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ON LIPSCHITZ STABILITY FOR F. D. E.

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Fozi M. Dannan and Saber Elaydi presented Lipschitz stability of O. D. E., and made a comparison between Lipschitz stability and Liapunov stability. In this paper, we will extend the concept of Lipschitz stability to the systems of functional differential equations (F.D.E.), and give some criteria via Liapunov's second method.

1. Definitions. We consider the system

$$x'(t) = f(t, x_t), \tag{1.1}$$

where $x \in R^n$, $f: R \times C([-r, 0], R^n) \mapsto R^n$, $f(t, 0) = 0$, f is continuous, $x_t = x(t + \theta)$, $\theta \in [-r, 0]$, $r > 0$. The initial value condition associated with (1.1) is

$$x(\theta) = \phi(\theta), \quad \theta \in [-r, 0], \quad \phi(\theta) \in C([-r, 0], R^n). \tag{1.2}$$

Set $\|\phi\| = \sup_{\theta\in[-r,0]} |\phi(\theta)|$, where $|\cdot|$ is a norm in R^n. We always assume that the solution of (1.1) with (1.2) is existent and unique.

DEFINITION 1. For the solution $x(t)$ of (1.1) through (t_0, ϕ), (see [2, p. 38]), $(t_0, \phi) \in R^+ \times C([-r, 0], R^n)$, $R^+ \overset{\text{def}}{=} [0, +\infty)$, if there exists a constant $\delta > 0$, which is independent of t_0, and another constant $M = M(\delta) > 0$, such that

$$|x_t| \le M\|\phi\|, \quad \text{for } t \ge t_0, \text{ and } \|\phi\| < \delta, \tag{1.3}$$

then the zero solution of (1.1) is said to be Lipschitz uniformly stable. This is denoted by $(1.1) \in \text{Lip. U. S.}$

DEFINITION 2. If in Definition 1, δ is allowed to be $+\infty$, then the zero solution of (1.1) is said to be Lipschitz globally uniformly stable. This is denoted by $(1.1) \in \text{Lip. G. U. S.}$

Obviously, if $r = 0$, each definition above reduces to a definition for O.D.E.

If on $[t_0, T]$, where T is large enough, the solution of (1.1) through (t_0, ϕ) satisfies Definitions 1 and 2, it is said to be Lipschitz uniformly or globally uniformly stable on the large interval $[t_0, T]$.

2. Main results.

THEOREM 1. *For linear F.D.E.*

$$x'(t) = L(t, x_t), \tag{2.1}$$

where L *is a linear operator, the Lipschitz uniform stability of the zero solution is equivalent to Liapunov uniform stability of the zero solution.*

Proof. If the zero solution (2.1) is Liapunov uniformly stable, from [2, p. 163], there exists a linear operator $T(t, t_0)$, such that the solution of (2.1) through (t_0, ϕ) can be represented by

$$x_t(t_0, \phi) = T(t, t_0)\phi,$$

and there exists a constant $M > 0$, such that

$$|T(t, t_0)| \leq M, \qquad \text{for } t \geq t_0.$$

This implies that the zero solution of (2.1) is Lipschitz uniformly stable.

On the other hand, it follows from Definition 1 that Lipschitz uniform stability implies Liapunov uniform stability.

The proof is complete.

THEOREM 2. *If there exists a continuous functional* $V(t, \psi) \geq 0$, $(t, \psi) \in [t_0, +\infty) \times C([-r, 0], R^n)$, *for which*:

(i) *There exist nondecreasing continuous nonzero functions* u, v, $u(0) = 0$, $v(0) = 0$, *and* $v(s) \leq u(Ms)$ *for all* $s > 0$, *where* $M \geq 1$ *is a constant, and*

$$u(|\psi|) \leq V(t, \psi) \leq v(|\psi|), \qquad \text{for } \psi \in C([-r, 0], R^n) \text{ and } t \geq t_0.$$

(ii) *For the solution* $x(t)$ *of* (1.1) *through* (t_0, ϕ), *we suppose*

$$V'_{(1.1)}(t, x_t) \leq 0, \quad t \geq t_0,$$

where

$$V'_{(1.1)}(t, x_t) = \limsup_{h\to 0^+} \frac{1}{h}(V(t+h, x_{t+h}(t, \phi)) - V(t, \phi)).$$

Then $(1.1) \in Lip.G.U.S.$

Proof. For the solution $x(t)$ of (1.1) through (t_0, ϕ), from (ii) we have

$$V(t, x_t) \leq V(t_0, \phi), \qquad t \geq t_0.$$

From (i) we obtain

$$u(|x_t|) \leq V(t, x_t) \leq V(t_0, \phi) \leq v(|\phi|) \leq u(M|\phi|), \qquad t \geq t_0.$$

Hence, $|x_t| \leq M\|\phi\|$ holds for $t \geq t_0$ and $\|\phi\| < +\infty$. This completes the proof.

