'')^*).$$

By the same way as for usual differential forms, we have

$$\overline{\partial}_b^{(p)} : \Gamma(M \cap V(p), \Lambda^p({}^\circ T'')^*) \to \Gamma(M \cap V(p), \Lambda^{p+1}({}^\circ T'')^*)$$

and so

$$\overline{\partial}_b^{(p+1)} \circ \overline{\partial}_b^{(p)} = 0.$$

2. Kuranishi's L^2-estimate. Let $(M, {}^\circ T'')$ be a strongly pseudo convex CR manifold, embedded as a real hypersurface in C^n. Let p be a reference point of M. Then by a change of coordinates, we can assume that there is a neighborhood $W(p)$ of p in C^n, satisfying:

$$M \cap W(p) = \{(z_1, \dots, z_n) : (z_1, \dots, z_n) \in W(p),\ \operatorname{Im} z_n = h(z_1, \dots, z_{n-1}, \operatorname{Re} z_n)\},$$

where $z_i(p) = 0$, $1 \le i \le n-1$, and h is a real valued C^∞ function, and

$$(\partial^2 h/\partial z_i \partial \overline{z}_j)(0) = \delta_{ij}, \qquad 1 \le i, j \le n-1,$$
$$(\partial^2 h/\partial z_i \partial z_j)(0) = \delta_{ij}, \qquad 1 \le i, j \le n-1,$$
$$dh(0, \dots, 0) = 0.$$

In this set up, we introduce a neighborhood $M \cap U_\varepsilon(p)$ of p as follows:

$$M \cap U_\varepsilon(p) = \{(z_1, \dots, z_n) : (z_1, \dots, z_n) \in W(p),\ \operatorname{Im} z_n = h(z_1, \dots, z_{n-1}, \operatorname{Re} z_n),\ 2\operatorname{Re}\{(1/2\sqrt{-1}) z_n + z_n^2\} < \varepsilon\}.$$

Now we briefly sketch Kuranishi's L^2-estimate over $M \cap U_\varepsilon(p)$. Obviously by the above assumption, our $M \cap U_\varepsilon(p)$ is diffeomorphic to the real $2n-1$ dimensional ball. We denote this diffeomorphism map by h and we fix this. If ε is chosen sufficiently small, there is a system of bases $Y'_1, Y'_2, \dots, Y'_{n-1}$ of ${}^\circ T''$ over $M \cap U_\varepsilon(p)$, where ${}^\circ T''$ means the CR structure over $M \cap U_\varepsilon(p)$ induced from C^n. In our case, we can define a real vector field ζ, dual to

$$\sqrt{-1}\partial \rho,$$

where $\rho = \operatorname{Im} z_n - h(z_1, \dots, z_{n-1}, \operatorname{Re} z_n)$. And by using this ζ, we have a C^∞ vector bundle decomposition and so we have the Levi form. By the Schmidt orthogonal process, form $Y'_1, Y'_2, \dots, Y'_{n-1}$, we have a system of bases $Y_1, Y_2, \dots, Y_{n-1}$ of ${}^\circ T''$ satisfying

$$-\sqrt{-1}[Y_i, \overline{Y}_j]_\zeta = \delta_{ij},$$

where $-\sqrt{-1}[Y_i, \overline{Y}_j]_\zeta$ means the coefficient of the ζ part of $[Y_i, \overline{Y}_j]$ according to the above C^∞ vector bundle decomposition. By using this $Y_1, Y_2, \dots, Y_{n-1}$, we put an L^2-norm on

$$\Gamma(M \cap U_\varepsilon(p), \Lambda^p({}^\circ T'')^*).$$

Namely for u in $\Gamma(M \cap U_\varepsilon(p), \Lambda^p({}^\circ T'')^*)$, we have C^∞ functions u_I by

$$u_I = u(Y_{i_1}, \dots, Y_{i_p}), \qquad I = (i_1, \dots, i_p).$$

By using these u_I, we set

$$\|u\|^2_{M\cap U_\varepsilon(p)} = \sum_I \int_{B_1(0)} |u_I \circ h|^2 \, dx_1 \cdots dx_{2n-1},$$

