PACIFIC JOURNAL OF MATHEMATICS
Vol. 152, No. 1, 1992

ON A CHARACTERIZATION OF VELOCITY MAPS IN THE SPACE OF OBSERVABLES

B. V. RAJARAMA BHAT

Motivated by Heisenberg's picture of quantum dynamics the notion of a velocity map is introduced and its properties are investigated. The main theorem in the present exposition strengthens the well-known result that every derivation on the algebra of all bounded operators on a complex separable Hilbert space is inner. A constructive proof leads to an inversion formula for the observables inducing the derivation.

1. Introduction. Let $\mathscr{A}$ be a von Neumann algebra. Then a derivation δ on $\mathscr{A}$ is a linear map $\delta: \mathscr{A} \to \mathscr{A}$ satisfying $\delta(XY) = X\delta(Y) + \delta(X)Y$ for every X, Y in $\mathscr{A}$. Inner derivations are the derivations of the form $\delta(X) = [D, X]$ for some D in $\mathscr{A}$. It is a well-known result of Sakai and Kadison (cf. **[1]**, **[2]**) that every derivation δ on a von Neumann algebra $\mathscr{A}$ is inner.

In Heisenberg's picture of quantum dynamics maps of the form $\delta(X) = i[H, X]$, where H, X are self-adjoint operators, determine the rate of change (or velocity) of observables. However, in this case, we are interested in the action of δ only on the real linear space $\mathscr{O}$ of observables (self-adjoint elements) of the algebra and not on the full algebra $\mathscr{A}$. Keeping this in mind K. R. Parthasarathy suggested the following "axioms" for a velocity map which measures rate of change of observables:

Let $\mathscr{O}$ be the real linear space of all self-adjoint elements of a von Neumann algebra $\mathscr{A}$. Then a map $\delta: \mathscr{O} \to \mathscr{O}$ is called a velocity map if it satisfies the following conditions.

(1.1) $\delta(aX) = a\delta(X) \quad \forall a \in \mathbb{R}, \forall X \in \mathscr{O}$,

(1.2) $\delta(X + Y) = \delta(X) + \delta(Y) \quad \forall X, Y \in \mathscr{O}$ with $[X, Y] = 0$,

(1.3) $\delta(XY) = X\delta(Y) + \delta(X)Y \quad \forall X, Y \in \mathscr{O}$ with $[X, Y] = 0$.

It should be noted that the requirement $[X, Y] = 0$ in (1.3) is an algebraic necessity to define $\delta(XY)$. We insist on the same requirement in (1.2) for the purely physical reason that the observables X, Y and $X + Y$ are simultaneously measurable if and only if $[X, Y] = 0$.

In this paper we study continuous velocity maps. (Here and throughout this paper by continuity we mean norm continuity.) Under the assumption of continuity we show that if a map $\delta : \mathscr{O} \to \mathscr{O}$ satisfies (1.1) and (1.3) then it automatically satisfies (1.2) and hence becomes a velocity map.

A priori, it is not clear whether such a velocity map can be extended to a derivation on $\mathscr{A}$. We expect that such derivations are also inner in the sense $\delta(X) = i[H, X]$ for some H in $\mathscr{O}$ and hence can be extended to a derivation on $\mathscr{A}$ in a unique way.

In this paper we have an elementary constructive proof that this is indeed so for linear velocity maps on the von Neumann algebra of all bounded operators on a complex separable Hilbert space. In fact we have an explicit inversion formula for H in terms of δ.

2. Velocity maps. Let $\mathscr{O}$ be the real linear space of observables of a von Neumann algebra $\mathscr{A}$. For non-zero real numbers c define the map $\delta_c : \mathscr{O} \to \mathscr{O}$ by

$$\delta_c(X) = cX \log|X| \quad \forall X \in \mathscr{O}. \tag{2.1}$$

As the function $f_c(x) = cx \log|x|$ (which is defined to be 0 at the origin) is a continuous function on the real line δ_c is well-defined. δ_c clearly satisfies the condition (1.3), that is,

$$\delta_c(XY) = X\delta_c(Y) + \delta_c(X)Y \qquad \forall X, Y \text{ in } \mathscr{O} \text{ with } [X, Y] = 0.$$

However δ_c does not satisfy conditions (1.1) and (1.2). In contrast to this we have the following theorem which shows that if a continuous map $\delta : \mathscr{O} \to \mathscr{O}$ satisfies (1.3) and a weakened (1.1) namely,

$$\delta(aI) = 0 \quad \forall a \in \mathbb{R} \tag{2.2}$$

then δ satisfies both (1.1) and (1.2).

THEOREM 2.1. *Let $\mathscr{O}$ be the real linear space of all self-adjoint elements of a von Neumann algebra $\mathscr{A}$. If $\delta : \mathscr{O} \to \mathscr{O}$ is a continuous map satisfying* (1.3) *and* (2.2) *then it is a velocity map.*

Proof. Condition (1.3) implies

$$\begin{aligned} \delta(aX) &= \delta(aI \cdot X) \\ &= aI \cdot \delta(X) + \delta(aI)X \\ &= a\delta(X) \quad \forall a \in \mathbb{R}, X \in \mathscr{O}. \end{aligned}$$

This proves (1.1).