THEOREM 3. *Assume that*

(i) *There exists a functional* $V: [t_0, +\infty) \times C([-r, 0], R^n) \mapsto R^+$, *such that*

$$u(|\psi|) \leq V(t, \psi) \leq v(|\psi|), \quad \text{for } \psi \in C([-r, 0], R^n), t \geq t_0,$$

and

$$\limsup_{s \to 0^+} \frac{u^{-1}(v(s))}{s} \leq M,$$

where u, v *is defined as in Theorem* 2, *respectively, and* $M \geq 1$ *is a constant.*

(ii) *Condition* (ii) *of Theorem* 2 *is satisfied.*

Then $(1.1) \in Lip.U.S.$

THEOREM 4. *Assume that there exists a continuous function* g: $[t_0, +\infty) \times R^1 \mapsto R^1$, *and* $V: [t_0, +\infty) \times C([-r, 0], R^n) \mapsto R^+$, *for which*

(i) $V'_{(1.1)}(t, x_t) \leq g(t, V(t, x_t))$, $t \geq t_0$, *and* $h(|\psi|) \leq V(t, \psi) \leq v(|\psi|)$, $\psi \in C([-r, 0], R^n)$,

where $x(t)$ *is the solution of* (1.1) *through* (t_0, ϕ), $h(s)$, $v(s)$ *are continuous and nondecreasing nonzero functions for* $s \geq 0$, *satisfying* $h(0) = 0$, $v(0) = 0$, $V(t, 0) = 0$.

(ii) *The zero solution of comparison scalar O.D.E.*

$$u' = g(t, u), \quad (g(t, 0) = 0), \tag{2.2}$$

is Liapunov uniformly stable.

Then $(1.1) \in Lip.U.S.$

Proof. For any $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$|u(t)| \leq h(\varepsilon), \quad \text{for } |u_0| < \delta.$$

Taking $u_0 = v(\|\phi\|)$, where $\frac{\varepsilon}{N} \leq \|\phi\| \leq v^{-1}(\delta)$, $\phi \in C([-r,0],R^n)$, $N = \text{const.} \geq 1$, we find that

$$V(t, x_t) \leq u(t), \quad \text{for } t \geq t_0 \quad \text{and} \quad V(t_0, \phi) \leq v(\|\phi\|) = u_0,$$

by the theory of differential inequality.

Thus, we have

$$h(|x_t|) \le V(t, x_t) \le u(t) \le h(N\|\phi\|), \quad t \ge t_0, \frac{\varepsilon}{N} \le \|\phi\| \le v^{-1}(\delta),$$

where $x(t)$ is the solution of (1.1) through (t_0, ϕ).

The proof is complete.

3. Examples. (1) We consider a scalar model of infectious diseases

$$x'(t) = f(t, x(t)) - f(t, x(t-r)), \quad x \ge 0, \ t \in [t_0, T], \tag{3.1}$$

where T is large enough, $t_0 \ge 1$, $r > 0$, $f(t, x)$ is continuous and nonnegative, $f(t, 0) = 0$.

Assume that

$$0 \le \frac{\partial f}{\partial x} \le \frac{1}{2N}, \quad \text{for a constant } N \ge T.$$

Constructing

$$V(t, \psi) = \psi^2(0)\frac{1}{t}, \quad u(s) = s^2\frac{1}{N}, \quad v(s) = s^2,$$

we have

$$\begin{aligned} V'_{(3.1)}(t, x_t) &= \frac{-x}{t}\left(\frac{x}{t} - 2\frac{\partial f(t, \xi)}{\partial x}(x(t) - x(t-r))\right) \\ &= \frac{-x}{t}\left(\left(\frac{x}{t} - 2\frac{\partial f(t, \xi)}{\partial x}x\right) + 2\frac{\partial f(t, \xi)}{\partial x}x(t-r)\right) \end{aligned}$$

$t \in [t_0, T]$ and $\xi \in [x(t), x(t-r)]$, and

$$\frac{x}{t} - 2\frac{\partial f(t, \xi)}{\partial x}x \ge \frac{x}{t} - 2\frac{1}{2N}x \ge 0,$$

for $x \ge 0$ and $t_0 \le t \le T$, this implies that $V'_{(3.1)}(t, x_t) \le 0$.

It follows from Theorem 2 that $(3.1) \in$ Lip. G. U. S. on the large interval $[t_0, T]$.

(2) In (1.1), suppose that

$$\psi^T(0)f(t, \psi) \le K(t)\,\mathrm{Ln}(\psi^T(0)\psi(0) + 1),$$

for $\psi \in C([-r, 0], R^n), K(t) > 0, \int^{+\infty} K(t)\,dt < +\infty$.