where I runs through all ordered indices of length p and h is a diffeomorphism map from $M \cap U_\varepsilon(p)$ to $B_1(0)$ defined as above. Furthermore we must introduce several notations. Namely $\overline{\partial}_1^*$ denotes the adjoint operator of $\overline{\partial}_1$ with respect to the above L^2-norm. And we set

$$b = \sqrt{\sum_{i=1}^{n-1} |Y_i t|^2},$$

where $t = 2\operatorname{Re}\{1/2\sqrt{-1})z_n + z_n^2\}$. And we set the characteristic curve C by

$$C = \{(z_1, \dots, z_n), (z_1, \dots, z_n) \in M \cap U_\varepsilon(p), \\ Y_i t = 0, \ 1 \le i \le n-1\}.$$

Then in **[K]**, Kuranishi obtained

$$\|(1/b)v\|^2_{M\cap U_\varepsilon(p)} \le c\{\|\overline{\partial}_b v\|^2_{M\cap U_\varepsilon(p)} + \|\overline{\partial}_b^* v\|^2_{M\cap U_\varepsilon(p)}\}$$

for v in $\Gamma(M \cap U_\varepsilon(p) - C, ({}^\circ T'')^*)$ satisfying:

$$v(Y^0) = 0 \quad \text{on } \{(z_1, \dots, z_n) \colon (z_1, \dots, z_n) \in M \cap U_\varepsilon(p) - C, \\ t = \varepsilon\},$$

where

$$Y^0 = \sum_{i=1}^{n-1} (\overline{Y}_i t/b) Y_i ,$$

if $\dim_R M = 2n - 1 \geq 7$. Actually, Kuranishi obtained the estimate more precisely. However, in this paper, we discuss this estimate. Then, the L^2-solution operator $\overline{\partial}_b^* N_b$ satisfies

$$\|(1/b)(\overline{\partial}_b^* N_b v)\|_{M\cap U_\varepsilon(p)} \leq c\|v\|_{M\cap U_\varepsilon(p)}$$

for v in $\Gamma(M \cap U_\varepsilon(p) - C, ({}^\circ T'')^*)$, which is of L^2. We show that an explicit solution obtained by Henkin and Harvey-Polking satisfies the similar estimate.

3. Rigid hypersurfaces in C^n. In this paper, we study the $\overline{\partial}_b$-operator over a special kind of real hypersurfaces in C^n. Namely let

$$M = \{(z_1, \dots, z_n) \colon \operatorname{Im} z_n = k(z_i, \overline{z}_j), \ 1 \leq i, j \leq n-1\},$$

where k is a real valued C^∞ function which depends only on z_i, $\overline{z}_j$, and not on z_n, $\overline{z}_n$ satisfying:

$$k(0, 0) = 0 \quad \text{and} \quad dk(0, 0) = 0.$$

We call M satisfying these relations a rigid hypersurface. Let M be a rigid hypersurface. And let M be strongly pseudo convex near the origin. Then by a change of coordinates, the defining equation of M becomes

$$\operatorname{Im} z_n'' = \sum_{i=1}^{n-1} |z_i''|^2 + \text{ terms of higher order in } z_j'', \overline{z}_j'',$$

where $1 \leq j \leq n-1$.

4. Integral formula for $\overline{\partial}_b$ and the Romanov kernel. Let u, v be C^∞ functions from $C^n \times C^n$ to C^n,

$$u(\zeta, z) = (u_1(\zeta, z), \dots, u_n(\zeta, z)),$$
$$v(\zeta, z) = (v_1(\zeta, z), \dots, v_n(\zeta, z)).$$