Now for any natural number n if we have n mutually orthogonal projections $P_1, P_2, \ldots, P_n$ in $\mathscr{O}$, we claim,

$$\delta\left(\sum_j a_j P_j\right) = \sum_j a_j \delta(P_j) \quad \text{for } a_j \in \mathbb{R}\ \forall j. \tag{2.3}$$

We have proved this for $n = 1$. For $n \geq 1$, put

$$P_{n+1} = I - \sum_{j=1}^{n} P_j, \quad a_{n+1} = 0.$$

Then we have

$$\sum_j P_j = I \quad \text{and} \quad \sum_{j=1}^{n+1} a_j P_j = \sum_{j=1}^{n} a_j P_j.$$

By (1.3), for every $k \geq 1$

$$\begin{aligned}\delta\left(\left(\sum_j a_j P_j\right) P_k\right) &= \delta(a_k P_k)\\ &= \left(\sum_j a_j P_j\right)\delta(P_k) + \delta\left(\sum_j a_j P_j\right) P_k,\end{aligned}$$

$$\begin{aligned}\delta\left(\sum_j a_j P_j\right) P_k &= \delta(a_k P_k) - \sum_j a_j P_j \delta(P_k)\\ &= a_k \delta(P_k) + \sum_{j\neq k} a_j \delta(P_j) P_k - a_k P_k \delta(P_k)\\ &= a_k (I - P_k)\delta(P_k) + \sum_{j\neq k} a_j \delta(P_j) P_k.\end{aligned}$$

Adding over k and using (1.3) we get

$$\begin{aligned}\delta\left(\sum_j a_j P_j\right) &= \sum_k a_k (I - P_k)\delta(P_k) + \sum_j a_j \delta(P_j)(I - P_j)\\ &= \sum_j a_j \{(I - P_j)\delta(P_j) + \delta(P_j)(I - P_j)\}\\ &= \sum_j a_j \{\delta(P_j) - P_j\delta(P_j) + \delta(P_j) - \delta(P_j)P_j\}\\ &= \sum_j a_j \delta(P_j).\end{aligned}$$

Let Z, W be commuting elements in $\mathscr{O}$ with finite number of points in the spectrum. We know that spectral projections of elements of $\mathscr{O}$ are in $\mathscr{O}$. So we can write Z, W in the form

$$Z = \sum_{i=1}^{n} a_i P_i , \quad W = \sum_{i=1}^{n} b_i P_i$$

where a_i's and b_i's are real numbers and P_i's are mutually orthogonal projections.

By (2.3)

$$\begin{aligned}(2.4) \quad \delta(Z+W) &= \delta\left(\sum_i a_i P_i + \sum_i b_i P_i\right) \\ &= \delta\left(\sum_i (a_i + b_i) P_i\right) = \sum_i (a_i + b_i)\delta(P_i) \\ &= \sum_i a_i \delta(P_i) + \sum_i b_i \delta(P_i) = \delta(Z) + \delta(W).\end{aligned}$$

Let X, Y be any two commuting elements in $\mathscr{O}$. Using spectral theorem we can approximate X, Y by commuting finite spectrum elements of $\mathscr{O}$. By (2.4) and continuity of δ we get

$$\delta(X+Y) = \delta(X) + \delta(Y). \qquad \square$$

REMARK 2.2. In Theorem 2.1 if $\mathscr{O}$ is finite dimensional then we need not assume that δ is continuous.

This is clear from (2.4).

3. Main result. Let $\mathscr{H}$ be a complex separable Hilbert space with inner product $\langle \ \rangle$ which is conjugate linear in the first variable and linear in the second variable. For any two vectors x, y in $\mathscr{H}$, $|x\rangle\langle y|$ is the operator defined by

$$(3.1) \qquad |x\rangle\langle y| z = \langle y , z\rangle x \quad \forall z \in \mathscr{H}.$$

Observe that $|\ \rangle\langle\ |$ is linear in the first variable and conjugate linear in the second variable and for any unit vector x, $|x\rangle\langle x|$ is the projection on the one dimensional subspace generated by x.

Let $\mathscr{B}(\mathscr{H})$ be the von Neumann algebra of all bounded operators on $\mathscr{H}$ and $\mathscr{O}(\mathscr{H})$ be the real linear space of bounded self-adjoint operators on $\mathscr{H}$. Let $\delta\colon \mathscr{O}(\mathscr{H}) \to \mathscr{O}(\mathscr{H})$ be a linear map satisfying the condition (1.3), that is,

$$\delta(XY) = X\delta(Y) + \delta(X)Y \quad \forall X, Y \text{ in } \mathscr{O}(\mathscr{H}) \text{ with } [X, Y] = 0.$$

Now we would like to obtain a self-adjoint operator H such that $\delta(X) = i[H, X]$ $\forall X \in \mathscr{O}(\mathscr{H})$. To recover the H from δ we study the action of δ on various rank one projections. To avoid trivialities, assume $\dim \mathscr{H} \geq 3$.

