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SOME REMARKS ON ORDERINGS UNDER FINITE FIELD EXTENSIONS

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Let X_K denote the space of orderings of a field K, and $r_{L/K}: X_L \to X_K$ the restriction mapping, when L/K is a field extension. Fixing K, the image sets $r_{L/K}(X_L)$ for finite extensions L/K are investigated. If K is hilbertian, any clopen subset $U \subset X_K$ has the form $U = r_{L/K}(X_L)$ for some finite L/K, and $[L:K]$ can be bounded in terms of U. This bound is even sharp in some cases, but not always. A second construction gives the same qualitative result for a much larger class of fields. It is based on iterated quadratic extensions. The bounds on $[L:K]$ obtained here are weaker than in the hilbertian case.

Let K be a field, and let X_K be the (topological) space of its orderings. It is known to be compact and totally disconnected. If L/K is a finitely generated field extension, then Elman, Lam and Wadsworth showed that the natural restriction mapping $r = r_{L/K}: X_L \to X_K$ is (not only closed but also) open [**ELW**, Theorem 4.9]. In particular, the set $r_{L/K}(X_L)$ of those orderings of K which extend to L is clopen (:= closed and open) in X_K. This means that it is a union of finitely many *basic* clopen subsets, i.e. sets of the form

$$X_K(a_1, \dots, a_t) := \{x \in X_K : a_1, \dots, a_t \text{ are non-negative in } x\}$$

with $a_i \in K$. Conversely, given a clopen subset U of X_K, it is not hard to find explicitly a finitely generated extension L/K such that $U = r_{L/K}(X_L)$. For example, if the complement of U is presented as

$$X_K \backslash U = \bigcup_{i=1}^{s} X_K(a_1^i, \dots, a_{t_i}^i),$$

then one may take $L = K(\phi_1, \dots, \phi_s)$ where ϕ_i is the Pfister form $\langle 1, a_1^i\rangle \otimes \cdots \otimes \langle 1, a_{t_i}^i\rangle$ [**ELW**, Theorem 4.18].

The question becomes somewhat harder when one tries to realize U by a *finite* extension L/K. In fact, this is not always possible [**ELW**, §5]. On the other hand, Prestel has shown [**Pr**, p. 904] that it is possible if the field K is hilbertian. In fact, this is merely a corollary to a much stronger theorem by Prestel and Bröcker which essentially states that the set of trace forms of finite extensions of a (fixed) hilbertian field K is closed under addition in the Witt ring WK ([**Pr**]; see also [**K**, Kapitel 1(b)] and [**KS**]).

Prestel's proof, however, gives little hint about the possible degree $[L:K]$ of such an L (which one would like to keep bounded in terms of the prescribed set $U \subset X_K$). Such information is provided implicitly, in the special case of function fields over a real closed field R, by work of Andradas and Gamboa [**AG2**]. Here, too, the field-theoretic statement is a corollary to a more general theorem, in this case a geometric one, about realizing closed semi-algebraic sets as images of finite morphisms between irreducible R-varieties.

This little note has two aims. First, we obtain a quantitative version of Prestel's corollary for hilbertian fields, i.e. for any clopen subset $U \subset X_K$ a bound is given for the degree of a finite extension L/K which realizes U. These bounds are even best possible in some cases, but not always. Second, we present a different approach, based on the construction of iterated quadratic extensions. It yields weaker bounds for $[L:K]$, but has the advantage of applying to a larger class of fields than only the hilbertian ones. Besides it is fairly constructive, and could probably be used for an algorithmic procedure to find L/K (for a given $U \subset X_K$), e.g. in the case of function fields over some base field.

After this work was done, I learned about recent work by D. Pecker [**Pe1–Pe3**] in which he improves the results of Andradas and Gamboa about real varieties. Part of the construction in [**Pe3**] has some similarity to ours in Lemma 1. It is interesting that he also arrives at similar bounds in the "geometric case".

I would like to thank J. Königsmann and M. Krüskemper for their helpful comments and suggestions on this subject.