We use the following notations:

$$u(\zeta, z)(\zeta - z) = \sum_{j=1}^{n} u_j(\zeta, z)(\zeta_j - z_j),$$

$$u(\zeta, z)\, d(\zeta - z) = \sum_{j=1}^{n} u_j(\zeta, z)\, d(\zeta_j - z_j),$$

$$\overline{\partial} u(\zeta, z)\, d(\zeta - z) = \sum_{j=1}^{n} \overline{\partial} u_j(\zeta, z) \wedge d(\zeta_j - z_j),$$

and we define the following kernels:

$$\text{(4-1-1)} \quad \Omega^u(\zeta, z) = (2\pi i)^{-n}((u(\zeta, z)\, d(\zeta - z))/(u(\zeta, z)(\zeta - z))) \wedge ((\overline{\partial} u(\zeta, z)\, d(\zeta - z))/(u(\zeta, z)(\zeta - z)))^{n-1},$$

$$\text{(4-1-2)} \quad \Omega^v(\zeta, z) = (2\pi i)^{-n}((v(\zeta, z)\, d(\zeta - z))/(v(\zeta, z)(\zeta - z))) \wedge ((\overline{\partial} v(\zeta, z)\, d)(\zeta - z))/(v(\zeta, z)(\zeta - z)))^{n-1},$$

$$\begin{aligned} \text{(4-1-3)} \quad & \Omega^{u,v}(\zeta, z) \\ &= (2\pi i)^{-n}((u(\zeta, z)\, d(\zeta, z))/(u(\zeta, z)(\zeta - z))) \\ &\quad \wedge ((\overline{\partial} v(\zeta, z)\, d(\zeta - z))/(v(\zeta, z)(\zeta - z))) \\ &\quad \wedge \sum_{j+k=n-2} ((\overline{\partial} u(\zeta, z)\, d(\zeta - z))/(u(\zeta, z)(\zeta - z)))^j \\ &\quad \wedge ((\overline{\partial} v(\zeta, z)\, d(\zeta - z))/(v(\zeta, z)(\zeta - z)))^k. \end{aligned}$$

Then as is well known, in **[B]** and **[BS]**, we have

$$\overline{\partial}\Omega^{u,v}(\zeta, z) = \Omega^v(\zeta, z) - \Omega^u(\zeta, z),$$
$$\overline{\partial}\Omega^v(\zeta, z) = 0.$$

Let M be as in §1 in this paper. Then we can define *formally*

$$R_M(u, v)(\phi)(z) := \left\{ \int_{\zeta \in M} \Omega^{u,v}(\zeta, z) \wedge \phi(\zeta) \right\}_{T_M},$$

$$L(u)(\phi)(z) := \int_{\zeta \in M} \Omega^u(\zeta, z) \wedge \phi(\zeta),$$

for $\phi \in \mathscr{D}^{0,1}(M \cap U)$, where $\{\ \}_{T_M}$ means the tangential part of $\{\ \}$. Of course without any assumption for u, v and M, the operators R_M, L do not make sense. However if we assume that u is a local support function for (M, D) at a point p (for the definition, see 2.4 Definition in **[BS]**), then $R_M(u, v)(\phi)$, $L(u)(\phi)$ make sense. And furthermore, the boundary value of $L(u)(\phi)$ from D^- and D^+ exists respectively, where D means U and

$$D^+ = \{z \colon z \in C^n, \rho(z) > 0\},$$
$$D^- = \{z \colon z \in C^n, \rho(z) > 0\}.$$

And for $\phi \in \mathscr{D}^{0,1}(M \cap U)$,

$$\phi = -(\overline{\partial}_b R_M(u, v)(\phi) + R_M(u, v)\overline{\partial}_b \phi) + L_M^+(v)(\phi) - L_M^-(u)(\phi) \quad \text{on } M \cap U.$$