LEMMA 3.1. *Let u be a unit vector in $\mathscr{H}$. Then there is a unique vector $\varphi(u)$ such that $\delta(|u\rangle\langle u|) = i(|\varphi(u)\rangle\langle u| - |u\rangle\langle\varphi(u)|)$ and $\langle u, \varphi(u)\rangle = 0$. Moreover if v is a unit vector orthogonal to u then $\langle \varphi(u), v\rangle = \langle u, \varphi(v)\rangle$.*

Proof. Let u be a unit vector in $\mathscr{H}$. Define $\varphi(u)$ by

$$\varphi(u) = -i\delta(|u\rangle\langle u|)u. \tag{3.2}$$

As $|u\rangle\langle u|$ is a projection we have

$$\begin{aligned} \delta(|u\rangle\langle u|) &= \delta(|u\rangle\langle u|)|u\rangle\langle u| + |u\rangle\langle u|\delta(|u\rangle\langle u|) \qquad (3.3)\\ &= |\delta(|u\rangle\langle u|)u\rangle\langle u| + |u\rangle\langle\delta(|u\rangle\langle u|)u| \\ &= i(|\varphi(u)\rangle\langle u| - |u\rangle\langle\varphi(u)|). \qquad (3.4) \end{aligned}$$

To prove the second assertion use (3.3) to get

$$\langle u, \delta(|u\rangle\langle u|)u\rangle = \langle u, \delta(|u\rangle\langle u|)u\rangle + \langle u, \delta(|u\rangle\langle u|)u\rangle.$$

Then

$$\langle u, \delta(|u\rangle\langle u|)u\rangle = 0$$

which implies, by the definition of $\varphi(u)$,

$$\langle u, \varphi(u)\rangle = 0. \tag{3.5}$$

Uniqueness is obvious as whenever $\langle u, \varphi(u)\rangle = 0$ we have,

$$i(|\varphi(u)\rangle\langle u| - |u\rangle\langle\varphi(u)|)u = i\varphi(u).$$

Again by (1.3)

$$\delta(|u\rangle\langle u|)|v\rangle\langle v| + |u\rangle\langle u|\delta(|v\rangle\langle v|) = 0.$$

Now using the formula (3.4) for $\delta(|u\rangle\langle u|)$ and $\delta(|v\rangle\langle v|)$ we get

$$-i\langle\varphi(u), v\rangle|u\rangle\langle v| + i\langle u, \varphi(v)\rangle|u\rangle\langle v| = 0$$

which means

$$\langle\varphi(u), v\rangle = \langle u, \varphi(v)\rangle. \qquad \square$$

Analysing the action of δ on some more projections we have

LEMMA 3.2. *Let u, v and w be three mutually orthogonal unit vectors in $\mathscr{H}$. Then the following equalities hold:*

(i) $\langle w, \delta(|u\rangle\langle v| + |v\rangle\langle u|)w\rangle = 0$;
(ii) $\langle w, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle = i\langle w, \varphi(u)\rangle$;
(iii) $\mathrm{Re}\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle = 0$;
(iv) $\langle u, \delta(|u\rangle\langle iv| + |iv\rangle\langle u|)iv\rangle = \langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle$;
(v) $\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)u\rangle = i\langle u, \varphi(v)\rangle - i\langle v, \varphi(u)\rangle$;
(vi) $\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle = \langle u, \delta(|u\rangle\langle w| + |w\rangle\langle u|)w\rangle + \langle w, \delta(|w\rangle\langle v| + |v\rangle\langle w|)v\rangle$.

Proof. By linearity,

$$\delta(|u\rangle\langle v| + |v\rangle\langle u|) = \delta\left(\left|\frac{u+v}{\sqrt{2}}\right\rangle\left\langle\frac{u+v}{\sqrt{2}}\right|\right) - \delta\left(\left|\frac{u-v}{\sqrt{2}}\right\rangle\left\langle\frac{u-v}{\sqrt{2}}\right|\right).$$

Then (i) is obvious. To show (ii) we consider

$$\begin{aligned}
&\langle w, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle \\
&= \left\langle w, \delta\left(\left|\frac{u+v}{\sqrt{2}}\right\rangle\left\langle\frac{u+v}{\sqrt{2}}\right|\right)v\right\rangle \\
&\quad - \left\langle w, \delta\left(\left|\frac{u-v}{\sqrt{2}}\right\rangle\left\langle\frac{u-v}{\sqrt{2}}\right|\right)v\right\rangle \\
&= \frac{i}{\sqrt{2}}\left\langle w, \varphi\left(\frac{u+v}{\sqrt{2}}\right)\right\rangle + \frac{i}{\sqrt{2}}\left\langle w, \varphi\left(\frac{u-v}{\sqrt{2}}\right)\right\rangle \\
&= \frac{i}{\sqrt{2}}\left\langle \varphi(w), \frac{u+v}{\sqrt{2}}\right\rangle + \frac{i}{\sqrt{2}}\left\langle \varphi(w), \frac{u-v}{\sqrt{2}}\right\rangle \\
&= i\langle\varphi(w), u\rangle = i\langle w, \varphi(u)\rangle.
\end{aligned}$$

By (3.5)

$$\left\langle \varphi\left(\frac{u+v}{\sqrt{2}}\right), \frac{u+v}{\sqrt{2}}\right\rangle = 0 \quad \text{and} \quad \left\langle \varphi\left(\frac{u-v}{\sqrt{2}}\right), \frac{u-v}{\sqrt{2}}\right\rangle = 0.$$

