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CROSSED PRODUCTS AND GENERALIZED INNER ACTIONS OF HOPF ALGEBRAS

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This paper examines crossed products $R*H$ where the Hopf algebra H acts weakly on the algebra R and is twisted by a Hopf cocycle t. Invertible cocycles are discussed and a related sort of weak action which we call "fully invertible" is introduced. This condition allows us to undo the action of H in a useful way and allows reasonable behavior of ideals in crossed products. Many crossed products of interest are of this type, including crossed products of cocommutative Hopf algebras with invertible cocycles, crossed products of irreducible Hopf algebras, and all smash products with bijective antipode. We construct the quotient ring Q of an H-prime ring and discuss actions which become inner when extended to Q. This is then applied to describe prime ideals in crossed products over H-prime rings with extended inner actions and it is shown that some of these crossed products are semiprime.

Introduction. This paper involves crossed products $R * H$ where the Hopf algebra H acts on the algebra R and the image of H is twisted by a Hopf cocycle t. The ideas here build on some of those introduced in [**BCM**] and that paper serves as a foundation for what is done here. Under some fairly general technical hypotheses we examine the behavior of ideals and Martindale quotient rings in relation to the weak action of H. Using the quotient ring and results from [**Ch**] and [**BCM**] as main tools we focus on prime ideals in crossed products and then show that certain crossed products with extended inner actions are semiprime.

In the first section crossed products with invertible cocycles are discussed. We introduce a sort of weak action which we call "fully invertible". This condition allows us to undo the action of H in useful ways and allows for reasonable behavior of ideals in crossed products. Many crossed products of interest are fully invertible, including all crossed products with invertible cocycles and cocommutative Hopf algebras H, and all smash products with bijective antipode. Starting with Proposition 1.2, some basic facts concerning ideals in crossed products with fully invertible actions are established.

We introduce H-prime and H-invariant ideals and prove Lemma 1.7, establishing a useful construction of H-invariant ideals when the cocycle is invertible. As an application we have Theorem 1.9 which establishes a close link between H-primes and primes when H is finite dimensional and irreducible as a coalgebra.

We apply this preliminary material in §2, moving next to the construction of quotient rings of H-prime rings. The main result is Theorem 2.3 which shows that crossed products with invertible cocycles and fully invertible actions can be extended to quotient rings of H-prime rings. Here we assume an invertible cocycle and fully invertible action in order to construct and extend the action to the symmetric quotient ring Q. This section ends with a brief discussion of Q-inner actions, i.e., actions which are inner on Q.

In §3 we turn to the study of prime ideals in crossed products with Q-inner actions with H-prime coefficient rings. The section begins with a sequence of results concerning prime ideals from **[Ch]** and appropriate generalizations. The idea here is to lift to primes in $Q * H$ and then drop down to $C_\tau[H]$, a twisted product over the extended center. Here we use the fact **[BCM]** that the action can be trivialized in Q by altering the cocycle. As is the case for crossed products of restricted enveloping algebras **[Ch]** we can get a description of the primes having trivial intersection with the coefficient ring as the prime spectrum of a finite dimensional twisted product (Corollary 3.5). In fact we use this to show in Theorem 3.6 that this finite dimensional algebra is (semi)prime if and only if $R * H$ is. This is applied to obtain the last result, Corollary 3.7, which states that $R*H$ is semiprime provided R is H-prime, the action of H is Q-inner and H is finite dimensional semisimple. It is not known whether the Q-inner hypothesis is needed. It is also natural to ask if we can replace the H-prime condition with "H-semiprime".

The reader is assumed to be familiar with the elementary theory Hopf algebras and the sigma notation in **[S2]** and some of the basic material from **[BCM]**.

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1. Invertible actions and crossed products. We begin by mentioning some notation and some facts we shall need about the crossed product construction **[BCM]**.

R shall denote an algebra over the field k and H a Hopf algebra over k with structure maps μ, Δ, u, ε, S denoting the multiplication, comultiplication, unit, counit, and antipode, respectively. $\overline{S}$ shall denote the composition inverse of S (when it exists). We shall use the following abbreviated form of Sweedler's notation for comultiplication:

$$\Delta(h) = \sum h_1 \otimes h_2, \qquad h \in H.$$

A crossed product $R*H$ is an associative algebra with underlying k-space $R \otimes_k H$ and identity $1*1$. We shall assume throughout that an "action" ("weak" in the sense of [**BCM**]) of H on R yields a crossed product $R*H$ with implicit (left) action $\Phi \in \mathrm{Hom}_k(H, \mathrm{End}\, R)$ and cocycle $t \in \mathrm{Hom}_k(H \otimes H, R)$. Let $h, l, m \in H$ and $a, b \in R$. We shall sometimes write $\Phi_h(a) = h.a$ for the action. A subset $A \subset R$ is said to be H-invariant or Φ-invariant if $\Phi_h(a) \in A$ for all $a \in A$. The multiplication in $R*H$ is defined by

$$(a*h)(b*l) = \sum a(h_1, b)t(h_2, l_1) * h_3 l_2.$$

It turns out [**BCM**, p. 691] that $R*H$ is a crossed product with identity $1*1$ if and only if the map t is *normal*: $t(h, 1) = t(1, h) = \varepsilon(h)1$, satisfies the *cocycle conditions*:

$$\sum [(h_1.t(l_1, m_1))]t(h_2, l_2 m_2) = \sum t(h_1, l_1)t(h_2 l_2, m)$$

and satisfies the *twisted module condition*:

$$\sum h_1.(l_1, a)t(h_2, l_2) = \sum t(h_1, l_1)(h_2 l_2 \cdot a).$$

Invertibility. Let R be any algebra. Consider the vector space $\mathrm{Hom}_k(H, R)$. As usual we have the convolution multiplication $f \cdot g = \sum f(h_1)g(h_2)$. As in [**DT**] there is also an anti-convolution product $f \times g$ defined by

$$(f \times g)(h) = \sum f(h_2)g(h_1).$$

Both $\times$ and $\cdot$ have the same identity $u\varepsilon$. We say that $f \in \mathrm{Hom}_k(H, R)$ is *anti-invertible* if it has an inverse with respect to $\times$.

