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ON SURFACES IN THE 3-DIMENSIONAL LORENTZ-MINKOWSKI SPACE

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Let M_s^2 be a surface in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ and denote by H its mean curvature vector field. This paper locally classifies those surfaces verifying the condition $\Delta H = \lambda H$, where λ is a real constant.

The classification is done by proving that M_s^2 has zero mean curvature everywhere or it is isoparametric, i.e., its shape operator has constant characteristic polynomial.

0. Introduction. Let $x : M^2 \to \mathbb{R}^3$ be an isometric immersion of a surface M^2 in the 3-dimensional Euclidean space and denote by Δ its Laplacian. A well-known result due to Takahashi states that minimal surfaces and spheres are the only surfaces in $\mathbb{R}^3$ satisfying the condition $\Delta x = \lambda x$, for a real constant λ. From the formula $\Delta x = -2H$, where H is the mean curvature vector field, we know that those surfaces also verify the condition $\Delta H = \lambda H$. Thus, it is worthwhile to explore the existence of other surfaces satisfying that condition. The answer is given in **[FGL]**, where the authors (jointly with O. J. Garay), as a consequence of the main theorem there, get the following

THEOREM 0.1. *Let M^2 be a surface in $\mathbb{R}^3$. Then $\Delta H = \lambda H$ if and only if M^2 is minimal or an open piece of one of the following surfaces: a sphere $S^2(r)$ or a right circular cylinder $S^1(r) \times \mathbb{R}$.*

Now, if the ambient space is the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$, the surface M^2 can be endowed with a Riemannian metric (spacelike surface) or a Lorentzian metric (Lorentzian surface) and then it seems natural to hope that a richer classification can be achieved. Indeed, we state a first question:

Problem 1. Classify all surfaces in $\mathbb{L}^3$ satisfying the condition $\Delta H = \lambda H$, where H is the mean curvature vector field.

A maximal surface in $\mathbb{L}^3$ is a spacelike surface with zero mean curvature everywhere. Obviously, those surfaces satisfy $\Delta H = 0$. Then, another question arises naturally:

Problem 2. Are there any other surfaces, apart from maximal ones, satisfying $\Delta H = 0$?

In this paper, see §3, we solve the above two problems.

1. The Laplacian of the mean curvature vector field. Let M_s^2 be a surface in $\mathbb{L}^3$ with index $s = 0, 1$. Denote by σ, A, H, ∇ and $\overline{\nabla}$ the second fundamental form, the shape operator, the mean curvature vector field, the Levi-Civita connection of M_s^2 and the usual flat connection of $\mathbb{L}^3$, respectively. Let N be a unit vector field normal to M_s^2 and let α be the mean curvature with respect to N, i.e., $H = \alpha N$.

In order to compute ΔH at a point $p \in M_s^2$, choose a local orthonormal frame $\{E_1, E_2\}$ such that $\nabla_{E_i} E_j(p) = 0$. From the formula

$$\overline{\nabla}_{E_i}\overline{\nabla}_{E_i} H = E_i E_i(\alpha)N - 2E_i(\alpha)AE_i - \alpha(\nabla_{E_i}A)E_i - \alpha\sigma(AE_i, E_i),$$

one has

$$\Delta H = 2A(\nabla\alpha) + \alpha \operatorname{tr}\nabla A + \{\Delta\alpha + \varepsilon\alpha|A|^2\}N, \tag{1.1}$$

where $\operatorname{tr}\nabla A = \operatorname{trace}\{(X, Y) \to (\nabla_X A)Y\}$ and $\varepsilon = \langle N, N\rangle$.

To find a nice expression of $\operatorname{tr}\nabla A$, we distinguish three cases, according to the canonical form of A.