Making use of these equalities we obtain

$$\begin{aligned}
&\left\langle u, \delta\left(\left|\frac{u+v}{\sqrt{2}}\right\rangle\left\langle\frac{u+v}{\sqrt{2}}\right|\right)v\right\rangle \\
&= \frac{i}{\sqrt{2}}\left\langle u, \varphi\left(\frac{u+v}{\sqrt{2}}\right)\right\rangle - \frac{i}{\sqrt{2}}\left\langle \varphi\left(\frac{u+v}{\sqrt{2}}\right), v\right\rangle \\
&= \frac{i}{\sqrt{2}}\left\langle u, \varphi\left(\frac{u+v}{\sqrt{2}}\right)\right\rangle + \frac{i}{\sqrt{2}}\left\langle \varphi\left(\frac{u+v}{\sqrt{2}}\right), u\right\rangle \\
&= i\sqrt{2}\,\mathrm{Re}\left\langle u, \varphi\left(\frac{u+v}{\sqrt{2}}\right)\right\rangle
\end{aligned}$$

and

$$\begin{aligned}\left\langle u, \delta\left(\left|\frac{u-v}{\sqrt{2}}\right\rangle\left\langle\frac{u-v}{\sqrt{2}}\right|\right) v\right\rangle &= \frac{-i}{\sqrt{2}}\left\langle u, \varphi\left(\frac{u-v}{\sqrt{2}}\right)\right\rangle - \frac{i}{\sqrt{2}}\left\langle \varphi\left(\frac{u-v}{\sqrt{2}}\right), v\right\rangle \\ &= \frac{-i}{\sqrt{2}}\left\langle u, \varphi\left(\frac{u-v}{\sqrt{2}}\right)\right\rangle - \frac{i}{\sqrt{2}}\left\langle \varphi\left(\frac{u-v}{\sqrt{2}}\right), u\right\rangle \\ &= -i\sqrt{2}\,\mathrm{Re}\left\langle u, \varphi\left(\frac{u-v}{\sqrt{2}}\right)\right\rangle.\end{aligned}$$

So (iii) follows.

In order to show (iv) define the projection,

$$P_1 = \left|\frac{u}{\sqrt{2}} + \left(\frac{1+i}{2}\right) v\right\rangle\left\langle\frac{u}{\sqrt{2}} + \left(\frac{1+i}{2}\right) v\right|.$$

It is clear that

$$\begin{aligned}\langle u, \delta(P_1)v\rangle &= \frac{1}{2}\langle u, \delta(|u\rangle\langle u|)v\rangle + \frac{1}{2}\langle u, \delta(|v\rangle\langle v|)v\rangle \\ &\quad + \frac{1}{2\sqrt{2}}\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle \\ &\quad + \frac{1}{2\sqrt{2}}\langle u, \delta(|u\rangle\langle iv| + |iv\rangle\langle u|)v\rangle \\ &= \frac{-i}{2}\langle u, \varphi(v)\rangle + \frac{i}{2}\langle u, \varphi(v)\rangle \\ &\quad + \frac{1}{2\sqrt{2}}\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle \\ &\quad + \frac{(-i)}{2\sqrt{2}}\langle u, \delta(|u\rangle\langle iv| + |iv\rangle\langle u|)iv\rangle \\ &= \frac{1}{2\sqrt{2}}\{\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle \\ &\qquad - i\langle u, \delta(|u\rangle\langle iv| + |iv\rangle\langle u|)iv\rangle\}.\end{aligned} \tag{3.6}$$

As P_1 is a projection we have

$$\begin{aligned}\langle u, \delta(P_1)u\rangle &= \langle u, \delta(P_1)P_1u\rangle + \langle u, P_1\delta(P_1)u\rangle \\ &= 2\,\mathrm{Re}\langle u, \delta(P_1)P_1u\rangle \\ &= 2\,\mathrm{Re}\left\{\frac{1}{\sqrt{2}}\left\langle u, \delta(P_1)\left(\frac{u}{\sqrt{2}} + \left(\frac{1+i}{2}\right)v\right)\right\rangle\right\} \\ &= \langle u, \delta(P_1)u\rangle + \sqrt{2}\,\mathrm{Re}\left\langle u, \delta(P_1)\left(\frac{1+i}{2}\right)v\right\rangle.\end{aligned}$$

This means

$$\operatorname{Re}(1+i)\langle u, \delta(P_1)v\rangle = 0.$$

Then from earlier computation (3.6)

$$\operatorname{Re}(1+i)\{\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle - i\langle u, \delta(|u\rangle\langle iv| + |iv\rangle\langle u|)iv\rangle\} = 0$$

and from (iii)

$$\operatorname{Re}\{\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle\} = 0,$$
$$\operatorname{Re}\{\langle u, \delta(|u\rangle\langle iv| + |iv\rangle\langle u|)iv\rangle\} = 0;$$

combining these we obtain (iv).