It will be useful to notice the fact that f is anti-invertible if and only if it is invertible in the convolution algebra $\mathrm{Hom}(D, R)$ where D is the "opposite coalgebra" of the underlying coalgebra of H (with the order of the tensor factors in the comultiplication reversed). Also it is evident that these products are identical if H is cocommutative.

We shall generally denote the convolution inverse of a map f by f^{-1} and the anti-inverse of f by $f^{\frown}$ (if they exist). In this paper "(anti)-invertibility" will always be used in this sense of (anti-)convolution.

The following useful result is based on an argument of Sweedler **[S1]**. It guarantees that our cocycles t are often invertible.

PROPOSITION 1.1. *Let $t \in \mathrm{Hom}_k(D, R)$ where D is a pointed coalgebra which is the sum of its irreducible components, and R is a k-algebra. If $t(x)$ is invertible as an element of R for each group-like element $x \in D$, then t is convolution invertible. In particular if H is irreducible as a coalgebra then every cocycle t is invertible.*

REMARK. Inner cocycles [**BCM**, Example 4.11], and the twistings arising in the splitting theorems [**BCM**, **DT**] are both invertible.

Proof. Let $G = G(D)$ denote the set of group-like elements of D and let $kG = D_0 < D_2 < D_3 < \cdots$ be the coradical filtration of D so that $D = \bigcup D_n$. Write $H = \sum_{x \in G} D_x$, the direct sum decomposition of H into irreducible components. Note that $D_n = \sum_{x \in G} D_{nx}$.

We define an inverse for t by induction on n: For $\lambda \in k$, $x \in G$ let $t^{-1}(\lambda x) = \lambda t(x)^{-1}$. Assume that t^{-1} has been defined on D_i for all $i < n$. Let $h \in D_{nx} \cap \ker \varepsilon = D_{nx}^+$ so that we have

$$\Delta(h) = h \otimes x + x \otimes h + Y, \quad \text{for some } Y \in D_{n-1} \otimes D_{n-1}.$$

Let μ be the multiplication map of R and define

$$t^{-1}(h) = [\varepsilon(h)1_R - t(x)^{-1}t(h) - \mu((t^{-1} \otimes t)(Y))]t(x)^{-1}.$$

Since $D_{nx} = D_{nx}^+ + kx$ for all x and n, the map t^{-1} is now defined on all of D.

We now verify that the convolution t^{-1} is the inverse of t:

$$\begin{aligned}
t^{-1} \cdot t(h) &= \mu(t^{-1} \otimes t)\Delta(h) \\
&= \mu(t^{-1} \otimes t)(x \otimes h + h \otimes x + Y) \\
&= \mu\{[\varepsilon(h)1_R - t(x)^{-1}t(h) - \mu((t^{-1} \otimes t)(Y))]t(x)^{-1} \otimes t(x) \\
&\qquad\qquad + t(x)^{-1} \otimes t(h)\} \\
&\quad + \mu(t^{-1} \otimes t)(Y) \\
&= \varepsilon(h)1_R - t(x)^{-1}t(h) - \mu((t^{-1} \otimes t)(Y)) \\
&\quad + t(x)^{-1}t(h) + \mu(t^{-1} \otimes t)(Y) \\
&= \varepsilon(h)1_R.
\end{aligned}$$

Similarly t has a right inverse (necessarily equal to the left inverse). Thus t is invertible.

If H is irreducible then $H \otimes H$ is irreducible with unique group element $1 \otimes 1$. The second statement now holds since $t(1, 1) = 1_R$. $\square$

DEFINITION. Recall that the action $\Phi \in \mathrm{Hom}(H, \mathrm{End}\, R)$ is denoted by $\Phi_h r = h.r$. We say that H has an *anti-invertible* action on R if the left action $\Phi \in \mathrm{Hom}(H, \mathrm{End}\, R)$ has an *anti*-inverse Ψ. In other words for $h \in H$,

$$\sum \Psi_{h_2} \circ \Phi_{h_1} = \varepsilon(h)\, \mathrm{id}_R \quad \text{and} \quad \sum \Phi_{h_2} \circ \Psi_{h_1} = \varepsilon(h)\, \mathrm{id}_R .$$

We further say that the action of H is *fully invertible* if H-invariant ideals of R are also invariant under (the image of) Ψ. We say that Ψ *anti-measures* R if for $a, b \in R$ and $h \in H$, $\Psi_h(ab) = \sum(\Psi_{h_2} a)(\Psi_{h_1} b)$.

We do not know of any anti-invertible actions that are not fully invertible.

PROPOSITION 1.2. *Let $R*H$ be a crossed product with anti-invertible action Φ. Then the anti-inverse Ψ anti-measures R.*

Proof. Let $a, b \in R$ and $h \in H$. Then

$$\begin{aligned}\sum(\Psi_{h_2} a)(\Psi_{h_1} b) &= \sum \Psi_{h_4} \Phi_{h_3} (\Psi_{h_2} a)(\Psi_{h_1} b) \\ &= \sum \Psi_{h_5} (\Phi_{h_3} \Psi_{h_2} a)(\Phi_{h_4} \Psi_{h_1} b) = \Psi_h(ab).\end{aligned}$$ $\square$

When this work was originally done, we assumed that H was cocommutative and t invertible or that t was trivial and S bijective. As the referee has pointed out, computations can be simplified and generalized by using the following technical set-up: Let $\gamma \in \mathrm{Hom}(H, R*H)$ be defined by $\gamma(h) = 1 * h$. We shall use the recent result [**BM**, Proposition 1.8] which says that γ is invertible if and only if t is invertible.

