PACIFIC JOURNAL OF MATHEMATICS
Vol. 149, No. 2, 1991

A COMBINATORIAL MATRIX IN 3-MANIFOLD THEORY

KI HYOUNG KO AND LAWRENCE SMOLINSKY

In this paper we study a combinatorial matrix considered by W. B. R. Lickorish. We prove a conjecture by Lickorish that completes his topological and combinatorial proof of the existence of the Witten-Reshetikhin-Turaev 3-manifold invariants. We derive a recursive formula for the determinant of the matrix and discover some interesting numerical relations.

In this paper we study the matrix $A(n)$ which was defined by W. B. R. Lickorish [**3**]. We prove a result required by Lickorish which completes his topological and combinatorial approach to the 3-manifold invariants of Witten-Reshetikhin-Turaev [**4**], [**5**]. This matrix arises from a pairing on a set of geometric configurations. These are the configurations of n nonintersecting arcs in the disk with $2n$ specified boundary points. There are C_n such configurations where C_n is the nth Catalan number so the matrix increases in size very rapidly. The Catalan numbers were discovered by Euler who considered the ways to partition a polygon into triangles [**1**]. These two counting problems correspond naturally by considering "restricted sequences".

The matrix has entries in $\mathbf{Z}[\delta]$. Lickorish needed that $\det A(n) = 0$ if $\delta = \pm 2\cos\frac{\pi}{n+1}$. We find a recursive formula for $\det A(n)$ and show that all the roots are of the form $2\cos\frac{k\pi}{m+1}$ for $1 \leq m \leq n$ and $1 \leq k \leq m$ and verify the result. Using this formula, we derive a simple rule that allows one to recursively compute $\det A(n)$ by generating all of its factors.

There have been three approaches to study polynomial invariants of classical links: the topological and combinatorial approach considered by Kauffman, Lickorish and many other topologists; the study of quantized Yang-Baxter equations and related Lie algebras by Reshetikhin and Turaev; and the study of subfactors and traces of von Neumann and Hecke algebras by Jones. We took a topological and combinatorial viewpoint. The authors have been informed that the essential result needed by Lickorish could have been obtained by pursuing the two other approaches.

1. Combinatorial manipulation. Let D_n be the set of configurations of n non-intersecting arcs on a disk joining $2n$ points on the boundary of the disk. We draw these configurations by taking S^1 to be $[0, 1]/0 \sim 1$ as in Fig. 1.

The cardinality of D_n is equal to $(2n)!/n!(n+1)!$, known as the Catalan number, denoted here by C_n. It satisfies the recursive relation:

$$C_n = C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1}.$$

We can inductively represent the elements of D_n by sequences of n integers $(a_1, a_2, \ldots, a_n)$ where $1 \leq a_i \leq n - i + 1$. The first entry a_1 means that there is an innermost arc in the configuration joining the a_1th point and the $(a_1 + 1)$th point on the interval. One then deletes that arc and has an element of D_{n-1} remaining. The sequence $(a_2, a_3, \ldots, a_n)$ then represents this element of D_{n-1}. See Fig. 2 for an example.

Note that every configuration in D_n must contain an innermost arc between adjacent points among the first $n+1$ points. Thus this representation captures all possible configurations but with repetitions. For example the configuration in Fig. 3 has 12 distinct associated sequences.

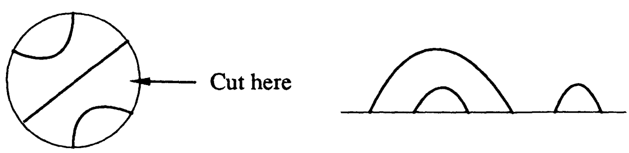


FIGURE 1

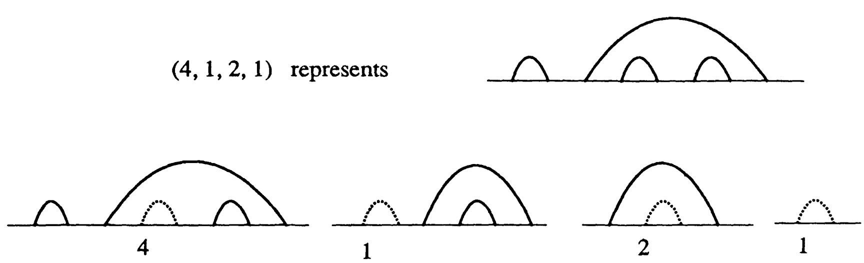


FIGURE 2

FIGURE 3

Given such a sequence $(a_1, a_2, \ldots, a_n)$ one may construct the unique configuration inductively. Into the configuration $(a_2, a_3, \ldots, a_n)$ one can insert two additional points between $(a_1 - 1)$st and a_1th points then joining these two new points by an innermost arc. Thus two distinct configurations cannot have the same sequence. To a given configuration in D_n, one can associate the unique sequence $(a_1, a_2, \ldots, a_n)$ in which a_1 indicates the initial position of the first occurring innermost arc and a_2 does the same for the configuration without the previous innermost arc and so on. Such a sequence is said to be *restricted*.

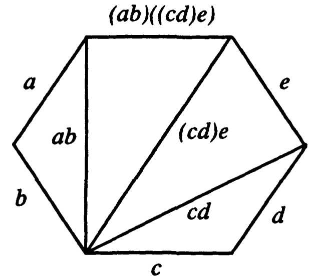
PROPOSITION 1.1. *A sequence $(a_1, a_2, \ldots, a_n)$ of a configuration is restricted if and only if $a_{i-1} - 1 \leq a_i$ for all $i = 2, \ldots, n$.*

Proof. For a restricted sequence $(a_1, a_2, \ldots, a_n)$, it is enough to prove $a_1 - 1 \leq a_2$ since $(a_2, \ldots, a_n)$ is also a restricted sequence. After removing the first innermost arc, either the second innermost arc or the arc joining the $(a_1 - 1)$th and the $(a_1 + 2)$th point in the original configuration will become the first innermost arc in the remaining configuration. Thus $a_1 - 1 \leq a_2$.