In order to show (v) define the projection

$$P_2 = \left|\frac{u}{2} + \frac{\sqrt{3}}{2}v\right\rangle\left\langle\frac{u}{2} + \frac{\sqrt{3}}{2}v\right|.$$

Evidently,

$$\text{(3.7)} \qquad \begin{aligned}\langle u, \delta(P_2)u\rangle &= \left\langle u, \delta\left(\left|\frac{u}{2} + \frac{\sqrt{3}}{2}v\right\rangle\left\langle\frac{u}{2} + \frac{\sqrt{3}}{2}v\right|\right)u\right\rangle \\ &= \frac{\sqrt{3}}{4}\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle\end{aligned}$$

and

$$\begin{aligned}\langle u, \delta(P_2)P_2u\rangle &= \frac{1}{2}\left\langle u, \delta(P_2)\left(\frac{u}{2} + \frac{\sqrt{3}}{2}v\right)\right\rangle \\ &= \frac{1}{4}\langle u, \delta(P_2)u\rangle \\ &\quad + \frac{\sqrt{3}}{4}\left\{\frac{1}{4}\langle u, \delta(|u\rangle\langle u|)v\rangle + \frac{3}{4}\langle u, \delta(|v\rangle\langle v|)v\rangle\right. \\ &\qquad\qquad \left. + \frac{\sqrt{3}}{4}\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle\right\} \\ &= \frac{1}{4}\langle u, \delta(P_2)u\rangle \\ &\quad + \frac{\sqrt{3}}{4}\left\{\left(\frac{-i}{4}\right)\langle u, \varphi(v)\rangle + \frac{3i}{4}\langle u, \varphi(v)\rangle\right. \\ &\qquad\qquad \left. + \frac{\sqrt{3}}{4}\langle u, \delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle\right\}.\end{aligned}$$

So by (iii)

$$\text{(3.8)} \qquad \begin{aligned}2\operatorname{Re}\langle u, \delta(P_2)P_2u\rangle &= \frac{1}{2}\langle u, \delta(P_2)u\rangle \\ &\quad + \frac{\sqrt{3}}{8}\{i\langle u, \varphi(v)\rangle - i\langle v, \varphi(u)\rangle\}.\end{aligned}$$

But by (3.7)

$$(3.9)\qquad 2\,\mathrm{Re}\langle u\,,\,\delta(P_2)P_2u\rangle = \langle u\,,\,\delta(P_2)u\rangle = \frac{\sqrt{3}}{4}\langle u\,,\,\delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle.$$

Combining (3.8) and (3.9) we get (v).

The relation (vi) is obtained in a similar way by considering the projection

$$P_3 = \left|\frac{u+v+w}{\sqrt{3}}\right\rangle\left\langle\frac{u+v+w}{\sqrt{3}}\right|$$

and the equation

$$\langle u\,,\,\delta(P_3)v\rangle = \langle u\,,\,\delta(P_3)P_3v\rangle + \langle P_3u\,,\,\delta(P_3)v\rangle. \qquad \square$$

Now we would like to exploit the linearity of δ by considering unit vectors of the form $cu + dv$ for some complex numbers c and d. For this purpose we extend Lemma 3.2 to get

LEMMA 3.3. *Let u, v and w be three mutually orthogonal unit vectors in $\mathscr{H}$. Let c and d be any two complex numbers. Then we have the following relations*:

(i) $\langle w\,,\,\delta(\,|cu\rangle\langle dv| + |dv\rangle\langle cu|)w\rangle = 0$;
(ii) $\langle w\,,\,\delta(|cu\rangle\langle dv| + |dv\rangle\langle cu|)v\rangle = c\bar{d}i\langle w\,,\,\varphi(u)\rangle$;
(iii) $\langle u\,,\,\delta(|cu\rangle\langle dv| + |dv\rangle\langle cu|)v\rangle = c\bar{d}\langle u\,,\,\delta(|u\rangle\langle v| + |v\rangle\langle u|)v\rangle$;
(iv) $\langle u\,,\,\delta(|cu\rangle\langle dv| + |dv\rangle\langle cu|)u\rangle = \bar{c}di\langle u\,,\,\varphi(v)\rangle - c\bar{d}i\langle v\,,\,\varphi(u)\rangle$.

Proof. Write c, d in $\mathbb{C}$ as

$$c = c_1 + ic_2\,,\quad c_1\,,\,c_2 \in \mathbb{R}\,,$$
$$d = d_1 + id_2\,,\quad d_1\,,\,d_2 \in \mathbb{R}.$$

Then we have

$$\delta(|cu\rangle\langle dv| + |dv\rangle\langle cu|) = (c_1d_1 + c_2d_2)\delta(|u\rangle\langle v| + |v\rangle\langle u|) + (c_1d_2 - c_2d_1)\delta(|u\rangle\langle iv| + |iv\rangle\langle u|).$$

Now note that (iv) is also a unit vector orthogonal to u and w. Then the result is immediate from Lemma 3.2 and linearity of δ. $\square$