The next result gives conditions involving γ which guarantee that the action is fully invertible. Here the action and its anti-inverse can be nicely expressed in terms of γ.

LEMMA 1.3. *Let $R*H$ be a crossed product. Suppose that the map γ is both invertible and anti-invertible. Then the action is fully invertible, with*

(i) $\Phi_h r = \sum \gamma(h_1) r \gamma^{-1}(h_2)$,
(ii) $\Psi_h r = \sum \gamma^{\frown}(h_2) r \gamma(h_1)$, *for all $h \in H$ and $r \in R$.*

Proof. The first equality is straightforward:

$$\begin{aligned}\sum \gamma(h_1)r\gamma^{-1}(h_2) &= \sum(1 * h_1)(r * 1)\gamma^{-1}(h_2)\\ &= \sum((h_1.r) * 1)\gamma(h_2)\gamma^{-1}(h_3) = h \cdot r * 1 = \Phi_h r.\end{aligned}$$

Let $\theta(r) = \sum \gamma^{\frown}(h_2)r\gamma(h_1)$. The second equality will follow once we show that $\theta(r) \in R$ for all $r \in R$. To this end let $T = R * H$ and $\rho: T \to T \otimes H$ denote the usual comodule structure map

$$\rho(r * h) = \sum (\rho * h_1) \otimes h_2.$$

Let $i: T \to T \otimes H$ and $j: H \to T \otimes H$ be the algebra maps defined by $i(\alpha) = \alpha \otimes 1$ and $j(h) = 1 * 1 \otimes h$. Now according to [**DT**, Proposition 5], the convolution inverse j^{-1} and anti-inverse $j^{\frown}$ exist and satisfy

$$\rho \circ \gamma^{\frown} = (i \circ \gamma^{\frown}) \times j^{\frown}.$$

Using this expression we now verify that θ has image in the "co-invariant" subalgebra of T (i.e., $\{\alpha \in T | \rho(\alpha) = \alpha \otimes 1\}$) which, by [**BCM**, p. 701, Lemma 5.10] is precisely $R * 1 = R$.

For $r \in R$ and $h \in H$,

$$\begin{aligned}\rho(\theta(r)) &= \sum \rho(\gamma^{\frown}(h_2)r\gamma(h_1))\\ &= \sum(i(\gamma^{\frown}(h_4))j^{\frown}(h_3))(r * h_1 \otimes h_2) \quad (\rho \text{ is an algebra map})\\ &= \sum i(\gamma^{\frown}(h_4))j^{\frown}(h_3)(1 * 1 \otimes h_2)(r * h_1 \otimes 1)\\ &= \sum i(\gamma^{\frown}(h_4))j^{\frown}(h_3)j(h_2)(r * h_1 \otimes 1)\\ &= \sum i(\gamma^{\frown}(h_2))(r * h_1 \otimes 1)\\ &= \sum(\gamma^{\frown}(h_2)r\gamma(h_1)) \otimes 1\\ &= \theta(r) \otimes 1.\end{aligned}$$

We conclude that θ is a k-linear endomorphism of R. Now one may directly verify that $\Psi_h = \theta$ defines the required anti-inverse for Φ.

Finally if A is an H-invariant ideal of R, then surely $\Psi_h(a) \in R \cap (A * H) = A$ for $a \in A$. Thus the action is *fully* invertible, as claimed. □

There are weak actions (yielding to crossed products) that are fully invertible without γ being invertible. For instance let H have a bijective antipode and suppose that R is an H-module and t is a non-invertible cocycle resulting in a crossed product $R * H$. Setting

$\Psi_h(r) = \overline{S}(h).r$ we see that the action is fully invertible. But since t is not invertible, neither is γ **[BM]**.

Generally, groups acting as automorphisms yield crossed products with non-invertible cocycles. **[HLS]** initiates a study of these crossed products when R is a Galois extension and G is the Galois group.

Using the lemma we show next that many actions are fully invertible.

PROPOSITION 1.4. *An action of H on R is fully invertible in the following cases:*

(i) *H is cocommutative and t is invertible,*
(ii) *t is trivial and H has a bijective antipode, or*
(iii) *H is irreducible (as a coalgebra).*

Proof. If (i) holds then γ is invertible by **[BM]**, Proposition 1.8. By cocommutivity γ is obviously anti-invertible as well. Thus the previous lemma yields the result.

If (ii) holds we can let $\gamma^{-1}(h) = 1 * Sh$ and $\gamma^{\wedge}(h) = 1 * \overline{S}h$. In case H is irreducible, then γ is invertible in $\operatorname{Hom}(H, R * H)$ by Proposition 1.1. The opposite coalgebra D of H is clearly also irreducible with unique group-like element 1_H. Also $\gamma(1) = 1 * 1$, which is obviously a unit in $R * H$. Therefore γ is invertible as an element of $\operatorname{Hom}(D, R * H)$ by Proposition 1.1. Thus γ is anti-invertible in $\operatorname{Hom}(H, R * H)$. □

We identify R with its image $R * 1$ in $R * H$. The next result says that invariant ideals extend naturally. The result is essentially contained in **[CF]** in the case where t is trivial and H has a bijective antipode.

LEMMA 1.5. *Let $R * H$ be a crossed product and let A be an H-invariant ideal of R.*

(a) *Then $(R * H)A \subset A * H$ and hence $A * H$ is an ideal of $R * H$.*
(b) *In addition if the action is fully invertible, then $A * H = (1 * H)A$.*

Proof. (a) follows directly from the formula for crossed product multiplication.

Now assume that the action is fully invertible. Let $b \in a$ and compute:

$$\sum (1 * h_2)(\Psi_{h_1} b * 1) = \sum h_2 \cdot (\Psi_{h_1} b) * h_3 = b * h .$$

(For the second equality the cocycle term is absorbed since t is normal.) Thus since A is Ψ-invariant, the reverse inclusion holds. □

H-prime ideals. In preparation for what follows we introduce H-primes and show how they many arise.