Conversely if $a_{i-1} - 1 \leq a_i$, then the newly inserted innermost arc into the configuration of $(a_{i-1}, \ldots, a_n)$ becomes the first innermost arc in the configuration of $(a_i, \ldots, a_n)$. □

REMARK. The number of ways to divide an $(n+2)$gon into triangles or the number of ways to interpret the product $x_1 x_2 \cdots x_{n+1}$ in a non-associative algebra is equal to the Catalan number C_n. Restricted sequences are useful to see the correspondence between these and configurations defined earlier. Label the vertex of the $(n+2)$gon counterclockwise 1 through n except fixed adjacent vertices. A triangle in a partition is said to be outermost if it has a vertex contained in no other triangle. To a partition of the $(n+2)$gon we assign the sequence $(a_1, a_2, \ldots, a_n)$ where a_1 is the vertex that is solely contained in the first occurring outermost triangle. Then the sequence $(a_2, \ldots, a_n)$ inductively represents the partition of the $(n+1)$gon obtained by deleting the vertex a_1 and its adjacent sides. See Fig. 4 for an example.

(1, 2, 2, 1) is the unique representation of



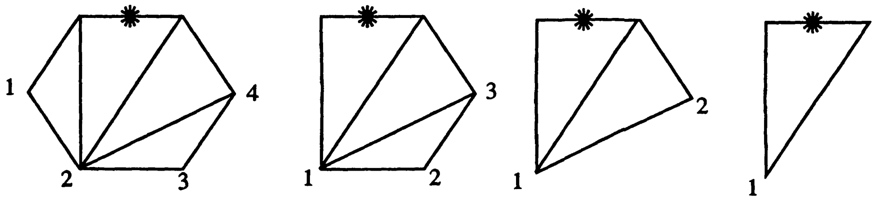


FIGURE 4

We give the lexicographic order to the set of all the sequences of configurations, i.e., $(a_1, a_2, \dots, a_n) < (b_1, b_2, \dots, b_n)$ if there is an index k such that $a_1 = b_1, \dots, a_{k-1} = b_{k-1}$ and $a_k < b_k$. If two distinct sequences α and β represent the same configuration and α is restricted, then clearly $\alpha < \beta$.

Let $B(n, k)$ be the set of restricted sequences of length n with initial entry k and let $b(n, k)$ be the cardinality $|B(n, k)|$ of the set $B(n, k)$. Since D_n can be identified with the set of all restricted sequences of length n, $C_n = \sum_{k=1}^{n} b(n, k)$. It is convenient to set $b(n, k) = 0$ for $k = 0$ or $k > n$.

PROPOSITION 1.2. *$b(n, k) = \sum_{i=k-1}^{n-1} b(n-1, i)$ for $k = 1, \dots, n$.*

Proof. Immediately follows from Proposition 1.1. □

It is interesting that $b(n, 1) = b(n, 2) = C_{n-1}$, $b(n, n-1) = n-1$, and $b(n, n) = 1$. The only element in $B(n, n)$ is $(n, n-1, \dots, 2, 1)$

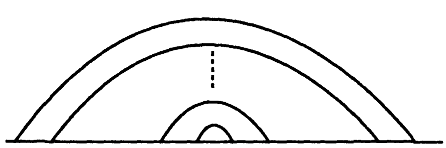


FIGURE 5

which represents the configuration in Fig. 5. In fact we have:

COROLLARY 1.3. *Using the binomial coefficients,*

$$b(n, k) = \frac{k}{n}\binom{2n-k-1}{n-1}.$$

Proof. By Proposition 1.2, $b(n+1, k) - b(n+1, k+1) = b(n, k-1)$. And this recursive formula together with initial conditions $b(2, 1) = b(n, n) = 1$ for all n generates all $b(n, k)$'s. But a computation shows that

$$\frac{k}{n+1}\binom{2n-k+1}{n} - \frac{k+1}{n+1}\binom{2n-k}{n} = \frac{k-1}{n}\binom{2n-k}{n-1}. \qquad \square$$

Let $\mathfrak{B}(n, k)$ be the set of sequences with initial entry k and the remaining terms forming a restricted sequence of length $n-1$. We will sometimes write (k, α) with α restricted for such a sequence. Note that $|\mathfrak{B}(n, k)| = C_{n-1}$.

Let V be the free $\mathbf{Z}[\delta]$ module generated by D_n where δ is a variable. We define a bilinear form on $V \times V$. If α, β are two configurations in D_n, we can form the union of their respective disks along the boundary to obtain a configuration of circles in the 2-sphere. We denote this configuration in S^2 by $\alpha \cup \beta$. Let c be the number of circles in $\alpha \cup \beta$; then $\langle \alpha, \beta \rangle = \delta^c$. Then we linearly extend this pairing to all elements in the free module. Lickorish first considered this symmetric bilinear form to give a more geometric and combinatorial proof of the existence of the 3-manifold invariant developed by Witten and Reshetikhin-Turaev. See [2], [3], [4] and [5]. So we call it *Lickorish's bilinear form.* We can also consider this a pairing of restricted sequences or of sequences since they correspond to configurations.

LEMMA 1.4. *For* $\alpha, \beta \in D_n$, $\langle \alpha, \beta \rangle = \delta^n$ *if and only if* $\alpha = \beta$.

Proof. If $\alpha = \beta$ then each component of $\alpha \cup \beta$ consists of one arc of α and one of β so $\alpha \cup \beta$ has n components. If $\langle \alpha, \beta \rangle = \delta^n$ then each arc of α is in a separate component of $\alpha \cup \beta$. But if $\alpha \neq \beta$ then some arc of β joins endpoints of two distinct arcs of α and these arcs are in the same component of $\alpha \cup \beta$. $\square$

THEOREM 1.5 (*Properties of Lickorish's bilinear form*). (1) *Let* S *be any subset of* D_n. *Then* $\langle\ ,\ \rangle$ *is nondegenerate over the free* $\mathbf{Z}[\delta]$ *module generated by* S.

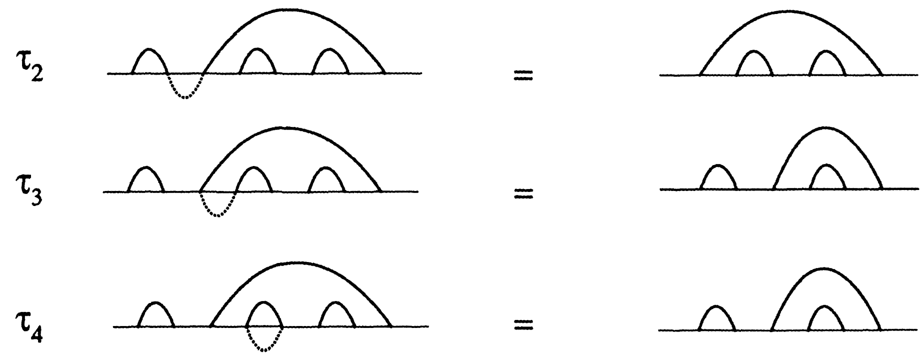


FIGURE 6

(2) *Suppose* α *is any configuration in* D_n. *Then for any* $b \in \{1, 2, \dots, n\}$ *with* $\alpha \notin \mathfrak{B}(n, b)$, *there is a* $\beta \in \mathfrak{B}(n, b)$ *such that* $\delta\langle\alpha, \gamma\rangle = \langle\beta, \gamma\rangle$ *for all* $\gamma \in \mathfrak{B}(n, b)$.