Now we are ready to recover H from δ. Note that if $\delta(X) = i[H\,,\,X]$ for every X in $\mathscr{O}(\mathscr{H})$ then for any real number a we have $\delta(X) = i[H + aI\,,\,X]$. This nonuniqueness of H is taken care of by insisting $\langle u_0, Hu_0\rangle = 0$ for some fixed unit vector u_0 in $\mathscr{H}$. So choose and fix a unit vector u_0 in H. Define $H : \mathscr{H} \to \mathscr{H}$ by

$$
\begin{aligned}
(3.10) \qquad & Hu_0 = -i\delta(|u_0\rangle\langle u_0|)u_0 = \varphi(u_0), \\
& Hv = -i\delta(|v\rangle\langle v|)v + i\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle v \\
& \qquad \text{for } v \in \mathscr{H} \text{ with } \langle v, u_0\rangle = 0 \text{ and } \|v\| = 1. \\
& H(au_0 + z) = aHu_0 + \|z\| H\left(\frac{z}{\|z\|}\right) \\
& \qquad \text{for } a \in \mathbb{C} \text{ and } z \in \mathscr{H} \text{ with } \langle z, u_0\rangle = 0.
\end{aligned}
$$

Note that we do have $\langle u_0, Hu_0\rangle = 0$. We use Lemma 3.2 and Lemma 3.3 to obtain the linearity of H.

LEMMA 3.4. *The map H defined above in* (3.10) *is linear.*

Proof. Let v, w be mutually orthogonal unit vectors in $\mathscr{H}$ which are also orthogonal to u_0. Let z be a vector in $\mathscr{H}$ orthogonal to v and w. Let c, d be any two complex numbers. Then we show

$$(3.11) \qquad \langle v, H(cv + dw)\rangle = c\langle v, Hv\rangle + d\langle v, Hw\rangle$$

and

$$(3.12) \qquad \langle z, H(cv + dw)\rangle = c\langle z, Hv\rangle + d\langle z, Hw\rangle.$$

From these linearity of H follows. From the definition of H,

$$
\begin{aligned}
H(iv) = & -i\delta(|iv\rangle\langle iv|)iv + i\langle u_0, \delta(|u_0\rangle\langle iv| + |iv\rangle\langle u_0|)iv\rangle iv \\
= & \; i\{-i\delta(|v\rangle\langle v|)v + i\langle u_0, \delta(|u_0\rangle\langle iv| + |iv\rangle\langle u_0|)iv\rangle v\}.
\end{aligned}
$$

By (iv) of Lemma 3.2 we get $H(iv) = iHv$.

Now a simple computation shows that $H(ax) = aHx$ for any complex number a and any vector x. So without loss of generality we can assume $|c|^2 + |d|^2 = 1$, while showing (3.11) and (3.12). As $(cv + dw)$ is now a unit vector, we have

$$
\begin{aligned}
(3.13) \quad & \langle v, H(cv + dw)\rangle \\
& = -i\langle v, \delta(|cv + dw\rangle\langle cv + dw|)(cv + dw)\rangle \\
& \quad + ic\langle u_0, \delta(|u_0\rangle\langle cv + dw| + |cv + dw\rangle\langle u_0|)(cv + dw)\rangle \\
& = S_1 + S_2 \quad \text{(say)}.
\end{aligned}
$$

Linearity of δ implies

$$
\begin{aligned}
S_1 = (-i)\{ & d|c|^2\langle v, \delta(|v\rangle\langle v|)w\rangle + d|d|^2\langle v, \delta(|w\rangle\langle w|)w\rangle \\
& + c\langle v, \delta(|cv\rangle\langle dw| + |dw\rangle\langle cv|)v\rangle \\
& + d\langle v, \delta(|cv\rangle\langle dw| + |dw\rangle\langle cv|)w\rangle\}.
\end{aligned}
$$

Using (3.4) in the first two terms and (iv) and (iii) of Lemma 3.3 in the next two terms we get

$$\begin{aligned}
S_1 &= (-i)\{d|c|^2(-i)\langle\varphi(v), w\rangle + d|d|^2(i)\langle\varphi(v), w\rangle \\
&\qquad + c\bar{c}di\langle\varphi(v), w\rangle - cc\bar{d}i\langle\varphi(w), v\rangle \\
&\qquad\qquad + dc\bar{d}\langle v, \delta(|v\rangle\langle w| + |w\rangle\langle v|)w\rangle\} \\
&= (-i)\{d|d|^2(i)\langle\varphi(v), w\rangle - c^2\bar{d}(i)\langle\varphi(w), v\rangle \\
&\qquad\qquad + c|d|^2\langle v, \delta(|v\rangle\langle w| + |w\rangle\langle v|)w\rangle\}, \\
S_2 &= ic\{c\langle u_0, \delta(|u_0\rangle\langle dw| + |dw\rangle\langle u_0|)v\rangle \\
&\qquad + d\langle u_0, \delta(|u_0\rangle\langle cv| + |cv\rangle\langle u_0|)w\rangle \\
&\qquad + c\langle u_0, \delta(|u_0\rangle\langle cv| + |cv\rangle\langle u_0|)v\rangle \\
&\qquad\qquad + d\langle u_0, \delta(|u_0\rangle\langle dw| + |dw\rangle\langle u_0|)w\rangle\}.
\end{aligned}$$

Using (ii) of Lemma 3.3 in the first two terms and (iii) of Lemma 3.3 in the last two terms we obtain

$$\begin{aligned}
S_2 &= ic\{c\bar{d}(-i)\langle\varphi(w), v\rangle + \bar{c}d(-i)\langle\varphi(v), w\rangle \\
&\qquad + |c|^2\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle \\
&\qquad\qquad + |d|^2\langle u_0, \delta(|u_0\rangle\langle w| + |w\rangle\langle u_0|)w\rangle\}.
\end{aligned}$$