DEFINITION. R is said to be an *H-prime* ring if the product of any two nonzero H-invariant ideals is again nonzero. H-prime ideals are H-invariant ideals with an H-prime factor ring.

The following simple and well-known example shows that H-prime does not imply semiprime. Let $R = k[t|t^p = 0]$ where k is a field of characteristic $p > 0$. Let $L = kx$ be the one-dimensional restricted Lie algebra with $x^p = 0$ and let x act as d/dt. Then, with $H = u(L)$, R is H-simple and hence H-prime, but R is not semiprime.

The following shows a way in which H-prime ideals arise naturally.

LEMMA 1.6. *Let $R * H$ be a crossed product and let P be a prime ideal of $R * H$. Then $P \cap R$ is an H-prime ideal of R.*

Proof. Suppose that A_1 and A_2 are H-invariant ideals of R such that $A_1 A_2 \subset P \cap R$. Then

$$(A_1 * H)(A_2 * H) = A_1(R * H)A_2(R * H) \subset A_1 A_2(R * H) \subset P,$$

where the next to last inclusion holds by Lemma 1.5(a). Now since P is prime, $A_i * H \subset P$ for some i; thus $(A_i * H) \cap R = A_i \subset P \cap R$. □

Let $R * H$ be a crossed product. Further let P be an ideal of R and define

$$(P : H) = \{r \in R | h.r \in P, \text{ all } h \in H\}.$$

Note that $(P : H) \subset P$ since $1 \in H$.

LEMMA 1.7. *$(A : H)$ is an H-invariant ideal of R if A is an ideal of R and t is an invertible cocycle. In this case $(A : H)$ is the largest invariant ideal contained in A.*

Proof. $(A : H)$ is an ideal of R contained in A since H measures R and H contains an identity. We need to show that $(A : H)$ is closed under the action of H. To do so we let $h, l \in H$ and $a \in (A : H)$. Then

$$\begin{aligned} h \cdot (l \cdot a) &= \sum t(h_1, l_1)(h_2 l_2 \cdot a)t^{-1}(h_3, l_3) \\ &\in R(H \cdot (A : H))R \subset A, \end{aligned}$$

using the twisted module condition for the first equality. Thus $l \cdot a \in (A : H)$ as required. □

REMARK. If R is an H-module (t not necessarily trivial) then the twisted module condition becomes irrelevant in the lemma above. It is then evident that the conclusions of the lemma hold.

Let H be a finite dimensional irreducible Hopf algebra acting on R. (For example $H = u(L)$). Let m denote the length of the coradical filtration

$$k = H_0 < H_1 < \cdots < H_m = H$$

of H. (If $H = u(L)$, this is the filtration by total degree; see **[S2]**.) Note that by Proposition 1.1, for any crossed product $R * H$ the cocycle t is actually invertible, so the preceding lemma applies in this situation. This fact is tacit in the following lemma and theorem.

LEMMA 1.8. *Let H be a finite dimensional irreducible Hopf algebra having coradical filtration of length m, and let $R * H$ be a crossed product. Let N be an ideal of R. Then*

(i) *if N is nilpotent and R is H-prime, then $(N : H) = 0$ and $N^{m+1} = 0$;*
(ii) *if $(N : H) = 0$, then N is nilpotent.*

Proof. Suppose $(N : H) = 0$. We show by induction on j that $H_j.N^n \subset N$ if $n > j$. If $j = 0$, $H_j = k$ so the result is clear. Assume $j > 0$ and note that

$$\begin{aligned} H_j \cdot N^n &= H_j \cdot (NN^{n-1}) \\ &\subset (H_0 \cdot N)(H_j \cdot N^{n-1}) + (H_j \cdot N)(H_0 \cdot N^{n-1}) \\ &\quad + (H_{j-1} \cdot N)(H_{j-1} \cdot N^{-1}) \subset N, \end{aligned}$$

where the second inclusion holds because H measures R. The last inclusion holds because $H_0 = k$ by induction. Let m be as above. We have shown that $H_m \cdot N^{m+1} = H \cdot N^{m+1} \subset N$; thus $N^{m+1} \subset (N : H) = 0$.

Next suppose N is a nilpotent ideal. Then, being H-prime, R certainly has no H-invariant nilpotent ideals, so $(N : H) = 0$. Finally, be the result of first paragraph, we also have that N is nilpotent of degree at most $m + 1$. □

H-Spec R denotes the set of H-prime ideals of R.

THEOREM 1.9. *Let H be a finite dimensional irreducible Hopf algebra and let $R * H$ be a crossed product. Let Q be an H-prime ideal of R and let P be a prime ideal. Then there exists a prime ideal $N = N(Q)$ which is maximal with respect to $(N : H) = Q$.*

(i) *N is nilpotent* mod Q.

(ii) *The maps defined by $P \to (P : H)$ and $Q \to N(Q)$ are inclusion preserving inverses, defining a bijection from* $\operatorname{Spec}(R)$ *to* H-$\operatorname{Spec}(R)$.

Proof. We just let $N(Q)$ be the sum of the nilpotent ideals containing Q. By the lemma, this is a sum of ideals of bounded nilpotency index and thus is nilpotent. The maximality property is immediate from part (ii) of the lemma.

The maps are easily seen to be inclusion preserving.

Being nilpotent $(\bmod(P : H))$ by part (i) of the lemma, the prime ideal P is the unique largest nilpotent ideal containing $(P : H)$, so $P = N((P : H))$. The following argument shows that $(P : H)$ is H-prime. Suppose we have H-invariant ideals A and B with $AB \subset (P : H)$. Then of course $AB \subset P$, and thus (say) $A \subset P$. Hence $A = (A : H) \subset (P : H)$.