(3) $\delta\langle\alpha, \varepsilon\rangle = \langle(a, \alpha), (a, \varepsilon)\rangle = \delta\langle(a \pm 1, \alpha), (a, \varepsilon)\rangle$ *for all sequences* ε, α *whenever* $a \pm 1$ *makes sense.*

(4) *Suppose* (a, α), (b, β) *are restricted sequences of length* n *and there is an* $\eta \in \mathbf{Z}[\delta]$ *such that* $\langle(a, \alpha), \gamma\rangle = \eta\langle(b, \beta), \gamma\rangle$ *for all* $\gamma \in \mathfrak{B}(n, a)$ *with* $\gamma \leq (a, \alpha)$.

(i) *If* $b = a, a \pm 1$ *then* $\alpha \leq \beta$.
(ii) *If* $b \neq a, a \pm 1$ *then* $\alpha < \beta$.

Before we begin the proof, we first define a set of maps

$$\tau_a : D_n \to D_{n-1} \quad \text{for } a = 1, 2, \dots, n.$$

These mappings eliminate the ath and the $(a+1)$st points in D_n by an inverse of a "finger move" as in Fig. 6.

Note that $\tau_a((a, \alpha)) = \alpha$ for any sequence α.

Proof of Theorem 1.5. (1) Suppose $\sum_{\alpha \in S} q_\alpha \alpha$ is an arbitrary element in the free $\mathbf{Z}[\delta]$ module generated by S. From among the q_α, pick a β so that the degree of q_β is maximal. Then by Lemma 1.4, the degree of $\langle\beta, \beta\rangle$ is strictly greater than the degree $\langle\alpha, \beta\rangle$ for all $\alpha \neq \beta$. Therefore $\langle\sum_{\alpha \in S} q_\alpha \alpha, \beta\rangle$ has a nonvanishing term of degree $(n + \deg q_\beta)$.

(2) If $\alpha \notin \mathfrak{B}(n, b)$, then

$$\delta\langle\alpha, \gamma\rangle = \langle(b, \tau_b(\alpha)), \gamma\rangle \quad \text{for } \gamma \in \mathfrak{B}(n, b)$$

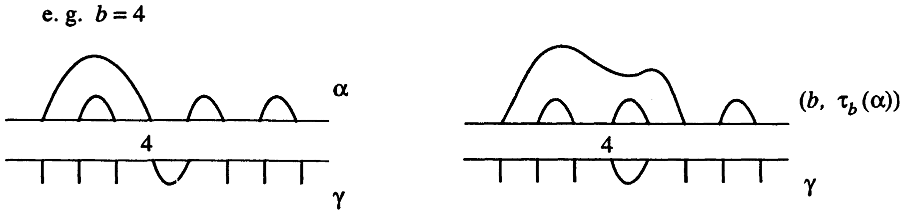


FIGURE 7

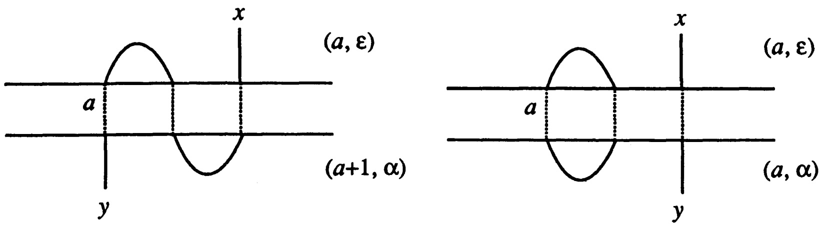


FIGURE 8

since the innermost arc at b performs τ_b when joined to α. See Fig. 7.

(3) $\tau_a(a, \alpha) = \tau_a(a+1, \alpha) = \tau_a(a-1, \alpha) = \alpha$. See Fig. 8.

(4) It follows from Lemma 1.4 that

$$\eta = \langle (a, \alpha), (a, \alpha) \rangle / \langle (b, \beta), (a, \alpha) \rangle = \delta^k$$

for some $k \geq 0$. First suppose $b = a$ and so $(b, \beta) \in \mathfrak{B}(n, a)$. Let $S = \{\varepsilon \in D_{n-1} | \varepsilon \leq \alpha\}$. If $(b, \beta) < (a, \alpha)$, i.e., $\beta < \alpha$ then $\delta\langle \alpha - \delta^k \beta, \varepsilon \rangle = \langle (a, \alpha) - \delta^k (b, \beta), (a, \varepsilon) \rangle = 0$ for all $\varepsilon \in S$. This contradicts property (1). Thus $(b, \beta) \geq (a, \alpha)$.

Suppose that $b = a \pm 1$. If $\beta < \alpha$ then this together with property (3) contradicts property (1). Thus $\beta \geq \alpha$.

Now suppose that $b \neq a$, $a \pm 1$. If $(b, \beta) \in \mathfrak{B}(n, a)$ then $b < a$ because (b, β) is a restricted sequence. So $(b, \beta) < (a, \alpha)$, which again contradicts property (1). Thus $(b, \beta) \notin \mathfrak{B}(n, a)$. We then have as in Fig. 9,

$$\langle (b, \beta), \gamma \rangle = \langle \tau_a(b, \beta), \tau_a \gamma \rangle \quad \text{for all } \gamma \in \mathfrak{B}(n, a) \text{ with } \gamma \leq (a, \alpha).$$

Then $\delta\langle \alpha, \tau_a \gamma \rangle = \langle (a, \alpha), \gamma \rangle = \delta^k \langle (b, \beta), \gamma \rangle = \delta^k \langle \tau_a(b, \beta), \tau_a \gamma \rangle$. Thus

$$\delta\langle \alpha, \varepsilon \rangle = \delta^k \langle \tau_a(b, \beta), \varepsilon \rangle \quad \text{for all } \varepsilon \in D_{n-1} \text{ with } \varepsilon \leq \alpha.$$

Thus we have $\tau_a(b, \beta) \geq \alpha$ by property (1). Let α_1, β_1, and β_1' be the first entry of restricted sequences α, β, and $\tau_a(b, \beta)$ respectively.