Note from this equality, the terms $L_M^+(v)(\phi)$ and $L_M^-(u)(\phi)$ are obstructions to solving the equations $\overline{\partial}_b g = \phi$. If we set

$$u_j(\zeta, z) = \partial\rho/\partial\zeta_j(\zeta), \quad v_j(\zeta, z) = -\partial\rho/\partial z_j(z), \qquad 1 \le j \le n,$$

then $u(\zeta, z) = (u_1(\zeta, z), \dots, u_n(\zeta, z))$ and $v(\zeta, z) = (v_1(\zeta, z), \dots, v_n(\zeta, z))$ are local support functions for (M, D^-) and (M, D^+) respectively. And in the case,

$$L_M^-(u)(\phi) = 0 \quad \text{unless } \phi \in \mathscr{D}^{p,0}(M \cap U),$$
$$L_M^+(v)(\phi) = 0 \quad \text{unless } \phi \in \mathscr{D}^{p,n-1}(M \cap U).$$

And so we have: for $\phi \in \mathscr{D}^{p,1}(M \cap U)$,

$$\phi = -\{\overline{\partial}_b R_M(u, v)(\phi) + R_M(u, v)(\overline{\partial}_b \phi)\},$$

if $n \ge 3$.

Henceforth, we abbreviate R for $R_M(u, v)$, where u and v are defined as above, and $R\phi$ stands for $R_M(u, v)(\phi)(z)$.

5. Kuranishi's L^2-estimate for the Romanov kernel. In §4, we see that the Romanov kernel R is a certain kind of the solution operator for $\overline{\partial}_b$. Concerning this R kernel, in this section, we show an L^2-estimate which the L^2 solution satisfies. Namely, we show

MAIN THEOREM. *For any ϕ in $\Gamma(M \cap U_\varepsilon(p) - C, ({}^\circ T'')^*)$, which is of L^2, and for any $\delta < 1$, we have*:

$$\|(1/b^\delta) R\phi\|_{M \cap U_\varepsilon(p)} \le C_\delta \|\phi\|_{M \cap U_\varepsilon(p)},$$

where C_δ depends only on δ.

In order to prove the main theorem, we first show

LEMMA 5.1.

$$C_1 \sqrt{\sum_{i=1}^{n-1} |z_i''|^2} \le b \le C_2 \sqrt{\sum_{i=1}^{n-1} |z_i''|^2},$$

where C_1, C_2 are positive constants, and b is defined by

$$b = \sqrt{\sum_{i=1}^{n-1} |Y_i''t|^2},$$

where $\{Y_i''\}_{1\le i\le n-1}$ is obtained from $\{Y_i\}_{1\le i\le n-1}$, by the Schmidt orthogonal process, and

$$Y_i = \partial/\partial \overline{z}_i'' - (\rho_{\bar{i}}/\rho_{\bar{n}})\partial/\partial \overline{z}_n'', \qquad 1 \le i \le n-1,$$

$$\rho = \operatorname{Im} z_n'' - \sum_{i=1}^{n-1} |z_i''|^2 - Q(z_j'', \overline{z}_j''),$$

where $\{z_i''\}_{1\le i\le n}$ means the coordinate obtained in §3 in this paper.

Proof of Lemma 5.1. By the construction of Y_i'', Y_i'' is a linear combination of Y_j, $1 \le j \le n$, satisfying:

$$Y_i'' := \sum_{j=1}^{n-1} a_{ji} Y_j,$$

where a_{ji} is a C^∞ function over $M \cap U_\varepsilon(p)$ and $a_{ji}(p) = 0$. So

$$Y_i''t = Y_i t + \sum_{j=1}^{n-1} a_{ji} Y_j t.$$

While

$$\begin{aligned} Y_j t &= (\partial/\partial \overline{z}_j'' - (\rho_{\bar{i}}/\rho_{\bar{n}})\partial/\partial \overline{z}_n'') 2 \operatorname{Re}\{(1/2\sqrt{-1}) z_n'' + z_n''^2\} \\ &= z_i''(1 + 4\sqrt{-1} z_n''). \end{aligned}$$

Therefore we have our lemma. □

And we have

LEMMA 5.2. *There is a constant c satisfying:*

$$\int_{\zeta \in M\cap U_\varepsilon(p)} (1/b^\delta)|\Omega^{u,v}(\zeta, z)|dV_\zeta \le c \quad \textit{for } z \textit{ in } U_\varepsilon(p).$$