Let us observe that $Q(N(Q) : H)$. We may assume that $Q = 0$ so R is H-prime. Since $N = N(Q)$ is nilpotent we obtain $Q = 0 = (N : H)$ from part (i) of the lemma.

To see that N is a prime ideal, notice that N is maximal subject to the condition $(N : H) = 0$. A standard sort of argument now applies. □

2. Crossed products and quotient rings. Next we establish a basic identity which shall be used to extend the action to the quotient ring.

LEMMA 2.1. *Let $R * H$ be a crossed product with an anti-invertible action. Then for $a, b \in R$ and $h \in H$,*

$$b(h.a) = \sum h_2.[(\Psi_{h_1} b)a].$$

Proof. This equation follows immediately using measuring and the fact that Ψ is the anti-inverse of Φ. □

LEMMA 2.2. *Let H act fully invertibly on R. Then right annihilators in R of H-invariant ideals are H-invariant ideals of R. If the cocycle t is invertible then the same is true of left annihilators.*

Proof. Let us show that right annihilators of H-invariant ideals are H-invariant. Let A be an H-invariant (and hence Ψ-invariant) ideal of R and let B denote its right annihilator on R. With $a \in A$ and $b \in B$, we use (i) in the lemma above:

$$\begin{aligned} a(h \cdot b) &= \sum h_2 \cdot [(\Psi_{h_1} a)b] \in h \cdot (Ab) \quad (A \text{ is } \Psi\text{-invariant}) \\ &= h \cdot \{0\} = \{0\}. \end{aligned}$$

Thus B is H-invariant.

Now if t^{-1} (and hence γ^{-1}) exists, let A be the left annihilator of the H-invariant ideal B. We now observe that

$$\begin{aligned} (h \cdot a)b &= \sum \gamma(h_1) a \gamma^{-1}(h_2) b \subset (R * H)a(R * H)B \\ &\subset (R * H)aB(R * H) = \{0\}, \end{aligned}$$

using Lemma 1.5. □

Let H act fully invertibly on R (with t invertible) and let R be an H-prime ring. Let $\mathscr{F} = \mathscr{F}_H(R)$ denote the set of nonzero H-invariant ideals of R. As right (and left) annihilators of elements of $\mathscr{F}$ are again H-invariant (Lemma 2.2), these annihilators are all zero. Also $\mathscr{F}$ is easily seen to be closed under finite intersections and products using Lemma 1.5. Hence we may form the left (and symmetric) Martindale quotient rings.

We denote the left and symmetric quotient rings by Q^l and Q, respectively. We shall assume that the reader is familiar with the basic properties of these quotient rings. A detailed account of quotient rings of prime rings may be found in [P].

Crossed products over quotient rings.

THEOREM 2.3. *Let R be H-prime. A fully invertible action of H on R extends uniquely to an action of H on Q^l. Consequently $R * H$ extends uniquely to a crossed product $Q^l * H$ (with the same cocycle t). If in addition the cocycle t is invertible then the same conclusions hold with Q in place of Q^l.*

REMARKS. These facts, except for uniqueness, are contained in [C] in the case that the cocycle t is trivial and H has a bijective antipode.

Proof. To prove that H extends to Q^l, given $A \in \mathscr{F}$, $h \in H$ and $f \colon {}_RA \to R$, define $H.F \colon {}_RA \to R$ by

$$a(h \cdot f) = \sum h_2 \cdot [(\Psi_{h_1} a) f] \quad (\text{writing } f \text{ on the right}).$$

One needs to show that $h \cdot f$ is actually a left R-module homomorphism and that $f \to h \cdot f$ induces a well-defined weak action of H on Q^l which extends the weak action of H on R. Most of the details are left to the reader. As a sample computation, the following argument shows that H measures Q^l. Let $B \in \mathscr{F}$, $h \in H$, and $g\colon {}_RB \to R$. We check that the composition $fg\colon BA \to R$ satisfies $h \cdot (fg) = \sum (h_1 \cdot f)(h_2 \cdot g)$. To do this let $ba \in BA$ and observe that

$$\begin{aligned}
ba(h \cdot (fg)) &= \sum h_2 \cdot [(\Psi_{h_1}(ba))fg] \\
&= \sum h_3 \cdot [(\Psi_{h_2} b)(\Psi_{h_1} a)fg] \\
&= \sum h_3 \cdot [(\Psi_{h_2} b)((\Psi_{h_1} a)f)g] \\
&= \sum h_5 \cdot \{(\Psi_{h_4} b)[\Psi_{h_3} \circ \Phi_{h_2} \cdot ((\Psi_{h_1} a)f)]g\} \\
&\qquad\qquad\qquad\qquad (\Phi \text{ and } \Psi \text{ are anti-inverses}) \\
&= \sum bh_4 \cdot \{[\Psi_{h_3}(h_2 \cdot ((\Psi_{h_1} a)f))]g\} \quad (\text{measuring}) \\
&= \sum b[h_2 \cdot ((\Psi_{h_1} a)f)]h_3 \cdot g \quad (\text{definition of } h.g) \\
&= \sum b(a(h_1 \cdot f))(h_2 \cdot g) \quad (\text{definition of } h.f) \\
&= ba \sum (h_1 \cdot f)(h_2 \cdot g).
\end{aligned}$$

Now one may check that H measures Q^l follows by passing to equivalence classes in the quotient ring.

Let us construct the twisted smash product $Q^l \#_t H$ with the same cocycle t. To show that this product is associative and hence a crossed product, we shall embed it in an associative quotient ring of $R * H$.