Case 1. A is diagonalizable. Let $\{X_1, X_2\}$ be a local orthonormal basis of eigenvectors of A, i.e., $AX_i = \mu_i X_i$. Then from the connection equations one has

$$\begin{aligned} \operatorname{tr}\nabla A &= \sum_{i=1}^{2} \varepsilon_i(\nabla_{X_i}A)X_i \\ &= \{\varepsilon_1 X_1(\mu_1) + \varepsilon_2(\mu_2 - \mu_1)\omega_2^1(X_2)\}X_1 \\ &\quad + \{\varepsilon_2 X_2(\mu_2) + \varepsilon_1(\mu_1 - \mu_2)\omega_1^2(X_1)\}X_2, \end{aligned} \tag{1.2}$$

where $\varepsilon_i = \langle X_i, X_i\rangle$.

Now, from the Codazzi's equation $(\nabla_{X_1}A)X_2 = (\nabla_{X_2}A)X_1$, we get

$$\varepsilon_i(\mu_i - \mu_j)\omega_i^j(X_i) = \varepsilon_j X_j(\mu_i),$$

which, jointly with (1.2), yields

$$\operatorname{tr}\nabla A = \varepsilon_1 X_1(\mu_1 + \mu_2)X_1 + \varepsilon_2 X_2(\mu_1 + \mu_2)X_2 = 2\varepsilon\nabla\alpha . \tag{1.3}$$

Case 2. A is not diagonalizable and its minimal polynomial is $(x-\beta)^2$. Now, it has to be $\varepsilon = 1$ and we can choose a local null frame $\{X_1, X_2\}$, i.e., $\langle X_1, X_1\rangle = \langle X_2, X_2\rangle = 0$ and $\langle X_1, X_2\rangle = -1$, such that $AX_1 = \beta X_1 + X_2$ and $AX_2 = \beta X_2$. Now we have

$$\operatorname{tr}\nabla A = -(\nabla_{X_1} A)X_2 - (\nabla_{X_2} A)X_1 = -2X_1(\beta)X_2 . \tag{1.4}$$

By using again the Codazzi's equation we get $X_2(\beta) = 0$ and considering that β is the mean curvature, one obtains from (1.4) that

$$\operatorname{tr}\nabla A = -2X_1(\alpha)X_2 = 2\nabla\alpha . \tag{1.5}$$

Case 3. A is not diagonalizable and its minimal polynomial is $(x-\beta)^2 + \gamma^2$, $\gamma \neq 0$. Here, it must be $\varepsilon = 1$ and we can choose a local orthonormal frame $\{X_1, X_2\}$ such that $AX_1 = \beta X_1 - \gamma X_2$, $AX_2 = \gamma X_1 + \beta X_2$. Writing $\varepsilon_i = \langle X_i, X_i\rangle$, we have

$$\begin{aligned} \operatorname{tr}\nabla = {} & \{\varepsilon_1 X_1(\beta) - \varepsilon_1\gamma\omega_2^1(X_1) - \varepsilon_1\gamma\omega_1^2(X_1) + \varepsilon_2 X_2(\gamma)\}X_1 \\ & + \{\varepsilon_2 X_2(\beta) + \varepsilon_2\gamma\omega_2^1(X_2) + \varepsilon_2\gamma\omega_1^2(X_2) - \varepsilon_1 X_1(\gamma)\}X_2 . \end{aligned} \tag{1.6}$$

From the Lorentzian structure of the tangent space, one has $\omega_1^2 = \omega_2^1$, and using once more the Codazzi's equation we find

$$\operatorname{tr}\nabla A = 2\{\varepsilon_1 X_1(\beta)X_1 + \varepsilon_2 X_2(\beta)X_2\} = 2\nabla\alpha . \tag{1.7}$$

Summarizing, for a surface M_s^2, $s = 0, 1$, in $\mathbb{L}^3$ we have got the desired formula

$$\Delta H = 2A(\nabla\alpha) + \varepsilon\nabla\alpha^2 + \{\Delta\alpha + \varepsilon\alpha|A|^2\}N , \tag{1.8}$$

where $\nabla\alpha$ is the gradient of α and $\varepsilon = \langle N, N\rangle$.

2. Some examples of surfaces with $\Delta H = \lambda H$. We are going to describe some examples of surfaces in $\mathbb{L}^3$ satisfying the condition $\Delta H = \lambda H$ for a real constant λ.