This lemma is proved in [HP]. So we briefly sketch the proof. For a system of coordinates of $M \cap U_\varepsilon(p)$, we can adopt $(z_1'', \dots, z_{n-1}'', t)$, which we constructed in §3 in this paper, where $t = \operatorname{Re} z_n''$. Then over $M \cap U_\varepsilon(p)$,

$$c_1 \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right) \le |z_n''| \le c_2 \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right),$$

where c_1, c_2 are positive constants. So over $M \cap U_\varepsilon(p)$,

$$c_3\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right) \le |u(\zeta - z'')| \le c_4\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right),$$

where c_3, c_4 are positive constants. And

$$c_5\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right) \le |v(\zeta - z'')| \le c_6\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right),$$

where c_5, c_6 are positive constants. And

$$u\, d(\zeta - z) \wedge v\, d(\zeta - z) = Q(|\zeta - z|).$$

So each coefficient of $(1/b^\delta)R$ is dominated by

$$\left(\sum_{i=1}^{n-1} |z_i''|^2\right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2}\right) \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right)^{-n}.$$

And this is locally integrable on $C^{n-1} \times R$ if $\delta < 1$. In fact, by using polar coordinates, we compute the following integral. We set

$$\begin{aligned}
x_1 &= r\cos\theta_1 \cos\theta_2 \cdots \cos\theta_{2n-3} \cos\theta_{2n-2}, \\
y_1 &= r\cos\theta_1 \cos\theta_2 \cdots \cos\theta_{2n-3} \sin\theta_{2n-2}, \\
x_2 &= r\cos\theta_1 \cdots \cos\theta_{2n-4} \sin\theta_{2n-3}, \\
y_2 &= r\cos\theta_1 \cdots \sin\theta_{2n-4}, \\
&\cdots \\
x_{n-1} &= r\cos\theta_1 \sin\theta_2, \\
y_{n-1} &= r\sin\theta_1,
\end{aligned}$$

where $z_j'' = x_j + \sqrt{-1}y_j$, $1 \le j \le n-1$. Then

$$\left(\sum_{i=1}^{n-1} |z_i''|^2\right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2}\right) \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right)^{-n}$$
$$= r^{-\delta}(t + r)(t + r^2)^{-n}.$$

So

$$\int_{M\cap U_\varepsilon(p)} \left(\sum_{i=1}^{n-1} |z_i''|^2\right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2}\right)$$
$$\times \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right) dV_{z,t}$$
$$\leq \int_0^\varepsilon \int_0^\infty r^{-\delta}(t+r)(t+r^2)^{-n} r^{2n-3}\, dt\, dr$$
$$= \int_0^\varepsilon \int_0^\infty \{(1/(t+r^2)^{n-1}) r^{2n-3-\delta}$$
$$+ (1/(t+r^2)^n)((r-r^2)/r^\delta) r^{2n-3}\}\, dt\, dr.$$

While

$$\int_0^\infty (1/(t+r^2)^{n-1}) r^{2n-3-\delta}\, dt$$
$$= -(1/(n-2))[(1/(t+r^2)^{n-2}) r^{2n-3-\delta}]_0^\infty$$
$$= (1/(n-2)) r^{1-\delta},$$

$$\int_0^\infty (1/(t+r^2)^n)((r-r^2) r^{2n-3}\, dt$$
$$= -(1/(n-1))[(1/(t+r^2)^{n-1}(1-r)) r^{2n-2-\delta}]_0^\infty$$
$$= (1/(n-1))(1-r) r^{-\delta}.$$

Therefore

$$\int_{M\cap U_\varepsilon(p)} \left(\sum_{i=1}^{n-1} |z_i''|^2\right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2}\right) \left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right) dV_{z,t}$$
$$\leq \int_0^\varepsilon ((1/(n-2)) r^{1-\delta} + (1/(n-1)) r^{-\delta} - (1/(n-1)) r^{1-\delta})\, dr$$
$$= (1/((n-2)(2-\delta)))\varepsilon^{1-(\delta/2)}$$
$$+ (1/((n-1)(1-\delta)))\varepsilon^{(1/2)-(\delta/2)}$$
$$- (1/((n-1)(2-\delta)))\varepsilon^{1-(\delta/2)}.$$