Let $\mathscr{F}^\sim$ denote the set $\{A * H | A \in \mathscr{F}\}$, which consists of nonzero ideals of $R * H$ by Lemma 1.5. Because A has zero right annihilator in R and $R*H$ is a free left R-module, it follows immediately that A has zero right annihilator in $R * H$. Hence $A * H$ also has zero right annihilator. Further, since $\mathscr{F}^\sim$ is closed under finite intersections and products (by Lemma 1.5) we may form the left quotient ring with respect to $\mathscr{F}^\sim$, which we denote by $Q^\sim$. Next we show that Q^l embeds in $Q^\sim$ by extending $f\colon {}_RA \to R$ to $f^\sim\colon A * H \to R$ as follows. Let $(1 * h)(a * 1) \in A * H = (1 * H)(A * 1)$ and define

$$(1 * h)(a * 1)f^\sim = (1 * h)(af * 1).$$

This is an additive map and obviously extends f. To see that $f^\sim$ is a left $R * H$-module homomorphism, observe that $l, h \in H$ and

$r \in R$,

$$\begin{aligned}((r * l)(1 * h)(a * 1))f^\sim &= \sum((rt(h_1, l_1) * l_2h_2)(a * 1))f^\sim \\ &= \sum((1 * l_3h_3)(\Psi_{l_2h_2}(rt(l_1, h_1))a * 1))f^\sim \\ &= \sum(1 * l_3h_3)(\Psi_{l_2h_2}(rt(l_1, h_1))af * 1) \\ &= \sum(1 * l_3h_3)(\Psi_{l_2h_2}(rt(l_1, h_1)) * 1)(af * 1) \\ &= ((r * l)(1 * h))(af * 1) \\ &= (r * l)((1 * h)(a * 1)f^\sim).\end{aligned}$$

Thus $f^\sim$ represents an element of $Q^\sim$. It follows that the map $f \to f^\sim$ induces an embedding of Q^l into $Q^\sim$. For instance to see that this embedding is one-one, suppose that $(A * H)f^\sim = 0$. This then yields $0 = (R * H)Af^\sim = (R * H)Af$ and therefore $Af = 0$. It is left to the reader to check further that the subalgebra of $Q^\sim$ generated by (the images of) Q^l and $R * H$ is isomorphic to Q^l and $R * H$ is isomorphic to $Q^l \#_t H$. Since $Q^\sim$ is an associative ring, so too is the product $Q^l \#_t H$. Thus $Q^l \#_t H = Q^l * H$ is a crossed product.

Now assume that t is invertible. Let $q \in Q$ and $h \in H$. It is readily checked that the action of H on Q^l defined above is given by $(h \cdot q) = \sum \gamma(h_1)q\gamma^1(h_2)$, using Lemma 1.3.

We show next that the action on Q^l restricts to an action on Q. It suffices to show that $(h \cdot q)A \in R$, for some $A \in \mathscr{F}$ (here we view Q as a subring of the left quotient ring of R). Note that with $q \in Q$, we have the inclusion $qA \subset R$, for some $A \in \mathscr{F}$. Now for all $a \in A$,

$$\begin{aligned}(h \cdot q)a &= \sum \gamma(h_1)q\gamma^1(h_2)a \\ &\in (R * H)qA(R * H) \quad (A \text{ is an invariant ideal}) \\ &\subset (R * H).\end{aligned}$$

Thus $(h \cdot q)A \subset (R * H) \cap Q = R$, as desired.

Finally we come to the uniqueness claims. Let $q \in Q$ be represented by the map $f\colon {}_RA \to R$ and let $a \in A$. By Lemma 2.1(ii),

$$a(h \cdot q) = \sum h_2 \cdot [(\Psi_{h_1} a)q].$$

Since this holds for all $a \in A$, and the right-hand side depends only on the values of f on A and the action on R, it follows from standard quotient ring properties that any extension to an action of H on Q (and Q^l) is uniquely determined by the fully invertible action on R. □

Inner actions. Let Q be any algebra containing R. A weak action of H on R is said to be *inner on* Q or Q*-inner* if there exists an invertible $u \in \mathrm{Hom}_k(H, Q)$ with

$$h \cdot r = \sum u^{-1}(h_1)ru(h_2), \quad \text{for all } r \in R,\, h \in H.$$

In this paper we shall assume that $u(1) = 1$. This is done without loss because any inner action can be implemented by such a map u [**BCM**, Lemma 1.13].

Examples of results described by the term "Noether-Skolem", where actions are forced to be inner, are contained in [**OQ**, **BM**, **S1**].

The following lemma is proved when t is trivial in [**C**], and extends [**Ch**, Proposition 8]. The proof is essentially the same.

LEMMA 2.4. *Let $R * H$ be a crossed product with invertible cocycle and fully invertible action. Suppose R is an H-prime ring and let Q be its symmetric quotient ring. If H is Q-inner, then Q is a centrally closed prime ring. In particular the center of Q is a field.*

Proof. Let M be the left quotient ring of the prime ring Q. We adopt the notation in the definition of inner action above. Notice that the natural extension to an action on M given by

$$h \cdot s = \sum u^{-1}(h_1)su(h_2), \qquad s \in M$$

extends the weak action of H on R. This action is obviously M-inner; thus H acts trivially on the extended center D of Q.

Let $z \in D$. Then for some nonzero ideal $I \subset Q$, $Iz \subset Q$ and thus for some $q \in Q$, we have $0 \neq qz \in Q$. Thus we see that there is an H-invariant ideal $A \subset R$ with $Aqz \subset R$ and Aq is nonzero. Since H is trivial on D and D is central, we can set

$$B = \{r \in R | rz \in R\},$$

and it follows that B is a nonzero H-invariant ideal of R with $Bz \subset R$. Let c be the element of the center C of Q represented by the R-R bimodule map $B \to R$ defined to be multiplication by z. Finally, as c and z both centralize Q, we obtain $BQ(c - z) = 0$, whence $c = z \in C$. □

3. Minimal primes and semiprime crossed products. In this section we develop prime ideal correspondences extending results in [**C**]. The ideal maps are then applied to obtain conditions for crossed products to be semiprime and prime.