The essential observation for our quantitative version of Prestel's corollary is the following lemma. The construction is inspired by ideas of Andradas and Gamboa [**AG1, AG2**]:

LEMMA 1. *For every* $n \geq 1$ *there is* $N \geq 1$ *and a polynomial* $f_n = f_n(t; x, y)$ *with integer coefficients (where* $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_N)$*) having the following properties*:

(1) f_n *is monic of degree* $2n$ *with respect to* t;

(2) *for any field* K *and any sequence* $a = (a_1, \dots, a_n)$ *of pairwise distinct non-zero elements of* K, $f_n(t; a, y)$ *is irreducible in* $K[t, y]$;

(3) *for any real closed field* R *and any* $a \in R^n$, $b \in R^N$, $f_n(t; a, b)$ *has a root in* R *if and only if* $a_i \geq 0$ *for some* $1 \leq i \leq n$.

The lemma implies

PROPOSITION 1. *Let* K *be a hilbertian field. Let* $U \subset X_K$ *be a clopen subset, and fix some presentation*

$$U = \bigcap_{i=1}^{m} \bigcup_{j=1}^{n_i} X_K(a_j^i)$$

with $a_j^i \in K$. *Then there is a field extension* $L \supset K$ *with* $r_{L/K}(X_L) = U$ *and* $[L:K] = 2^m n_1 \cdots n_m$.

Proof by induction on m. We may assume that the a_j^i are non-zero and pairwise distinct. First let $n = 1$, write $n := n_1$, $a_j := a_j^1$. Since $f_n(t; a, y)$ is irreducible in $K[t, y]$ ($y = (y_1, \ldots, y_N)$ as in the lemma), there is $b \in K^N$ such that $f_n(t; a, b)$ remains irreducible, by the hilbertian property of K. Let $L = K(\tau)$, where τ is a root of $f_n(t; a, b)$. From property (3) of f_n it follows that $r_{L/K}(X_L) = X_K(a_1) \cup \cdots \cup X_K(a_n) = U$; moreover, $[L:K] = 2n$. Passing to the general case now, we may assume there is an extension F/K such that

$$r_{F/K}(X_F) = \bigcap_{i=1}^{m-1} \bigcup_{j=1}^{n_i} X_K(a_j^i)$$

and $[F:K] = 2^{m-1} n_1 \cdots n_{m-1}$. Since also F is hilbertian [**FJ**, Prop. 11.11], we find L/F of degree $2n_m$ with

$$r_{L/F}(X_L) = \bigcup_{j=1}^{n_m} X_F(a_j^m).$$

Thus, $[L:K] = 2^m n_1 \cdots n_m$. Moreover,

$$\begin{aligned} r_{L/K}(X_L) = r_{F/K}(r_{L/F}(X_L)) &= r_{F/K}\left(\bigcup_{j=1}^{n_m} X_F(a_j^m)\right) \\ &= r_{F/K}(X_F) \cap \bigcup_{j=1}^{n_m} X_F(a_j^m) = U, \end{aligned}$$

as desired. □

It remains to prove the lemma. First let $n \geq 2$, and consider the rational function

$$r_n(t; x, y) := 1 - \sum_{i=1}^{n} (1 + y_i^2) \frac{x_i}{t^2 - x_i}$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. Let R be a real closed field and $a_1, \dots, a_n \in R^*$, $b \in R^n$. Then one of $a_1, \dots, a_n$ is positive iff there is $t \in R$ for which $r_n(t; a, b)$ is defined and $r_n(t; a, b) = 0$. Indeed, if all of the a_i's are negative, r_n takes strictly positive values everywhere. If $a_i > 0$, then r_n has a simple pole in $t = +\sqrt{a_i}$ and jumps from $+\infty$ to $-\infty$ there. Moreover, $\lim_{t\to\infty} r_n(t; a, b) = 1$. Hence, if a_i is the largest of the a_j's and is positive, r_n must have a zero $t > \sqrt{a_i}$, by the Mean Value Theorem.