Now coming back to (3.13) we have

$$\begin{aligned}
\langle v, H(cv + dw)\rangle &= S_1 + S_2 \\
&= d(|d|^2 + |c|^2)\langle\varphi(v), w\rangle \\
&\quad + (-i)c|d|^2\langle v, \delta(|v\rangle\langle w| + |w\rangle\langle v|)w\rangle \\
&\quad + ic|c|^2\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle \\
&\quad + ic|d|^2\langle u_0, \delta(|u_0\rangle\langle w| + |w\rangle\langle u_0|)w\rangle.
\end{aligned}$$

(iii) and (vi) of Lemma 3.2 imply

$$\begin{aligned}
&\langle v, H(cv + dw)\rangle \\
&\quad = d\langle\varphi(v), w\rangle + ic|d|^2\langle w, \delta(|w\rangle\langle v| + |v\rangle\langle w|)v\rangle \\
&\qquad + ic|c|^2\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle \\
&\qquad + ic|d|^2\langle u_0, \delta(|u_0\rangle\langle w| + |w\rangle\langle u_0|)w\rangle \\
&\quad = d\langle\varphi(v), w\rangle + ic(|c|^2 + |d|^2)\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle \\
&\quad = d\langle\varphi(v), w\rangle + ic\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle \\
&\quad = d\langle v, Hw\rangle + c\langle v, Hv\rangle.
\end{aligned}$$

This proves (3.11). To show (3.12) we consider

$$\begin{aligned}\langle z, H(cv+dw)\rangle &= (-i)\langle z, \delta(|cv+dw\rangle\langle cv+dw|)(cv+dw)\rangle \\ &= (-i)\{c|c|^2\langle z, \delta(|v\rangle\langle v|)v\rangle + d|d|^2\langle z, \delta(|w\rangle\langle w|)w\rangle \\ &\quad + c\langle z, \delta(|cv\rangle\langle dw| + |dw\rangle\langle cv|)v\rangle \\ &\quad + d\langle z, \delta(|cv\rangle\langle dw| + |dw\rangle\langle cv|)w\rangle\}.\end{aligned}$$

Then use (3.4) in the first two terms and (ii) of Lemma 3.3 in the last two terms to obtain

$$\begin{aligned}\langle z, H(cv+dw)\rangle &= (-i)\{c|c|^2 i\langle z, \varphi(v)\rangle + d|d|^2 i\langle z, \varphi(w)\rangle \\ &\quad + c\bar{c}d i\langle z, \varphi(w)\rangle + dc\bar{d} i\langle z, \varphi(v)\rangle\} \\ &= (-i)\{ci\langle z, \varphi(v)\rangle + di\langle z, \varphi(w)\rangle\} \\ &= c\langle z, Hv\rangle + d\langle z, Hw\rangle. \qquad \square\end{aligned}$$

Now we are ready to prove our main result.

THEOREM 3.5. *Let $\mathscr{H}$ be a complex separable Hilbert space. Let $\mathscr{O}(\mathscr{H})$ be the real linear space of all bounded self-adjoint operators on $\mathscr{H}$. If $\delta : \mathscr{O}(\mathscr{H}) \to \mathscr{O}(\mathscr{H})$ is a continuous linear map satisfying*

$$\delta(XY) = X\delta(Y) + \delta(X)Y \quad \forall X, Y \text{ in } \mathscr{O}(\mathscr{H}) \text{ with } [X, Y] = 0.$$

Define $H : \mathscr{H} \to \mathscr{H}$ by

$$\begin{aligned}H(au_0 + bv) &= a(-i)\delta(|u_0\rangle\langle u_0|)u_0 \\ &\quad + b\{-i\delta(|v\rangle\langle v|)v + i\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle v\}\end{aligned}$$

for a fixed unit vector u_0 in $\mathscr{H}$ and $a, b \in \mathbb{C}$, $v \in \mathscr{H}$ with $\langle v, v\rangle = 1$ and $\langle v, u_0\rangle = 0$. Then H is a bounded self-adjoint operator on $\mathscr{H}$ satisfying

$$\delta(X) = i[H, X] \quad \forall X \in \mathscr{O}(\mathscr{H}).$$

Proof. We have already shown that H is linear. For any unit vector v we have

$$\langle v, Hv\rangle = i\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle.$$