Taking $h(|\psi|) = v(|\psi|) = V(t, \psi) = \mathrm{Ln}(\psi^T(0)\psi(0) + 1)$, we have

$$V'_{(1.1)}t, x_t) = \frac{2x^T f(t, x_t)}{(x^T x + 1)} \le 2K(t)V(t, x_t),$$

in view of that the zero solution of $y' = 2K(t)y$ is Liapunov uniformly stable, it follows from Theroem 4 that $(1.1) \in$ Lip. U. S.

(3) For scalar D.D.E.

$$x'(t) = f(t, x(t-r)), \qquad t \geq 0, \tag{3.2}$$

assume that

$$|f(t, x(t-r))| \leq \frac{g(t)}{2}|x(t-r)|, \qquad g(t) > 0, \quad \text{and}$$

$$\int_0^{+\infty} g(s)\,ds < +\infty, \qquad r > 0.$$

Constructing $u(|\psi|) = \psi^2(0)\exp(-\int_0^{+\infty} g(s)\,ds)$, $v(|\psi|) = \psi^2(0)$, and

$$V(t, \psi) = \psi^2(0)\exp\left(-\int_0^t g(s)\,ds\right) + \frac{g(t)}{2}\left(\int_{-r}^0 \psi^2(\theta)\,d\theta\right)\exp\left(-\int_0^t g(s)\,ds\right),$$

we have

$$\begin{aligned} &V'_{(3.2)}(t, x_t) \\ &= -\frac{g(t)}{2}(x^2(t) + x^2(t-r))\exp\left(-\int_0^t g(s)\,ds\right) \\ &\quad + 2x(t)f(t, x(t-r))\exp\left(-\int_0^t g(s)\,ds\right) \\ &\quad - \frac{g^2(t)}{2}\left(\int_{-r}^0 x^2(t+\theta)\,d\theta)\exp(-\int_0^t g(s)\,ds\right) \\ &\leq \frac{g(t)}{2}(-x^2(t) - x^2(t-r) + 2|x(t)||x(t-r)|) \\ &\quad \times \exp\left(-\int_0^t g(s)\,ds\right) \leq 0, \end{aligned}$$

$t \geq t_0 \geq 0$, and

$$u(|x_t|) \leq V(t, x_t) \leq V(t_0, \phi) \leq v(|\phi|) \leq u(M|\phi|),$$

where $M = \exp(\int_0^{+\infty} g(s)\,ds) > 1$.

It follows from Theroem 2 that $(3.2) \in \mathrm{Lip.G.U.S.}$

(4) For scalar D.D.E.

$$x'(t) = -4x^3(t) + 2x^3(t-r), \qquad t \geq 0, \tag{3.3}$$

constructing

$$V(\psi) = \frac{\psi^4(0)}{8} + \int_{-r}^0 \psi^6(\theta)\,d\theta, \quad u(s) = s^4, \; v(s) = s^4 + s^6 r,$$

it is easy to conclude that

$$V'_{(3.3)}(x_t) \leq 0, \quad t \geq t_0 \geq 0,$$

and

$$u(|x_t|) \leq V(x_t) \leq V(|\phi|) \leq \|\phi\|^4 + \|\phi\|^6 r = v(\|\phi\|),$$

$$\|\phi\| = \sup_{\theta \in [-r, 0]} |\phi(\theta)|,$$

$$\limsup_{s \to 0} \frac{u^{-1}(v(s))}{s} = \limsup_{s \to 0} (1 + s^2 r)^{1/4} \leq M, \text{ where } M \geq 2.$$

It follows from Theorem 3 that $(3.3) \in$ Lip. U. S.

4. Remarks. (1) Obviously, Lipschitz stability implies Liapunov stability.

(2) THEOREM 5. *If the system* $(1.1) \in$ *Lip.U.S., then the zero solution of the system*

$$y' = D_\phi f(t, x_t(t, 0, f))y_t, \tag{4.1}$$

is Lipschitz uniformly stable.

In fact, by the results of [**2**, page 46] and the definition of the derivative operator D_ϕ, we obtain that the solution of (4.1) through (t_0, ψ) is in the form of

$$y = D_\phi x(t_0, 0, f)\psi = x(t_0, \psi, f) - x(t_0, 0, f) - w(0, \psi),$$

where

$$\psi \in C([-r, 0], R^n), \quad t_0 \geq 0, \quad x(t_0, 0, f) = 0,$$

$$\lim_{\|\psi\| \to 0} \frac{\|w(0, \psi)\|}{\|\psi\|} = 0.$$

Hence, there exists a constant $\eta > 0$, such that $D_\phi x(t_0, 0, f)$ is uniformly bounded, whenever $\|\psi\| < \eta$. This implies that $(4.1) \in$ Lip. U. S.

Theorem 5 means that $(4.1) \in$ Lip. U. S. is a necessary condition for $(1.1) \in$ Lip. U. S. We can conclude that Lipschitz uniform stability is not equivalent to Liapunov uniform stability for nonlinear systems [**1**].

References

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