Clearing denominators, define f_n to be the polynomial

$$f_n(t; x, y) := r_n(t; x, y) \cdot \prod_{i=1}^{n} (t^2 - x_i).$$

Then (1) clearly holds, and (3) follows from what has just been said. Concerning (2), observe that one can write

$$f_n(t; a, y) = \alpha(t) + \sum_{i=1}^{n} \beta_i(t) y_i^2$$

with non-zero polynomials $\alpha, \beta_1, \dots, \beta_n \in K[t]$. As a polynomial over $K(t)$ in the variables $y_1, \dots, y_n$, this is clearly irreducible. Thus, if $f_n(t; a, y)$ were reducible in $K[t, y]$, the polynomials α, $\beta_1, \dots, \beta_n$ would have to have a non-trivial common divisor. But since

$$\beta_i(t) = -a_i \prod_{j \neq i} (t^2 - a_j),$$

the assumption on the a_i's shows that this is not the case. So $f_n(t; a, y)$ is irreducible.

In the case $n = 1$ one has to modify the construction slightly. For example, take $N = 2$ and

$$f_1(t; x, y_1, y_2) := t^2 - x(1 + y_1^2 + y_2^2). \qquad \square$$

Bröcker [**B**] has shown that there is a function $t\colon \mathbf{N} \cup \{0\} \to \mathbf{N}$ such that, whenever K is a field of stability index $n < \infty$, any clopen subset U of X_K is a union of at most $t(n)$ basic clopen subspaces. For example, $t(1) = 1$, $t(2) = 2$, $t(3) \leq 8008$. Passing to complements, Proposition 1 gives us

COROLLARY. *Let K be a hilbertian field of finite stability index $n \geq 1$. Then for any clopen subset U of X_K there is an extension L/K with $r_{L/K}(X_L) = U$ and $[L:K] \leq (2n)^{t(n)}$.* □

It it interesting to note that, at least in certain cases, the bound on $[L:K]$ provided by Proposition 1 is best possible. For example, this is true if $m = 1$ or $n_i = 1$ for all i, i.e. if U or its complement is a *basic* clopen subspace of X_K:

PROPOSITION 2. *Let K be any field and $L \supset K$ a proper finite extension. Suppose there are $a_1, \dots, a_n \in K^*$ with*
(a) $r_{L/K}(X_L) = X_K(a_1) \cup \dots \cup X_K(a_n)$, *resp.*
(b) $r_{L/K}(X_L) = X_K(a_1, \dots, a_n)$,
and that no presentation of the same type is possible with less than n elements. Then (a) $[L:K] \geq 2n$, *resp.* (b) $[L:K] \geq 2^n$.

Proof. We may assume $r_{L/K}(X_L) \neq X_K$. Let $d = [L:K]$, and let τ be the trace form of L/K. So τ is a d-dimensional quadratic form over K with everywhere non-negative signature, and

$$r_{L/K}(X_L) = \{x \in X_K \colon \operatorname{sign}_x(\tau) > 0\}$$

[**S**, Theorem 3.4.4]. We have to assume a little bit of real algebra, namely the notion of a fan in a field and some of its properties. For this, one may consult [**L**].

(a) Let Z be the basic clopen subspace

$$X_K \backslash r_{L/K}(X_L) = X_K(-a_1, \dots, -a_n) = \{x \in X_K \colon \operatorname{sign}_x(\tau) = 0\}$$

of X_K. It is well known that there is a fan $Y \subset X_K$ with $\#Y = 2^n$ such that $Y \cap Z$ contains exactly one element, x say. (Compare [**Sch**, Lemma 2.1].) Diagonalizing τ we can assume (writing $d = 2m$)

$$\tau = \langle b_1, \dots, b_m, c_1, \dots, c_m \rangle$$

where the b_i are positive in x and the c_i are negative in x. If $m < n$, then

$$X_K(-c_1, \dots, -c_m) \cap Y,$$

being non-empty, would contain at least one element y different from x. But then necessarily also $\operatorname{sign}_y(\tau) = 0$, contradicting $\#(Y \cap Z) = 1$. Hence $m \geq n$.