EXAMPLE 2.1. Let $f : \mathbb{L}^3 \to \mathbb{R}$ be a real function defined by

$$f(x, y, z) = -\delta_1 x^2 + y^2 + \delta_2 z^2 ,$$

where δ_1 and δ_2 belong to the set $\{0, 1\}$ and they do not vanish simultaneously. Taking $r > 0$ and $\varepsilon = \pm 1$, the set $f^{-1}(\varepsilon r^2)$ is a surface in $\mathbb{L}^3$ provided that $(\delta_1, \delta_2, \varepsilon) \neq (0, 1, -1)$.

A straightforward computation shows that the unit normal vector field is written as $N = (1/r)(\delta_1 x, y, \delta_2 z)$; the principal curvatures are $\mu_1 = -\delta_1/r$ and $\mu_2 = -\delta_2/r$. Then, the mean curvature is given by $\alpha = (\varepsilon/2)(\mu_1 + \mu_2) = (-\varepsilon/2r)(\delta_1 + \delta_2)$ and it is easy to show that $|A|^2 = \mu_1^2 + \mu_2^2 = (1/r^2)(\delta_1 + \delta_2)$. Therefore, by using formula (1.8), we have $\Delta H = (\varepsilon/r^2)(\delta_1 + \delta_2)H$.

We have listed all possibilities in Table 1.

TABLE 1

δ_1	δ_2	ε	Equation	Surface	Shape operator	α	ΔH
0	1	1	$y^2 + z^2 = r^2$	$\mathbb{L} \times S^1(r)$	$\begin{pmatrix} 0 & 0 \\ 0 & -1/r \end{pmatrix}$	$-\frac{1}{2r}$	$\frac{1}{r^2}H$
1	0	-1	$-x^2 + y^2 = -r^2$	$H^1(r) \times \mathbb{R}$	$\begin{pmatrix} -1/r & 0 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2r}$	$-\frac{1}{r^2}H$
1	0	1	$-x^2 + y^2 = r^2$	$S_1^1(r) \times \mathbb{R}$	$\begin{pmatrix} -1/r & 0 \\ 0 & 0 \end{pmatrix}$	$-\frac{1}{2r}$	$\frac{1}{r^2}H$
1	1	-1	$-x^2 + y^2 + z^2 = -r^2$	$H^2(r)$	$\begin{pmatrix} -1/r & 0 \\ 0 & -1/r \end{pmatrix}$	$\frac{1}{r}$	$-\frac{2}{r^2}H$
1	1	1	$-x^2 + y^2 + z^2 = r^2$	$S_1^2(r)$	$\begin{pmatrix} -1/r & 0 \\ 0 & -1/r \end{pmatrix}$	$-\frac{1}{r}$	$\frac{2}{r^2}H$

EXAMPLE 2.2 (B-scroll, see **[Gr]**). Let $x(s)$ be a null curve in $\mathbb{L}^3$ with Cartan frame $\{A, B, C\}$, i.e., A, B, C are vector fields along $x(s)$ satisfying the following conditions:

$$\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = -1,$$
$$\langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = 1,$$

and

$$\dot{x} = A,$$
$$\dot{A} = k(s)C,$$
$$\dot{B} = w_0 C, \qquad w_0 \text{ being a non-zero constant},$$
$$\dot{C} = w_0 A + k(s)B.$$

If we consider the immersion $\Psi: (s, u) \to x(s) + uB(s)$, then Ψ defines a Lorentz surface that L. K. Graves calls *B-scroll*. An easy computation leads to $N(s, u) = -w_0 u B(s) - C(s)$ and $H = w_0 N$. Then, from the formula (1.8), we get $\Delta H = 2w_0^2 H$.