For any crossed product $R * H$, let

$$\mathrm{Spec}_0(R * H) = \{P \in \mathrm{Spec}(R * H) | P \cap S = 0\}.$$

THEOREM 3.1. *Suppose that $R*H$ is a crossed product with invertible cocycle and fully invertible action. Let R be an H-prime ring with symmetric quotient ring Q. Define the maps*

$$P \to P^U = \{\alpha \in Q * H | A\alpha B \subset P, \text{ some } A, B \in \mathscr{F}\}, \quad P \in \mathrm{Spec}_0(R * H),$$
$$I \to I^D = I \cap (R * H), \qquad I \in \mathrm{Spec}_0(Q * H).$$

Then $P^{UD} = P$ and $P^U \in \mathrm{Spec}_0(Q*H)$; thus $\mathrm{Spec}_0(R*H)$ embeds in $\mathrm{Spec}_0(Q * H)$ via the inclusion-preserving map ^{U}.

Proof. Let P be as in the statement. We proceed in a series of steps.

P^U is an ideal of $Q * H$: it is clear that it is an additive subgroup. Let $\alpha \in P^U$ and $\beta \in Q * H$. By the definition of ^{U}, there are ideals $A, B \in \mathscr{F}$ with $A\alpha B \subset P$. And by basic quotient ring properties, there exists $A' \in \mathscr{F}$ with both $A'\beta$ and $\beta A'$ contained in $R * H$. Now

$$A\alpha\beta(A'B) \subset A\alpha(R * H)B \subset A\alpha B(R * H) \subset P.$$

Thus $\alpha\beta \in P^U$ since $A'B \in \mathscr{F}$. Thus P^U is a right ideal. A parallel argument works on the left.

$P^{UD} = P$: Let $\alpha \in P$. Then since $R\alpha R \subset P$, we have $\alpha \in P^U$ and obviously $\alpha \in R * H$. Thus $\alpha \in P^{UD}$. For the other inclusion, let $\delta \in P^{UD}$ and let $A, B \in \mathscr{F}$ with $A\delta B \subset P$. Observe that

$$(A * H)\delta(B * H) = (R * H)A\delta B(R * H) \subset P,$$

by Lemma 1.5. Since P is prime and the ideals $(A * H)$ and $(B * H)$ both have nonzero intersection with R, we conclude that $\delta \in P$.

$P^U \in \mathrm{Spec}_0(Q * H)$: Suppose that I_1 and I_2 are ideals of $Q * H$ with $I_1 I_2 \subset P^U$. Then

$$\begin{aligned} I_1^D I_2^D &= (I_1 \cap (R * H))(I_2 \cap (R * H)) \\ &\subset I_1 I_2 \cap (R * H) \subset P^{UD} = P. \end{aligned}$$

Since P is prime we have (say) $I^D = I_1^D \subset P$. Let $\alpha \in I = I_1$. Note that since α has coefficients in Q, $A\alpha \subset I \cap (R * H)$ for some $A \in \mathscr{F}$. Thus $\alpha \in I^D \subset P$ and by the definition of ^{U}, we obtain $\alpha \in P^U$. Hence $I \subset P^U$. We have shown that P^U is prime.

To finish the proof of the theorem, observe that

$$\begin{aligned}(P^U \cap Q) \cap R \subset P^U \cap (R * H) \cap Q \\ \subset P^{UD} \cap R \subset P \cap R = 0.\end{aligned}$$

Since every nonzero ideal of $Q * H$ meets R nontrivially, it follows immediately that $P^U \cap Q = 0$. □

THEOREM 3.2 [**Ch**, Theorem 17]. *Let Q be a centrally closed prime ring with center C. Let E be a C-algebra. Then $\operatorname{Spec}_0(Q \otimes_C E)$ is in bijection with $\operatorname{Spec} E$ via the inclusion-preserving maps*

$$\begin{aligned}P &\to P \otimes E, \qquad P \in \operatorname{Spec}_0(Q \otimes E), \\ L &\to Q \otimes E, \qquad L \in \operatorname{Spec}(E).\end{aligned}$$ □

COROLLARY 3.3. *Let $R * H$ be a crossed product over the H-prime ring R, with invertible cocycle and fully invertible Q-inner action. Then $\operatorname{Spec}_0(R*H)$ embeds in $\operatorname{Spec}(C_\tau[H])$ via an inclusion-preserving map, where $C_\tau[H]$ is a crossed product with trivial action ("twisted product") and C is a field.*

Proof. By [**BCM**, Theorem 5.3(5)] we have $Q*H \cong Q_\tau[H]$ for some cocycle τ. Since $u(1) = 1$ (where u implements the inner action), the isomorphism $(q * h \to \sum qu(h_1) * h_2)$ restrict to the identity map on Q. Further, [**BCM**, Example 4.10] yields $\tau(H \otimes H) \subset C$, where $C = Z(Q)$ is a field by Lemma 2.4. Thus $Q_\tau[H] = Q \otimes_C C_\tau[H]$. The result now follows by composing the maps in the previous two theorems. □

Next we sharpen Theorem 3.1 when H is finite dimensional and the action is inner.

COROLLARY 3.4. *Suppose that $R * H$ is a crossed product with invertible cocycle and that H has a fully invertible Q-inner action on the H-prime ring R. If H is finite dimensional, the maps defined in Theorem 3.1 satisfy $P^{UD} = P$ and $I^{DU} = I$.*

Moreover these maps give an inclusion-preserving bijection between $\operatorname{Spec}_0(Q * H)$ and $\operatorname{Spec}_0(R * H)$.

Proof. This proof is essentially the same as [**Ch**, Theorem 20]. We need to show that $I = I^{DU}$ and that $I^D \in \operatorname{Spec}_0(R * H)$.

By Zorn's Lemma there exists an ideal J of $R*H$ maximal subject to the condition $I^D \subset J$ and $J \cap R = 0$. Using the fact that R is H-prime, it follows easily that J is a prime ideal.