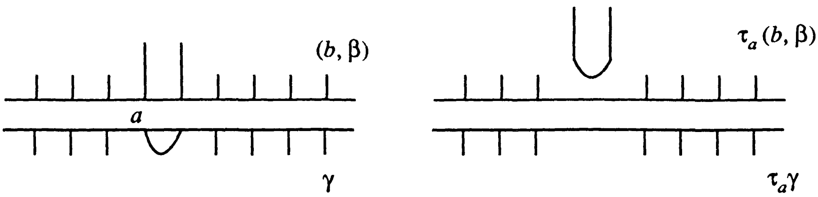


FIGURE 9

If $b < a-1$ then $\beta'_1 \le b < a-1 \le \alpha_1$ because (a, α) is a restricted sequence. So $\tau_a(b, \beta) < \alpha$ and this is a contradiction. Thus $b > a+1$. Since (b, β) has the first occurring innermost arc at b, $\beta'_1 = b-2 < \beta_1$ and so $\tau_a(b, \beta) < \beta$. Therefore $\alpha < \beta$. □

2. Matrix manipulation. Let T_n be the $(n \times n)$ tridiagonal matrix with δ in each diagonal element and 1 in each upper and lower superdiagonal. For example

$$T_5 = \begin{pmatrix} \delta & 1 & 0 & 0 & 0 \\ 1 & \delta & 1 & 0 & 0 \\ 0 & 1 & \delta & 1 & 0 \\ 0 & 0 & 1 & \delta & 1 \\ 0 & 0 & 0 & 1 & \delta \end{pmatrix}.$$

Let $\Delta_n = \det T_n$, then it is a polynomial in δ for $n \ge 1$.

PROPOSITION 2.1. (1) $\Delta_n = \delta\Delta_{n-1} - \Delta_{n-2}$ *for* $n \ge 3$.
(2) $\Delta_n = \prod_{k=1}^{n}(\delta - 2\cos\frac{k\pi}{n+1})$.

Proof. (1) Compute Δ_n by expanding along the first row.

(2) Note that Δ_n is of degree n and the coefficient of δ^n is 1 so that we must find the roots of Δ_n. Since $\Delta_n = n+1$ when $\delta = 2$ and $\Delta_n = (-1)^n(n+1)$ when $\delta = -2$, $\delta = \pm 2$ are not roots. We solve the recursion formula by a standard method. Let

$$\alpha = \frac{\delta + \sqrt{\delta^2 - 4}}{2} \quad \text{and} \quad \beta = \frac{\delta - \sqrt{\delta^2 - 4}}{2}$$

so that $\alpha\beta = 1$ and $\alpha + \beta = \delta$. From the recursion we get $\Delta_n - \alpha\Delta_{n-1} = \beta(\Delta_{n-1} - \alpha\Delta_{n-2}) = \beta^n$. Similarly $\Delta_n - \beta\Delta_{n-1} = \alpha^n$. Then $(\alpha - \beta)\Delta_n = \alpha^{n+1} - \beta^{n+1}$. Thus $\Delta_n = 0$ exactly when $\alpha \ne \beta$ and $\alpha^{n+1} = \beta^{n+1}$. Since $\beta < 1 < \alpha$ when $\delta > 2$ and $\beta < -1 < \alpha$ when $\delta < -2$, δ cannot be a root for $|\delta| > 2$. Thus we may assume $|\delta| < 2$ so $\delta = 2\operatorname{Re}\alpha = 2\operatorname{Re}\beta$. Also $\alpha^{n+1} = \beta^{n+1}$ is equivalent to $\alpha^{2n+2} = 1$. If we take α to be one among the first n of $(2n+2)$th roots of unity, then α is not equal to β which is now the conjugate of α. Thus $\delta = 2\cos(k\pi/(n+1))$ for $k = 1, \ldots, n$. Since they are all distinct, we found all of the roots of $\Delta_n = 0$. $\square$

LEMMA 2.2. *Let A be a symmetric matrix over a ring and A' be obtained by deleting the last row and column. If $\det A' \neq 0$, then a series of row operations and the corresponding column operations within the ring convert A into $\begin{pmatrix} A' & 0 \\ 0 & \det A' \det A \end{pmatrix}$.*

Proof. Let

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

be the last column of A. Let y be the solution of the system of equations:

$$A'x = \det A' \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}.$$

Define

$$E = \begin{pmatrix} I & -y \\ 0 & \det A' \end{pmatrix}.$$

Then

$$E^{\mathrm{tr}} A E = \begin{pmatrix} A' & 0 \\ 0 & \det A' \det A \end{pmatrix}. \qquad \square$$

REMARK. Applying row operations, one gets

$$E^{\mathrm{tr}} A = \begin{pmatrix} A' & \begin{matrix} v_1 \\ \vdots \\ v_{n-1} \end{matrix} \\ 0 & \det A \end{pmatrix}.$$

Let $A(n)$ be the matrix representation of Lickorish's bilinear form $\langle\ ,\ \rangle$ over the basis D_n ordered by restricted sequences. $A(n)$ consists of n^2 blocks of matrices M_{ij} such that M_{ij} represents $\langle\ ,\ \rangle$ on $B(n, i) \times B(n, j)$. So M_{ij} is a $b(n, i) \times b(n, j)$ matrix. Let $A(n, k)$ be the submatrix $(M_{ij})_{i,j=1,\ldots,k}$ of $A(n)$. Thus $A(n, n) = A(n)$ and $A(n, 1) = \delta A(n-1)$ in this notation. By Theorem 1.5(3),

$$A(n, 2) = \begin{pmatrix} \delta A(n-1) & A(n-1) \\ A(n-1) & \delta A(n-1) \end{pmatrix}.$$

Thus we have the following proposition.

Proposition 2.3. (1) $\det A(n, 1) = \Delta_1^{C_{n-1}} \det A(n-1)$.
(2) $\det A(n, 2) = \Delta_2^{C_{n-1}} (\det A(n-1))^2$.