Therefore we have our lemma. □

Now we prove our main theorem.

$$
\begin{aligned}
&\int_{M\cap U_\varepsilon(p)} (1/b^{2\delta})|R_M(u,v)(\phi)|^2\,dV \\
&\quad\le \int_{M\cap U_\varepsilon(p)} \left\{(1/b^{2\delta})\left(\int_{M\cap U_\varepsilon(p)} \Omega^{u,v}(\zeta,z)\phi(\zeta)\,dV_\zeta\right)^2\right\} dV_z \\
&\quad\le \int_{M\cap U_\varepsilon(p)} \left(\int_{M\cap U_\varepsilon(p)} |(1/b^{\delta})\Omega^{u,v}(\zeta,z)\phi(\zeta)|\,dV_\zeta\right)^2 dV_z \\
&\quad\le \int_{M\cap U_\varepsilon(p)} \left\{\left(\int_{M\cap U_\varepsilon(p)} |(1/b^{\delta})\Omega^{u,v}(\zeta,z)|\,dV_\zeta\right)\right. \\
&\qquad\qquad \left.\times\left(\int_{M\cap U_\varepsilon(p)} |(1/b^{\delta})\Omega^{u,v}(\zeta,z)|\,|\phi(\zeta)|^2\,dV_\zeta\right)\right\} dV_z \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the Schwarz lemma)} \\
&\quad\le c^2\int_{M\cap U_\varepsilon(p)} |\phi(\zeta)|^2\,dV_\zeta \qquad \text{(by Lemma 5.2 in this paper)} \\
&\quad\le c^2\|\phi\|^2_{M\cap U_\varepsilon(p)}.
\end{aligned}
$$

So we have our theorem. □

REFERENCES

[Ak 1] T. Akahori, *Intrinsic formula for Kuranishi's* $\overline{\partial}_b^{\phi}$, Publications of the Research Institute for Mathematical Sciences, Kyoto University, **14** (1978), 615–641.

[Ak 2] ——, *The new estimate for the subbundles* E_j *and its application to the deformation of the boundaries of strongly pseudo convex domains*, Invent. Math., **63** (1981), 311–334.

[Ak 3] ——, *A new approach to the local embedding theorem of* CR-*structures for* $n \ge 4$ (*the local solvability for the operator* $\overline{\partial}_b$ *in the abstract sense*), Mem. Amer. Math. Soc., **366** (1987).

[Am 1] H. Ameku, *A local solution operator for* $\overline{\partial}_b$ *over a certain weakly pseudo convex domain*, in preparation.

[B] A. Boggess, *Kernels for the tangential Cauchy-Riemann equation*, Trans. Amer. Math. Soc., **269** (1980), 1–49.

[BS] A. Boggess and M.-C. Shaw, *A kernel approach to the local solvablity of the tangential Cauchy Riemann equations*, Trans. Amer. Math. Soc., **289** (1980), 645–658.

[HP] R. Harvey and J. Polking, *Fundamental solutions in complex analysis*, I, II, Duke Math. J., **46** (1979), 253–300, 301–340.

[K] M. Kuranishi, *Strongly pseudo convex* CR *structures over small balls*, Ann. of Math., I, **115** (1982), 451–500, II, **116** (1982), 1–64, III, **116** (1982), 249–330.

[KS] N. Kerzman and E. M. Stein, *The Szegö kernel in terms of Cauchy-Fantappiè kernels*, Duke Math. J., **45** (1978), 197–224.

[LR] I. Lieb and R. M. Range, *The kernels of the $\bar{\partial}$-Neumann operator on strictly pseudo convex domains*, Math. Ann., **278** (1987), 151–173.

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