We wish to finish this section noticing that in the Riemannian case the sphere $S^2(r)$ and the right circular cylinder $S^1(r) \times \mathbb{R}$ are the only non-minimal surfaces satisfying the condition $\Delta H = \lambda H$, but in the Lorentzian situation we find a richer family of surfaces satisfying that condition, as the above examples show. In particular, we would like to point out the chief difference among the B-scroll example and the other ones. In fact, in the first five cases (see Table 1) the shape operator is diagonalizable, whereas in a B-scroll it takes, in the usual frame $\{\partial\Psi/\partial s, \partial\Psi/\partial u\}$, the following form

$$\begin{pmatrix} w_0 & 0 \\ k(s) & w_0 \end{pmatrix},$$

and its minimal polynomial is $(x - w_0)^2$.

3. Main results. This section is devoted to showing the following major result.

THEOREM 3.1. *Let M_s^2, $s = 0, 1$, be a surface in the Lorentz-Minkowski space $\mathbb{L}^3$. Then M_s^2 satisfies the equation $\Delta H = \lambda H$ if and only if one of the following statements is true*:

(1) *M_s^2 has zero mean curvature everywhere.*
(2) *M_s^2 is an open piece of a B-scroll.*
(3) *M_s^2 is an open piece of one of the surfaces exhibited in Table* 1.

Proof. Take in M_s^2 the open set $\mathscr{U} = \{p \in M_s^2 : \nabla\alpha^2(p) \neq 0\}$. Our first goal is showing that $\mathscr{U}$ is empty, i.e., α is constant. Otherwise, by assumption and (1.8) we have got

$$A(\nabla\alpha^2) = -\varepsilon\alpha\nabla\alpha^2, \tag{3.1}$$

and

$$\Delta\alpha + (\varepsilon|A|^2 - \lambda)\alpha = 0 \tag{3.2}$$

at the points of $\mathscr{U}$. Therefore $\nabla\alpha^2$ is a principal direction with principal curvature $-\varepsilon\alpha$ on $\mathscr{U}$. Now, we claim that the shape operator is diagonalizable at the points of $\mathscr{U}$. First, from (3.1), we see that the minimal polynomial of A cannot have the form $(x-\beta)^2+\gamma^2$, $\gamma \neq 0$. On the other hand, neither does $(x - \beta)$; otherwise, we would have $(1 + \varepsilon^2)\alpha(p) = 0$ and therefore $\nabla\alpha^2(p) = 0$, which cannot hold on $\mathscr{U}$. Finally, a similar reasoning shows that the minimal polynomial neither can have the form $(x - \beta)^2$.

Let us take, at a point of $\mathscr{U}$, a local orthonormal frame $\{E_1, E_2, E_3\}$ such that $E_3 = N$ and $\{E_1, E_2\}$ are eigenvectors of A and E_1 is parallel to $\nabla\alpha^2$. Then eigenvalues of A are given by $-\varepsilon\alpha$ and $3\varepsilon\alpha$.

Let $\{\omega^1, \omega^2, \omega^3\}$ and $\{\omega_i^j\}$ be the dual frame and the connection forms, respectively. It is easy to see that

$$\omega_3^1 = \varepsilon\alpha\omega^1, \tag{3.3}$$

$$\omega_3^2 = -3\varepsilon\alpha\omega^2, \tag{3.4}$$

$$d\alpha = E_1(\alpha)\omega^1. \tag{3.5}$$

Taking exterior differentiation in (3.3) and using the structure equations we obtain $d\omega^1 = 0$. Therefore, one locally has $\omega^1 = dv$, for some function v, that jointly with (3.5) yields $d\alpha \wedge dv = 0$. Then, α depends on v, $\alpha = \alpha(v)$, and therefore $d\alpha = \alpha' dv = \alpha'\omega^1$ and $E_1(\alpha) = \alpha'$.

Now, by exterior differentiation in (3.5) and using again the structure equations, one gets

$$4\alpha\omega_2^1 = 3\varepsilon_1\varepsilon_2\alpha'\omega^2, \tag{3.6}$$

where $\varepsilon_i = \langle E_i, E_i\rangle$. Then a direct computation from (3.6) allows us to write down the following differential equation

$$4\alpha\alpha'' - 7(\alpha')^2 + 16\varepsilon\varepsilon_1\alpha^4 = 0. \tag{3.7}$$

It is easy to see that a first integral is given by

$$(\alpha')^2 = C\alpha^{7/2} - 16\varepsilon\varepsilon_1\alpha^4, \tag{3.8}$$

where C is a constant.