We claim that $J^U = I$. Let $\alpha \in I$, and let $A \in \mathscr{F}$ be such that $A\alpha \subset R * H$. Then $A\alpha \subset I \cap (R * H)$ and therefore $\alpha \in I^{DU}$. Thus $I \subset I^{UD} \subset J^U$. By Theorem 3.1, $I \subset J^U$ is an inclusion in $\mathrm{Spec}_0(Q * H)$. Furthermore the previous corollary says that the chain corresponds to a chain of primes of $C_\tau[H]$, an artinian algebra; hence the inclusion cannot be proper. Thus the claim follows.

Finally by Theorem 3.1 we have $J = J^{UD} = I^D$, so $I^D \cap R = 0$ and I^D is prime. Thus $I^D \in \mathrm{Spec}_0(R * H)$ and $I = J^U = I^{DU}$. □

Assembling some of the foregoing information we have

COROLLARY 3.5. *Let $R * H$ be given with a fully invertible Q-inner action and invertible cocycle, where R is an H-prime ring and H is finite dimensional. Then $\mathrm{Spec}_0(R * H)$ is an inclusion-preserving bijection with $\mathrm{Spec}\, E$, where $E = C_\tau[H]$ is a twisted product where H acts trivially, and C, the center of Q, is a field. Explicitly, the bijection is given by the maps:*

$$P \to P^U \cap E, \qquad P \in \mathrm{Spec}_0(Q \otimes E),$$
$$L \to (Q \otimes L) \cap (R * H), \qquad L \in \mathrm{Spec}(E).$$

Proof. This follows by composing of the bijections given in Theorem 3.2 and Corollary 3.4. □

We conclude with some applications of the correspondences above. Parts (iii) and (iv) in the next result generalize [**M**, Theorem 7.1] when H is finite dimensional.

THEOREM 3.6. *Let $R * H$ be a crossed product with a fully invertible action and invertible cocycle, where R is H-prime, H is Q-inner and H is finite dimensional. Then $R * H$ has only finitely many minimal primes $P_1, \dots, P_n$, which satisfy*

(i) $\{P_1, \dots, P_n\} = \mathrm{Spec}_0(R * H)$.

(ii) $\bigcap P_i$ *is the unique largest nilpotent ideal of* $R * H$.

Furthermore:

(iii) *Let E be as in the previous corollary. $R * H$ is semiprime iff $Q * H$ is semiprime iff E is semiprime.*

(iv) *H is prime iff $Q * H$ is prime iff E is prime.*

Proof. As E is a finite dimensional algebra over C, it has finitely many (minimal) primes, say $L_1, \dots, L_n$. It is elementary that $\bigcap L_i$ is a nilpotent ideal. By the previous corollary, $R * H$ has finitely many minimal primes $P_1, \ldots, P_n \in \operatorname{Spec}_0(R * H)$ where $P_i = (Q \otimes L_i) \cap (R * H)$, and $P_i \cap R = 0$ for all i. Note that by the correspondence between $\operatorname{Spec} E$ and $\operatorname{Spec}_0(R*H)$, each P_i is a minimal prime. Now we check that

$$\begin{aligned}\left(\bigcap P_i\right)^n &\subset \bigcap (Q \otimes L_i)^n \cap (R * H) \\ &= \left(Q \otimes \left(\bigcap L_i\right)^n\right) \cap (R * H) = 0\end{aligned}$$

for some n. Thus if P is a minimal prime of $R*H$, $P = P_i$ for some i. This shows that the P_i are precisely the minimal primes of $R*H$, proving (i). Finally, being an intersection of primes, $\bigcap P_i$ contains every nilpotent ideal, proving (ii).

Now we prove (iii). With notation as above, we have by elementary linear algebra that $\bigcap (Q \otimes L_i) = (Q \otimes \bigcap L_i)$. Also by Theorem 3.2, together with (i) applied to the crossed product $Q * H$, we find that $\operatorname{Spec}_0(Q * H) = \{Q \otimes L_i\}$ is the set of minimal primes of $Q * H$. Therefore E is semiprime iff $Q * H$ is.

By (i), $\operatorname{Spec}_0(R * H) = \{(Q \otimes L_i) \cap (R * H)\}$ is the set of minimal primes of $R * H$. Therefore $R * H$ is semiprime if $Q * H$ is. Conversely, using the fact that nonzero ideals of $Q*H$ meet $R*H$ nontrivially, we deduce that $R*H$ semiprime implies $Q*H$ semiprime.

One uses the prime correspondences similarly to obtain (iv). This completes the proof of the Theorem. □

Our last result adds to known criteria for smash products to be semiprime (see [**BCM**, §6]). The proof relies on a recent result of Blattner and Montgomery which states that a crossed product $R * H$ is semiprime provided R is semiprime, H is finite dimensional semisimple, the cocycle is invertible and the action is R-inner.

COROLLARY 3.7. *Let $R * H$ be a crossed product with invertible cocycle and fully invertible action, where R is H-prime and H is Q-inner. Assume that H is finite dimensional and semisimple. Then $R * H$ is semiprime.*

Proof. Let $u \in \operatorname{Hom}(H, R)$ implement the inner action of H on Q. As in [**BCM**, p. 698], a cocycle τ is defined by

$$\tau(h, l) = \sum u^{-1}(l_1) u^{-1}(h_1) t(h_2, l_2) u(h_3 l_3),$$

and there it is shown that $Q * H \cong Q_\tau[H]$. Further, as in the proof of Corollary 3.3, we obtain $Q * H \cong Q \otimes_C C_\tau[H]$. Since u and t are both invertible, we see that τ^{-1} exists with

$$\tau^{-1}(h, l) = \sum u^{-1}(h_1 l_1) t^{-1}(h_2, l_2) u(h_3) u(l_3).$$

Thus, since τ is invertible, $C_\tau[H]$ is semiprime by [**BM**, Theorem 2.7]. Now the theorem applies to finish the proof. □

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