Proof. Just calculate. □

Lemma 2.4. *$A(n)$ and all of its principal minors have nonzero determinants.*

Proof. By Lemma 1.4, only the diagonal entries of $A(n)$ have the highest degree n. Thus the term δ^{nC_n} in the determinant of $A(n)$ has the coefficient 1. And the same argument applies to all principal minors. □

Given a matrix M, $M^{\langle p\rangle}$ denotes the matrix obtained from M by deleting the last p rows and columns. And $A(n, k)^{\langle \overline{p}\rangle}$ denotes the matrix $(M_{ij}^{\langle p\rangle})_{i,j=1,\dots,k}$.

Lemma 2.5. (1) *For $0 \le p \le b(n, k) - 1$, we have the following recursion formula*:

$$\det A(n, k)^{\langle \overline{p}\rangle} = \Delta_k\left(\tfrac{\det A(n-1)^{\langle p\rangle}}{\det A(n-1)^{\langle p+1\rangle}}\right)^k \det A(n, k)^{\langle \overline{p+1}\rangle}.$$

(2) *For $2 \le j \le k-1$ and $b(n, j+1) \le p \le b(n, j) - 1$, we have the following recursion formula*:

$$\det A(n, k)^{\langle \overline{p}\rangle} = \Delta_j\left(\frac{\det A(n-1)^{\langle p\rangle}}{\det A(n-1)^{\langle p+1\rangle}}\right)^j \det A(n, k)^{\langle \overline{p+1}\rangle}.$$

In order to help the understanding of the proof given below, we will describe some of the properties of $A(n)$ that reflect the properties of Lickorish's bilinear form in Theorem 1.5. Let $\mathfrak{M}_{ij}$ be the matrix representing $\langle\ ,\ \rangle$ on $\mathfrak{B}(n, i) \times \mathfrak{B}(n, j)$. Then property (2) in Theorem 1.5 means that each column of $\mathfrak{M}_{ij}$ is equal to either one of columns of $\mathfrak{M}_{ii}$ or δ^{-1} times one of the columns. Property (3) implies that $\mathfrak{M}_{ii} = \delta A(n-1)$ and $\mathfrak{M}_{i(i\pm 1)} = A(n-1)$. Furthermore the last column of $\mathfrak{M}_{ii}$ is independent of every column in the blocks $\mathfrak{M}_{ij}$ for $j \ne i, i \pm 1$. This can be seen through property (4) since unrestricted sequences (i.e., repeated configurations) always appear first in the sets $\mathfrak{B}(n, k)$ for $k \ge 3$. Then row operations as in Lemma 2.2 with $A = A(n-1)$ convert the last row of $\mathfrak{M}_{ii}$ into $(0, \dots, 0, \delta \det A(n-1))$, $M_{i(i\pm 1)}$ into $(0, \dots, 0, \det A(n-1))$, and $\mathfrak{M}_{ij}$ for $j \ne i, i \pm 1$ into $(0, \dots, 0, 0)$.

The matrix $(\mathfrak{M}_{ij})_{i,j=1,\dots,n}$ of the blocks has repeated rows due to the presence of unrestricted sequences. $A(n)$ then is obtained from this matrix by deleting repeated rows and corresponding columns. From $\mathfrak{M}_{ij}$ one would delete the first $\sum_{k=1}^{i-2} b(n-1, k)$ rows and $\sum_{k=1}^{j-2} b(n-1, k)$ columns so an undeleted column in $\mathfrak{M}_{ij}$ does not change its position when counted from the rear. Let $\mathfrak{B}(n, i)^{\langle p\rangle}$ and $B(n, i)^{\langle p\rangle}$ denote $\mathfrak{B}(n, i)$ and $B(n, i)$ with the last p configurations deleted. Consider the matrix given by $\langle\ ,\ \rangle$ on $\mathfrak{B}(n, i)^{\langle p\rangle} \times \bigcup_{j=1}^{n} B(n, j)^{\langle p\rangle}$. Any multiple of the column corresponding to the last element of $B(n, i)^{\langle p\rangle}$ appears only at the spots corresponding to the last elements of $B(n, i\pm 1)^{\langle p\rangle}$. By property (4) any other multiples were eliminated in the p deletions since they occur nearer the rear of their respective $\mathfrak{B}(n, i)^{\langle p\rangle} \times B(n, j)^{\langle p\rangle}$ block.

In the matrix $A(n)$ there is still a minor which is $\mathfrak{M}_{ij}$; however it does not appear as a solid block since some of its configurations have innermost arcs which occur before the ith spot. However one can perform the desired row operations by borrowing the missing rows from the blocks above. One may do similar operations on $A(n)^{\langle\overline{p}\rangle}$.

Proof of Lemma 2.5. (1) Let E be the matrix as in the proof of Lemma 2.2 such that

$$E^{\mathrm{tr}} A(n-1)^{\langle p\rangle} E = \begin{pmatrix} A(n-1)^{\langle p+1\rangle} & 0 \\ 0 & \det A(n-1)^{\langle p+1\rangle} \det A(n-1)^{\langle p\rangle} \end{pmatrix}.$$

We may assume that the entries of E are indexed by the first $C_{n-1}-p$ elements in D_{n-1} that is ordered by the restricted sequences. Consider the set $\mathfrak{S}$ of sequences (i, α) for $i = 1, \dots, k$ and the first $C_{n-1}-p$ restricted sequences α in D_{n-1}. There is an obvious equivalence relation in which two sequences are equivalent if they represent the same configuration. Mod out $\mathfrak{S}$ by this relation and we obtain a subset S of D_n. For $i = 1, \dots, k$ define a matrix E_i whose entries are indexed by S. The $([(i, \alpha)], [(i, \beta)])$th entry of E_i is equal to the (α, β)th entry of E for all elements $[(i, \alpha)]$, $[(i, \beta)]$ of S. All other diagonal entries of E_i are 1 and all other off-diagonal entries are 0. Hence E_1 is the identity except in the upper $(C_{n-1}-p)\times(C_{n-1}-p)$ corner where it is E. And E_i is obtained from E_1 by permuting rows and corresponding columns. Perform row operations E_1^{tr} to $A(n, k)^{\langle\overline{p}\rangle}$ and denote the blocks of $E_1^{\mathrm{tr}} A(n, k)^{\langle\overline{p}\rangle}$ by (G_{ij}). Then the last row of G_{12} consists of zeros except the last entry because the $(1, 2)$th block of $A(n, k)^{\langle\overline{p}\rangle}$ is exactly equal to $A(n-1)^{\langle p\rangle}$. And by Theorem 1.5(3), $G_{11} = \delta G_{12}$. Theorem 1.5(2) and (4) say that every column of the $(1,3)$th, $\ldots$, $(1,k)$th blocks of $A(n,k)^{\langle\overline{p}\rangle}$ is equal to one of the columns of the $(1,2)$th block of $A(n,k)^{\langle\overline{p}\rangle}$ which is not the last. Thus the last rows of $G_{13}, \ldots, G_{1k}$ are all zero. We now perform additional row operations $E_2^{\mathrm{tr}}, \ldots, E_k^{\mathrm{tr}}$ and all the corresponding column operations. Then the resulting matrix $E_k^{\mathrm{tr}} \cdots E_1^{\mathrm{tr}} A(n,k)^{\langle\overline{p}\rangle} E_1 \cdots E_k$ looks like