(b) Let $Z = r_{L/K}(X_L)$. As before, there is a 2^n-element fan $Y \subset X_K$ such that $Y \cap Z$ contains exactly one element x. Since 2^n divides

$$\sum_{y \in Y} \operatorname{sign}_y(\tau) = \operatorname{sign}_x(\tau) > 0,$$

it follows that $d = \dim \tau \geq 2^n$. □

However, there are certainly cases when the bound of Proposition 1 is not best possible. For example, consider an extension L/K of degree 4 such that neither $U := r_{L/K}(X_L)$ nor its complement is basic in X_K. Then in any presentation $U = \bigcap_{i=1}^m \bigcup_{j=1}^{n_i} X_K(a_j^i)$ one must have $m \geq 2$ and $n_i \geq 2$ for at least one i; hence $2^m n_1 \cdots n_m \geq 8$. As an example of such an extension take $K = \mathbf{R}(x, y)$, the 2-dimensional rational function field over the reals, and $L = K(\sqrt{y(1+\sqrt{x})})$; this gives $U = r_{L/K}(X_L) = X_K(x, y) \cup X_K(x-1)$.

We now start out for a different approach to the problem of realizing a given clopen set in X_K by a finite extension of K. It will be based only on iterated quadratic extensions.

Recall that a subset of X_K is called basic clopen if it is of the form $X_K(a_1, \ldots, a_r)$ for suitable $r \geq 1$ and $a_i \in K$. The following principle has already been used before.

LEMMA 2. *Let K be a field. Assume that for any finite extension F/K and any basic clopen subset Y of X_F there is a finite extension E/F such that $r_{E/F}(X_E) = X_F \backslash Y$. Then for any finite extension F/K and any clopen subset U of X_F there is a finite extension E/F with $r_{E/F}(X_E) = U$.*

Proof. Write $X_F \backslash U = Y_1 \cup \cdots \cup Y_t$, where the Y_i are basic clopen in X_F. Induction on t, the case $t = 1$ being settled by hypothesis. There is E'/F finite with $r_{E'/F}(X_{E'}) = X_F \backslash Y_1$. Let $Z_i := (r_{E'/F})^{-1}(Y_i)$, a basic clopen subset of $X_{E'}$ $(i = 1, \ldots, t)$, and $V := (r_{E'/F})^{-1}(U)$. Then $X_{E'} \backslash V = Z_2 \cup \cdots \cup Z_t$. By induction hypothesis we find E/E' finite with $r_{E/E'}(X_E) = V$. Since $r_{E'/F}(V) = U$, we get $r_{E/F}(X_E) = U$. □

LEMMA 3. *Assume that for any finite extension F/K of K and any $a, b \in F^*$ there is a finite extension E/F and $\lambda \in E^*$ such that $r_{E/F}\colon X_E \to X_F$ is surjective and $r_{E/F}(X_E(\lambda)) = X_F(a) \cup X_F(b)$. Then K satisfies the assumption (and hence the conclusion) of Lemma 2.*

Proof. Given F/K finite and a basic clopen $Y \subset X_F$ we have to find E/F finite with $r_{E/F}(X_E) = X_F \backslash Y$. Write $Y = X_F(a_1, \dots, a_s)$ with $a_i \in F^*$. If $s = 1$, take $E = F(\sqrt{-a_1})$. If $s > 1$, there is, by hypothesis, E'/F finite and $\lambda \in E'^*$ such that $r_{E'/F}$ is surjective and $r_{E'/F}(X_{E'}(\lambda)) = X_F(-a_1) \cup X_F(-a_2)$. Putting $Z := X_{E'}(-\lambda, a_3, \dots, a_s)$ we get

$$r_{E'/F}(X_{E'} \backslash Z) = \bigcup_{i=1}^{s} X_F(-a_i) = X_F \backslash Y.$$

So the claim follows by induction on s. □

Let $L = K(\sqrt{d})$ be a quadratic extension of K. If $\lambda = a + b\sqrt{d} \in L^*$ with $a, b \in K$, then

$$r_{L/K}(X_L(\lambda)) = r_{L/K}(X_L) \cap (\{a \geq 0\} \cup \{a^2 - b^2 d < 0\}),$$

i.e.

$$r_{L/K}(X_L(\lambda)) = X_K(d) \cap (X_K(\mathrm{tr}(\lambda)) \cup X_K(-\mathrm{N}(\lambda))),$$

tr and N denoting trace and norm of L/K. Writing $\sum K^2$ for the set of sums of squares in K and $\sum K^{*2} = \sum K^2 \backslash \{0\}$, we see:

LEMMA 4. *Let $a, b \in K$ such that $a^2 - 4b \in \sum K^{*2}$, but is not a square. Then $L = K(\sqrt{a^2 - 4b})$ is a quadratic extension of K for which $r_{L/K}$ is surjective and for which there is $\lambda \in L^*$ with*

$$r_{L/K}(X_L(\lambda)) = X_K(-a) \cup X_K(-b).$$

Proof. Take for λ a root of $T^2 + aT + b$. □

Conversely, let $a, b \in K^*$ be given. What does it mean to find $a', b' \in K^*$ such that $a'^2 - 4b' \in \sum K^{*2} \backslash K^{*2}$ and

$$X_K(-a) \cup X_K(-b) = X_K(-a') \cup X_K(-b')?$$

Putting $b' := \frac{1}{4} a^2 b (1 + b^2)^{-1}$, one has $X_K(b) = X_K(b')$, and

$$a^2 - 4b' = a^2 \left(1 - \frac{b}{1 + b^2}\right)$$

is a sum of squares since it is positive under every ordering (Artin's Theorem). If it is still a square, say $a^2 - 4b' = c^2$, put $b'' := b'(1 + c^2 t)$ with $t \in \sum K^2$. Then $X_K(b) = X_K(b'')$ and

$$a^2 - 4b'' = a^2 - 4b' - 4b'c^2 t = c^2(1 - 4b't).$$

So it suffices to find $t \in \sum K^2$ with $1 - 4b't \in \sum K^{*2} \backslash K^{*2}$. For ease of reference, let us single out this property of a field K:

($*$) Either K is non-real, or for every $u \in K^*$ there exists $t \in \sum K^2$ with $1 - tu \in \sum K^{*2} \backslash K^{*2}$.

By an *iterated quadratic extension* of K we mean an extension F/K for which there is a finite chain

$$K = F_0 \subset F_1 \subset \cdots \subset F_n = F$$

of intermediate fields with $[F_i : F_{i-1}] = 2$ for all i. Summarizing the above discussion, we get

PROPOSITION 3. *Assume K is a field such that every iterated quadratic extension of K satisfies* ($*$). *Then*

(a) *for any clopen $U \subset X_K$ there is an iterated quadratic extension F/K with $r_{F/K}(X_F) = U$;*

(b) *if $U = \bigcap_{i=1}^{n} \bigcup_{j=1}^{m_i} X_K(a_j^i)$ with $a_j^i \in K$, then there is such an F with $[F : K] = 2^{m_1 + \cdots + m_n}$.*

Proof. The statement about $[F : K]$ follows from the construction of F using Lemmas 2–4. □

EXAMPLES. 1. Every hilbertian field K satisfies ($*$). To see this, let $u \in K^*$ and consider

$$p(x, y) := y^2 - \left(1 - \frac{x^2 u}{1 + x^4 u^2}\right) \in K(x)[y].$$

Then $p(x, y)$ is irreducible over $K(x)$, so there is $a \in K^*$ with $1 + a^4 u^2 \neq 0$ such that

$$1 - \frac{a^2}{1 + a^4 u^2} \cdot u$$

is not a square in K. But this element is positive under every ordering of K.

Since finite extensions of hilbertian fields are hilbertian, every hilbertian field meets the hypotheses of Proposition 3. But observe that the bound given there is generally weaker than that obtained in Proposition 1.

2. Another class of examples is provided by the next result. An ordered abelian group Γ is said to be *n-regular* (where $n \geq 1$ is a given integer) if $S \cap n\Gamma \neq \varnothing$ for every infinite convex subset S of Γ. This notion is due to E. Zakon [Z], who proved together with A. Robinson that Γ is n-regular for all $n \geq 1$ precisely iff Γ satisfies the model theory of all archimedean ordered groups. (The latter

condition means that any sentence which holds in every archimedean ordered group also holds in Γ.) We are using only 2-regularity, which can be thought of as a far generalization of being archimedean.

Originally I only had a proof of the following proposition in the archimedean case. Then J. Königsmann pointed out to me the notion of n-regular groups and showed me how to prove the result for all 2-regular groups. I am indebted to him for his kind permission to include (a modified version of) his proof in this paper, as well as for other helpful comments and remarks.