(iii) of Lemma 3.2 implies

$$\langle v, Hv\rangle \in \mathbb{R}.$$

As H is defined on the whole of $\mathscr{H}$ we conclude that H is a bounded self-adjoint operator. It remains to show that

$$\delta(X) = i[H, X] \quad \forall X \in \mathscr{O}(\mathscr{H}).$$

First, we prove this for rank one projections, then for all projections and finally use the continuity of δ to prove the result for all X in $\mathscr{O}(\mathscr{H})$. It is clear that if $X = |v\rangle\langle v|$ is a rank one projection, then

$$\begin{aligned} i[H, X] &= i[H, |v\rangle\langle v|] \\ &= i(|Hv\rangle\langle v| - |v\rangle\langle Hv|) \\ &= i\{|(-i)\delta(|v\rangle\langle v|)v\rangle\langle v| + i\{\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle\}|v\rangle\langle v| \\ &\qquad - |v\rangle\langle(-i)\delta(|v\rangle\langle v|)v| \\ &\qquad\qquad + (-i)\langle u_0, \delta(|u_0\rangle\langle v| + |v\rangle\langle u_0|)v\rangle|v\rangle\langle v|\} \\ &= i(|\varphi(v)\rangle\langle v| - |v\rangle\langle\varphi(v)|) \\ &= \delta(|v\rangle\langle v|). \end{aligned}$$

Let P be a projection and v be a unit vector in the range of P. We have

$$P|v\rangle\langle v| = |v\rangle\langle v|P = |v\rangle\langle v|,$$
$$\delta(|v\rangle\langle v|) = \delta(P)|v\rangle\langle v| + P\delta(|v\rangle\langle v|).$$

Applying on v and using $g(|v\rangle\langle v|) = i[H, |v\rangle\langle v|]$ we have

$$i[H, |v\rangle\langle v|]v = \delta(P)v + P(i[H, |v\rangle\langle v|])v,$$
$$iHv - i\langle Hv, v\rangle v = \delta(P)v + iPHv - i\langle Hv, v\rangle v.$$

So

$$\delta(P)v = iHv - iPHv = iHPv - iPHv = i[H, P]v. \tag{3.14}$$

On the other hand if w is a unit vector orthogonal to the range of P,

$$P|w\rangle\langle w| = |w\rangle\langle w|P = 0.$$

So

$$\delta(P)(|w\rangle\langle w|) + P(\delta(|w\rangle\langle w|)) = 0,$$
$$\delta(P)(|w\rangle\langle w|) + P(i[H, |w\rangle\langle w|]) = 0,$$
$$|\delta(P)w\rangle\langle w| + i|PHw\rangle\langle w| = 0.$$

This means

$$\langle w, w\rangle\delta(P)w + \langle w, w\rangle iPHw = 0,$$

$$\delta(P)w = -iPHw = iHPw - iPHw = i[H, P]w. \tag{3.15}$$

Combining (3.14) and (3.15) $\delta(P) = i[H, P]$. That is $\delta(X) = i[H, X]$ whenever X is a projection. By linearity $\delta(X) = i[H, X]$ whenever X is a finite linear combination of projections. Now an application of the spectral theorem combined with the continuity of δ completes the proof. □

The proof in [1] of the fact that every derivation on a C^* algebra is bounded can be imitated to show that every linear velocity map is continuous. So we have

REMARK 3.6. Theorem 3.5 is true even without the assumption of continuity of δ.

Let $\mathscr{O}_1$, $\mathscr{O}_2$ be the spaces of self-adjoint elements of von Neumann algebras $\mathscr{A}_1$, $\mathscr{A}_2$ respectively. Then $\mathscr{O}_1 \oplus \mathscr{O}_2$ is the space of self-adjoint elements of the von Neumann algebra $\mathscr{A}_1 \oplus \mathscr{A}_2$. If δ is a linear velocity map on $\mathscr{O}_1 \oplus \mathscr{O}_2$ then we can write δ as $\delta_1 \oplus \delta_2$ where δ_1 and δ_2 are linear velocity maps on $\mathscr{O}_1$, $\mathscr{O}_2$ respectively. If δ_1 and δ_2 are inner in the sense $\delta_1(X) = i[H_1, X]$ for some H_1 in $\mathscr{O}_1$ and $\delta_2(Y) = i[H_2, Y]$ for some H_2 in $\mathscr{O}_2$, where X, Y are elements of $\mathscr{O}_1$, $\mathscr{O}_2$ respectively then δ is also inner as we have

$$\delta(X \oplus Y) = i[H_1 \oplus H_2, X \oplus Y].$$

As a corollary we have the following generalisation of Theorem 3.5.

THEOREM 3.7. *Let $\mathscr{A}$ be a subalgebra with identity of $M_n(\mathbb{C})$ for some natural number n. If δ is a linear velocity map on the space $\mathscr{O}$ of all self-adjoint elements in $\mathscr{A}$, then $\delta(X) = i[H, X]$ $X \in \mathscr{O}$, for some H in $\mathscr{O}$.*

Proof. This is clear from the discussion above as $\mathscr{A}$ is isomorphic to $M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ for some natural numbers $n_1, n_2, \ldots,$ n_k and we can use Theorem 3.5. □

Acknowledgment. The author thanks Professor K. R. Parthasarathy for suggesting and formulating the problem and also for several useful conversations.

REFERENCES

[1] G. K. Pederson, *C^*-Algebras and their Automorphism Groups*, Academic Press, London (1979).

[2] J. Dixmier, *Von Neumann Algebras*, North-Holland Publishing Company, Amsterdam, New York, Oxford (1981).

Received August 9, 1990. This research is funded by the National Board for Higher Mathematics (NBHM), India.

INDIAN STATISTICAL INSTITUTE
DELHI CENTRE
7, S.J.S. SANSANWAL MARG
NEW DELHI-16, INDIA