$$\left(\begin{array}{cccccccc}
 & 0 & & 0 & & 0 & & 0 \\
M_{11}^{\langle p+1\rangle} & \vdots & M_{12}^{\langle p+1\rangle} & \vdots & M_{13}^{\langle p+1\rangle} & \vdots & & \vdots \\
 & 0 & & 0 & & 0 & & 0 \\
0\cdots0 & \delta\xi & 0\cdots0 & \xi & 0\cdots0 & 0 & 0\cdots & 0 \\
 & 0 & & 0 & & 0 & & 0 \\
M_{21}^{\langle p+1\rangle} & \vdots & M_{22}^{\langle p+1\rangle} & \vdots & M_{23}^{\langle p+1\rangle} & \vdots & & \vdots \\
 & 0 & & 0 & & 0 & & 0 \\
0\cdots0 & \xi & 0\cdots0 & \delta\xi & 0\cdots0 & \xi & 0\cdots & 0 \\
 & 0 & & 0 & & 0 & & 0 \\
M_{31}^{\langle p+1\rangle} & \vdots & M_{32}^{\langle p+1\rangle} & \vdots & M_{33}^{\langle p+1\rangle} & \vdots & & \vdots \\
 & 0 & & 0 & & 0 & & 0 \\
0\cdots0 & 0 & 0\cdots0 & \xi & 0\cdots0 & \delta\xi & \cdots & 0 \\
 & 0 & & 0 & & & & 0 \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
0\cdots0 & 0 & 0\cdots0 & 0 & 0\cdots0 & 0 & 0\cdots & \delta\xi
\end{array}\right)$$

and $\xi = \det A(n-1)^{\langle p+1\rangle} \det A(n-1)^{\langle p\rangle}$. By permuting rows and corresponding columns, the matrix becomes

$$\begin{pmatrix} \xi T_k & 0 \\ 0 & A(n,k)^{\langle\overline{p+1}\rangle} \end{pmatrix}.$$

But

$$\det E_i = \det A(n-1)^{\langle p+1\rangle}$$

and

$$\det(\xi T_k) = \Delta_k (\det A(n-1)^{\langle p+1\rangle} \det A(n-1)^{\langle p\rangle})^k.$$

(2) The proof is similar. The only difference is that $A(n,k)^{\langle\overline{p}\rangle}$ now has j^2 blocks so we try to factor the tridiagonal matrix T_j out from it. $\square$

LEMMA 2.6. *For $3 \le k \le n$, we have the following recursion formulae*:

$$\det A(n-1) = \Delta_k^{b(n,k)} \left(\frac{\det A(n-1)}{\det A(n-1, k-2)}\right)^k \det A(n-1)^{\langle\overline{b(n,k)}\rangle},$$

and when $2 \le j \le k-1$,

$$\det A(n-1)^{\langle\overline{b(n,j+1)}\rangle} = \Delta_j^{b(n,j)-b(n,j+1)} \left(\frac{\det A(n-1,j-1)}{\det A(n-1,j-2)}\right)^j \det A(n-1)^{\langle\overline{b(n,j)}\rangle}.$$

Proof. We successively apply the recursion formula (1) in Lemma 2.5. Then

$$\begin{aligned}\det A(n,k) &= \Delta_k \left(\frac{\det A(n-1)}{\det A(n-1)^{\langle 1\rangle}}\right)^k \det A(n-1)^{\langle\bar{1}\rangle}\\ &= \Delta_k^2 \left(\frac{\det A(n-1)}{\det A(n-1)^{\langle 1\rangle}}\right)^k \left(\frac{\det A(n-1)^{\langle 1\rangle}}{\det A(n-1)^{\langle 2\rangle}}\right)^k \det A(n,k)^{\langle\bar{2}\rangle}\\ &\cdots\\ &= \Delta_k^{b(n,k)} \left(\frac{\det A(n-1)}{\det A(n-1)^{\langle 1\rangle}}\right)^k \left(\frac{\det A(n-1)^{\langle 1\rangle}}{\det A(n-1)^{\langle 2\rangle}}\right)^k\\ &\qquad \cdots \left(\frac{\det A(n-1)^{\langle b(n,k)-1\rangle}}{\det A(n-1)^{\langle b(n,k)\rangle}}\right)^k \det A(n,k)^{\langle\overline{b(n,k)}\rangle}\\ &= \Delta_k^{b(n,k)} \left(\frac{\det A(n-1)}{\det A(n-1)^{\langle b(n,k)\rangle}}\right)^k \det A(n,k)^{\langle\overline{b(n,k)}\rangle}.\end{aligned}$$

But $A(n-1)^{\langle b(n,k)\rangle} = A(n-1,k-2)$ because

$$b(n,k) = \sum_{i=k-1}^{n-1} b(n-1,i).$$

The other formula can be shown by using the formula (2) in Lemma 2.5. □

Theorem 2.7. *For* $3 \le k \le n$, *we have the following recursion formula*:

$$\det A(n,k) = \frac{\Delta_k^{b(n,k)} \Delta_{k-1}^{b(n,k-1)-b(n,k)} \cdots \Delta_2^{b(n,2)-b(n,3)} (\det A(n-1))^k}{(\det A(n-1,k-2))(\det A(n-1,k-3)) \cdots (\det A(n-1,1))}.$$

Proof. We recursively use the formulae in Lemma 2.6.