From the fact that E_1 is parallel to $\nabla\alpha^2$ and equation (3.6) we find

$$4\alpha\Delta\alpha = -4\varepsilon_1\alpha\alpha'' + 3\varepsilon_1(\alpha')^2. \tag{3.9}$$

On the other hand, since $|A|^2 = 10\alpha^2$, by using equations (3.2) and (3.9) we have

$$-4\alpha\alpha'' + 3(\alpha')^2 = 4\varepsilon_1\{\lambda - 10\varepsilon\alpha^2\}\alpha^2,$$

that jointly with (3.7) gives

$$(\alpha')^2 = -\varepsilon_1\lambda\alpha^2 + 14\varepsilon\varepsilon_1\alpha^4. \tag{3.10}$$

We deduce, by using (3.8) and (3.10), that α is locally a constant on $\mathscr{U}$, which is a contradiction with the definition of the set $\mathscr{U}$. As a consequence, α is a constant on M_s^2. Then, again from (3.2), we deduce that either M_s^2 has zero mean curvature everywhere or $|A|^2 = \varepsilon\lambda$ and therefore $|A|^2$ is also a constant. That implies that the characteristic polynomial of A is constant and therefore M_s^2 is isoparametric. Then, if $s = 0$, M_0^2 is an open piece of $H^2(r)$ or

$H^1(r) \times \mathbb{R}$. When $s = 1$, it follows from **[Ma]** that M_1^2 is an open piece of one of the following surfaces: $S_1^2(r)$, $S_1^1(r) \times \mathbb{R}$, $\mathbb{L} \times S^1(r)$ and the B-scroll of Example 2.

Theorem 3.1 gives the best solution to the stated Problem 1, so that additional hypothesis should be given in order to characterize those surfaces with zero mean curvature everywhere.

The following results can be deduced from Theorem 3.1.

COROLLARY 3.2. *Let M_s^2 be a surface in $\mathbb{L}^3$. Then $\Delta H = 0$ if and only if M_s^2 has zero mean curvature everywhere.*

This is the solution of the Problem 2. It is worth noticing that this solution is quite similar from that given for surfaces in the 3-dimensional Euclidean space, because according to Theorem 0.1 minimal surfaces in $\mathbb{R}^3$ are the only ones satisfying the condition $\Delta H = 0$.

COROLLARY 3.3. *Let M_s^2 be a surface in $\mathbb{L}^3$ with non-zero mean curvature. Then $\Delta H = \lambda H$, $\lambda > 0$, if and only if M_s^2 is an open piece of a Lorentzian cylinder, a De Sitter space $S_1^2(r)$ or a B-scroll.*

COROLLARY 3.4. *Let M^2 be a spacelike surface in $\mathbb{L}^3$. Then $\Delta H = \lambda H$ if and only if one of the following statements holds:*

(a) *M^2 is maximal.*

(b) *M^2 is an open piece of the hyperbolic plane $H^2(r)$.*

(c) *M^2 is an open piece of the hyperbolic cylinder $H^1(r) \times \mathbb{R}$.*

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Added in proof. We have known from Professor B. Y. Chen that Problem 2 has also been studied for Euclidean submanifolds by I. Dimitric in his thesis (MSU).

REFERENCES

[FGL] A. Ferrández, O. J. Garay and P. Lucas, *On a certain class of conformally flat Euclidean hypersurfaces*, to appear in the Proceedings of the Conference in Global Analysis and Global Differential Geometry, Berlin, June 1990.

[Gr] L. K. Graves, *Codimension one isometric immersions between Lorentz spaces*, Trans. Amer. Math. Soc., **252** (1979), 367–392.

[Ma] M. A. Magid, *Lorentzian isoparametric hypersurfaces*, Pacific J. Math., **118** (1985), 165–197.

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