PROPOSITION 4. *Let K be a field which has a valuation ring B satisfying*

(1) *the residue field of B is non-real of characteristic $\neq 2$;*

(2) *the value group Γ of B is 2-regular and not 2-divisible.*

Then every finitely generated extension of K satisfies $(*)$. In particular, the conclusions of Proposition 3 hold for all finitely generated extensions of K.

Proof. If K'/K is a finite extension and B' is any valuation ring of K' lying over B, then also B' satisfies (1) and (2). This is immediate for (1). As for (2), one has an exact sequence

$$0 \to \operatorname{Tor}(\Delta, \mathbf{Z}/2) \to \Gamma/2 \to \Gamma'/2 \to \Delta/2 \to 0,$$

where Γ' is the value group of B' and $\Delta := \Gamma'/\Gamma$. Since Δ is finite, $\operatorname{Tor}(\Delta, \mathbf{Z}/2) \cong {}_2\Delta$ is isomorphic to $\Delta/2$. This shows that actually $\Gamma/2 \cong \Gamma'/2$ (non-canonically). It is also easy to see that Γ 2-regular implies Γ' 2-regular.

It is clear anyway that every positive-dimensional function field (over any field of characteristic $\neq 2$) has a valuation ring satisfying (1) and (2). So we are reduced to show: Given K as in Proposition 4, and given $u \in K^*$, there is $t \in \sum K^2$ with $1 - tu \in \sum K^{*2} \backslash K^{*2}$.

By assumption, there are units $b_1, \dots, b_n \in B^*$ such that $v(1+b) > 0$ for $b := \sum b_i^2$. Given $c \in B$ with $0 < v(c) < v(1+b)$, we have $v(1+b') = v(c)$ for

$$b' := (b_1 + c)^2 + b_2^2 + \cdots + b_n^2 = b + 2b_1c + c^2.$$

Since by (2), any interval $[0, \alpha] \subset \Gamma$ with $\alpha > 0$ contains an odd element, it is hence possible to choose the b_i such that $\beta := v(1+b) > 0$ is odd. From 2-regularity of Γ it follows that every coset of Γ mod 2Γ is represented by an element in $[0, \beta]$. (In fact, this is immediate if the interval $[0, \beta]$ is infinite. If it is finite one may assume that β

is the smallest positive element. Looking at $\alpha + \mathbf{Z}\beta \subset \Gamma$ one sees that either α or $\alpha+\beta$ is even, for any $\alpha \in \Gamma$.) Consequently, any $\gamma \in \Gamma$ is of the form $\gamma = v(a)$, for some $a \in \sum K^{*2}$.

Given $u \in K^*$, we may hence assume $v(u) = 0$. Now observe that

$$1 - \frac{4u}{s+(u+1)^2} = \frac{s+(u-1)^2}{s+(u+1)^2}$$

is a sum of squares for any $s \in \sum K^{*2}$. Taking $s := b(u+1)^2$, where b is as above, we find that $v(s+(u+1)^2) = \beta + 2v(u+1)$ is positive and odd. On the other hand, $s+(u-1)^2 = (1+b)(u+1)^2 - 4u$ has valuation zero. So, for

$$t := \frac{4}{s+(u+1)^2} = \frac{4}{(u+1)^2(1+b)} \in \sum K^{*2},$$

$v(1-tu)$ is odd; hence $1 - tu \in \sum K^{*2} \backslash K^{*2}$. □

REMARKS. 1. As noted before, every rank one place of K with a non-2-divisible value group satisfies (2) of Proposition 4. However, the given formulation covers a larger class of fields K, since a 2-regular group Γ with $\Gamma/2 \neq 0$ need not have an *archimedean* ordered factor group Γ' with $\Gamma'/2 \neq 0$.

2. The class of fields covered by Proposition 3 is strictly larger than the class of all hilbertian fields. For example, a henselian valued field can never be hilbertian [**FJ**, p. 181].

3. The proof of Propositions 3 and 4 has the advantage of being more constructive than that of Proposition 1 (which yielded better bounds for hilbertian fields). Specifically, if K is a (positive-dimensional) function field over some base field k, and $U \subset X_K$ is given explicitly, this proof can be used to produce a concrete finite extension L/K with $r_{L/K}(X_L) = U$.

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