$$\begin{aligned}
&\det A(n, k)\\
&= \Delta_k^{b(n,k)}\left(\frac{\det A(n-1)}{\det A(n-1, k-2)}\right)^k \det A(n, k)^{\langle\overline{b(n,k)}\rangle}\\
&= \Delta_k^{b(n,k)}\Delta_{k-1}^{b(n,k-1)-b(n,k)}\left(\frac{\det A(n-1)}{\det A(n-1, k-2)}\right)^k\\
&\quad\cdot\left(\frac{\det A(n-1, k-2)}{\det A(n-1, k-3)}\right)^{k-1}\det A(n, k)^{\langle\overline{b(n,k-1)}\rangle}\\
&\quad\dots\\
&= \Delta_k^{b(n,k)}\Delta_{k-1}^{b(n,k-1)-b(n,k)}\cdots\Delta_3^{b(n,3)-b(n,4)}\left(\frac{\det A(n-1)}{\det A(n-1, k-2)}\right)^k\\
&\quad\cdot\left(\frac{\det A(n-1, k-2)}{\det A(n-1, k-3)}\right)^{k-1}\\
&\quad\cdots\left(\frac{\det A(n-1, 2)}{\det A(n-1, 1)}\right)^3\det A(n, k)^{\langle\overline{b(n,3)}\rangle}\\
&= \frac{\Delta_k^{b(n,k)}\Delta_{k-1}^{b(n,k-1)-b(n,k)}\cdots\Delta_3^{b(n,3)-b(n,4)}(\det A(n-1))^k}{(\det A(n-1, k-2))(\det A(n-1, k-3))\cdots(\det A(n-1, 2))}\\
&\quad\cdot\frac{\det A(n, k)^{\langle\overline{b(n,3)}\rangle}}{(\det A(n-1, 1))^3}.
\end{aligned}$$

But

$$\begin{aligned}
A(n, k)^{\langle\overline{b(n,3)}\rangle} &= A(n, 2)^{\langle\overline{b(n,3)}\rangle}\\
&= \begin{pmatrix}\delta A(n-1)^{\langle b(n,3)\rangle} & A(n-1)^{\langle b(n,3)\rangle}\\ A(n-1)^{\langle b(n,3)\rangle} & \delta A(n-1)^{\langle b(n,3)\rangle}\end{pmatrix}.
\end{aligned}$$

Since $A(n-1)^{\langle b(n,3)\rangle} = A(n-1, 1)$,

$$\begin{aligned}
\det A(n, k)^{\langle\overline{b(n,3)}\rangle} &= \Delta_2^{b(n-1,1)}(\det A(n-1, 1))^2\\
&= \Delta_2^{b(n,2)-b(n,3)}(\det A(n-1, 1))^2. \qquad\square
\end{aligned}$$

REMARK. By inserting the factor $\Delta_1^{b(n,1)-b(n,2)}$, which is 1, into the formula in Theorem 2.7, we obtain a recursion formula that works for all $k = 1, \dots, n$. See Proposition 2.3.

COROLLARY 2.8. *The* $\det A(n)$ *vanishes at twice the real part of any primitive* $2(n+1)$*st root of unity and* $\det A(m, k)$ *for* $1 \le m \le n-1$ *and* $1 \le k \le n-1$ *never vanishes at these values.*

Proof. The recursion formula of Theorem 2.7 shows that the determinants $\det A(m, k)$ for $1 \leq m \leq n$ and $1 \leq k \leq m$ can be written as a product of positive or negative powers of $\Delta_1, \ldots, \Delta_n$. It also shows that $\det A(n)$ contains the factor Δ_n exactly once and all the other determinants of lower indexes do not contain the factor Δ_n. Therefore $2\cos\frac{k\pi}{n+1}$ must be a root of $\det A(n)$ if k is relatively prime to $n+1$. □

COROLLARY 2.9. *After setting δ to be twice the real part of any primitive $2(n+1)$st root of unity, Lickorish's pairing can be considered as a symmetric bilinear form over the real (or complex) vector space with a basis D_n. Then the basis element $\alpha = (n, n-1, \ldots, 2, 1)$ has the property that there is a linear combination $\sum_{\beta\neq\alpha} q_\beta \beta$ of basis elements other than α such that $\langle\alpha, \gamma\rangle = \langle\sum_{\beta\neq\alpha} q_\beta\beta, \gamma\rangle$ for all γ in the vector space.*

Proof. The last row of $A(n)$ corresponds to α and $A(n)^{\langle 1\rangle} = A(n, n-1)$. By Corollary 2.8, $A(n)$ is singular but $A(n, n-1)$ is nonsingular. Thus it follows from Lemma 2.2. In fact, q_β's are equal up to sign to the (α, β)th cofactor of $A(n)$ divided by $\det A(n, n-1)$. □

REMARK. In fact the last elements of each block $B(n, k)$ of D_n as well as the rotations of the configuration $(n, n-1, \ldots, 1)$ have the property of Corollary 2.9.

COROLLARY 2.10. *We have the following recursive formula:*

$$\det A(n) = \prod_{i=1}^{[(n+1)/2]} (\det A(n-i))^{(-1)^{i-1}\binom{n-i+1}{i}} \prod_{i=0}^{[(n-1)/2]} \left(\frac{\Delta_{n-i}}{\Delta_i}\right)^{b(n-i,\, n-2i)}$$

where $\Delta_0 = 0$.

Proof. One can derive this from the formula in Theorem 2.7 using the following identities:

$$\binom{n-i+1}{i} = \sum_{k_1=1}^{n-2i+2} \sum_{k_2=1}^{k_1} \cdots \sum_{k_i=1}^{k_{i-1}} 1$$

and

$$b(n, k) - b(n, k+1) = b(n-1, k-1). \qquad \square$$

Let $d(n, j)$ denote the exponent of Δ_j in $\det A(n)$. It is not hard to see that $d(n, j)$ is well defined for $j \geq 1$.

COROLLARY 2.11. *For $j \geq 1$, we have that*

$$\sum_{i=0}^{[(n+1)/2]} (-1)^i \binom{n-i+1}{i} d(n-i, j) = b(j, 2j-n)$$

where $b(n, k) = -b(n-k, -k)$ for $k < 0$ and $b(n, 0) = b(n, k) = 0$ for $k > n$.

Proof. Immediate from Corollary 2.10. □

REMARK. It is interesting to note that the Catalan numbers satisfy the similar formula:

$$\sum_{i=0}^{[(n+1)/2]} (-1)^i \binom{n-i+1}{i} C_{j-i+1} = b(j+2, n+2) \quad \text{for } j \geq \left[\frac{n+1}{2}\right].$$

This formula can be proved by recalling that $b(n, k)$ is the number of configurations in D_n that the first innermost arc occurs at the kth point and by applying the inclusion-exclusion principle.

The following theorem shows that $\det A(n)$ is generated by a simple rule.

THEOREM 2.12. *For $j \geq 1$, we have that*

$$d(n, j) = d(n-1, j-1) + 2d(n-1, j) + d(n-1, j+1)$$

where $d(n, 0) = 2C_n - C_{n+1} = -\frac{4}{n+2}\binom{2n-1}{n+1}$.

Proof. By the remark following Corollary 2.11, the formula in Corollary 2.11 holds for $j = 0$ if we set $b(0, k) = -b(-k, -k) = -1$ for $k < 0$. Use an induction on (n, j) with lexicographic order. Since $d(1, 0) = 2C_1 - C_2 = 0$, $d(2, 1) = d(1, 0) + 2d(1, 1) + d(1, 2)$.

From the formula in Corollary 2.11,

$$d(n, j) = b(j, 2j-n) + \sum_{i=1}^{[(n+1)/2]} (-1)^{i-1} \binom{n-i+1}{i} d(n-i, j).$$

By the induction hypothesis and the identity $\binom{n-i+1}{i} = \binom{n-i}{i} + \binom{n-i}{i-1}$,

$$\begin{aligned}
&\sum_{i=1}^{[(n+1)/2]} (-1)^{i-1} \binom{n-i+1}{i} d(n-i, j) \\
&= \sum_{i=1}^{[(n+1)/2]} (-1)^{i-1} \binom{n-i}{i} (d(n-i-1, j-1) + 2d(n-i-1, j) \\
&\qquad\qquad + d(n-i-1, j+1)) \\
&\quad + \sum_{i=1}^{[(n+1)/2]} (-1)^{i-1} \binom{n-i}{i-1} (d(n-i-1, j-1) + 2d(n-i-1, j) \\
&\qquad\qquad + d(n-i-1, j+1)) \\
&= \sum_{i=1}^{[n/2]} (-1)^{i-1} \binom{n-i}{i} (d(n-i-1, j-1) + 2d(n-i-1, j) \\
&\qquad\qquad + d(n-i-1, j+1)) \\
&\quad + \sum_{i=0}^{[(n-1)/2]} (-1)^{i} \binom{n-i-1}{i} \\
&\qquad \times (d(n-i-2, j-1) + 2d(n-i-2, j) \\
&\qquad\qquad + d(n-i-2, j+1)) \\
&= d(n-1, j-1) - b(j-1, 2j-n-1) \\
&\quad + 2d(n-1, j) - 2b(j, 2j-n+1) + d(n-1, j+1) \\
&\quad - b(j+1, 2j-n+3) + b(j-1, 2j-n) \\
&\quad + 2b(j, 2j-n+2) + b(j+1, 2j-n+4) \\
&= -b(j, 2j-n) + d(n-1, j-1) + 2d(n-1, j) \\
&\quad + d(n-1, j+1).
\end{aligned}$$

The last equality is achieved by several uses of the identity

$$b(n, k) - b(n, k+1) = b(n-1, k-1)$$

for all integer k and all $n \geq 2$. □

One can now easily generate $\det A(n)$ by using the rule in Theorem 2.12 as in the following table. The term $\Delta_0 = 1$ is inserted for a computational purpose.

$$\begin{aligned}
\det A(1) &= \Delta_0^{0}\ \Delta_1\\
\det A(2) &= \Delta_0^{-1}\ \Delta_1^{2}\ \Delta_2\\
\det A(3) &= \Delta_0^{-4}\ \Delta_1^{4}\ \Delta_2^{4}\ \Delta_3\\
\det A(4) &= \Delta_0^{-14}\ \Delta_1^{8}\ \Delta_2^{13}\ \Delta_3^{6}\ \Delta_4\\
\det A(5) &= \Delta_0^{-48}\ \Delta_1^{15}\ \Delta_2^{40}\ \Delta_3^{26}\ \Delta_4^{8}\ \Delta_5\\
\det A(6) &= \Delta_0^{-165}\ \Delta_1^{22}\ \Delta_2^{121}\ \Delta_3^{100}\ \Delta_4^{43}\ \Delta_5^{10}\ \Delta_6\\
\det A(7) &= \Delta_0^{-572}\ \Delta_1^{0}\ \Delta_2^{364}\ \Delta_3^{364}\ \Delta_4^{196}\Delta_5^{64}\ \Delta_6^{12}\Delta_7\\
\det A(8) &= \Delta_0^{-2002}\Delta_1^{-208}\Delta_2^{1092}\Delta_3^{1288}\Delta_4^{820}\Delta_5^{336}\Delta_6^{89}\Delta_7^{14}\Delta_8
\end{aligned}$$

REMARK. Notice that the exponents of Δ_i may be negative; however $\det A(n)$ is a polynomial in δ. The negative exponents arise since the Δ_i's are not relatively prime to each other. In fact the factor $\delta - 2\cos\frac{k\pi}{i+1}$ of Δ_i is also a factor of Δ_j if $i+1$ divides $j+1$. Moreover, if k is relatively prime to $i+1$, then the converse holds. For example δ is a factor of Δ_{2i+1} for all i and $\delta^2 - 1$ is a factor of Δ_{3i+2} for all i. R. A. Litherland has shown that the exponent of δ in $\det A(n)$ is C_n and that the exponent of $\delta^2 - 1$ is $C_n - 1$.

REFERENCES

[1] R. A. Brualdi, *Introductory Combinatorics*, North-Holland, New York, 1977.

[2] R. C. Kirby and P. Melvin, *Evaluations of the* 3*-manifold invariants of Witten and Reshetikhin-Turaev for* $\mathrm{sl}(2, \mathbf{C})$, (to appear).

[3] W. B. R. Lickorish, 3*-manifold invariants from the combinatorics of the Jones polynomial*, preprint.

[4] N. Yu. Reshetikhin and V. G. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math., (to appear).

[5] E. Witten, *Quantum field theory and Jones' polynomial*, Comm. Math. Phys., **121** (1989), 351–399.

Received February 23, 1990 and in revised form April 5, 1990. The first author was partially supported by a Korea Science and Engineering Foundation Postdoctoral Fellowship, and the second author partially supported by a Louisiana Education Quality Support Fund grant LEQSF(87-89)-RD-A-8.

KOREA INSTITUTE OF TECHNOLOGY
TAEJON, 305-701, KOREA

Currently visiting: Louisiana State University
Baton Rouge, LA 70803

AND

LOUISIANA STATE UNIVERSITY
BATON ROUGE, LA 70803