PACIFIC JOURNAL OF MATHEMATICS
Vol. 152, No. 2, 1992

HOMOGENEOUS DIOPHANTINE APPROXIMATION IN S-INTEGERS

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In this paper we generalize classical results in Diophantine approximation to the setting of an arbitrary number field in the context of the ring of S-integers. Specifically, we present theorems pertaining to simultaneous approximations of linear forms and develop the notion of badly approximable S-systems. In addition, we expand the subject of the geometry of numbers over the adèle ring of a number field by developing the concept of the adelic polar body. This theory is then used to produce transference theorems in this general situation.

1. Introduction. Let $\mathbb{R}^N$ denote the vector space of $N \times 1$ column vectors over $\mathbb{R}$. For

$$\vec{x} = \begin{pmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{pmatrix} \in \mathbb{R}^N,$$

we define

$$|\vec{x}| = \max_{1 \leq n \leq N} \{|x_n|\}$$

to be the supremum norm.

Suppose A is an $M \times N$ matrix over $\mathbb{R}$. Dirichlet proved that if X is a real number greater than 1 then there exist $\vec{x} \in \mathbb{Z}^N \backslash \{\vec{0}\}$ and $\vec{y} \in \mathbb{Z}^M$ satisfying

$$|\vec{x}| \leq X$$

and

$$|A\vec{x} - \vec{y}| \leq X^{-N/M}.$$

If we further assume that $A\vec{x} \notin \mathbb{Q}^M$ for all $\vec{x} \in \mathbb{Z}^N \backslash \{\vec{0}\}$, then there exist infinitely many pairs of vectors $(\vec{x}, \vec{y}) \in \mathbb{Z}^N \times \mathbb{Z}^M$ with $\vec{x} \neq \vec{0}$ and the components of $\vec{x}$ and $\vec{y}$ forming a relatively prime $(M+N)$-tuple such that

$$|\vec{x}|^N |A\vec{x} - \vec{y}|^M \leq 1. \tag{1.1}$$

Traditionally one asks if (1.1) is sharp. That is, are there examples where the quantity $|\vec{x}|^N |A\vec{x} - \vec{y}|^M$ cannot be made arbitrarily small?

The answer is yes: these are known as badly approximable systems. Specifically, we say that the $M \times N$ matrix A is a badly approximable system of linear forms if there exists a constant $\tau = \tau(A) > 0$ such that

$$\tau < |\vec{x}|^N |A\vec{x} - \vec{y}|^M$$

for all $\vec{x} \in \mathbb{Z}^N \backslash \{\vec{0}\}$ and $\vec{y} \in \mathbb{Z}^M$.

We pause momentarily to make some remarks about the $M = N = 1$ situation. An irrational number α is badly approximable if there is a constant $\tau = \tau(\alpha) > 0$ such that

$$\tau < |x|\,|\alpha x - y|$$

for all $x \in \mathbb{Z}\backslash\{0\}$ and $y \in \mathbb{Z}$. It turns out that the issue of being badly approximable is related to the subject of continued fractions. Specifically, α is badly approximable if and only if the partial quotients in its continued fraction expansion are bounded. This implies the existence of uncountably many badly approximable numbers and uncountably many numbers which are not badly approximable. It is well-known that a real number α has a periodic simple continued fraction expansion if and only if α is a quadratic irrational (see, for example, [7]). Therefore all quadratic irrationals are badly approximable numbers. It remains an open question as to whether or not real algebraic numbers of degree greater than two are badly approximable. The situation in which α is transcendental remains equally mysterious.

We return back to the badly approximable $M \times N$ matrix A. An interesting issue which now arises is the relationship between A and its transpose A^T. By the application of a transference theorem, it follows that A is a badly approximable system of linear forms if and only if A^T is a badly approximable system of linear forms. A quantitative version of this is known as Khintchine's transference principle.

Our primary objective is to recast the previous results to the more general setting of an algebraic number field. In §2 we carefully describe the relevant objects which will occur and define our notation. But briefly, let k be an algebraic number field of degree d over $\mathbb{Q}$. We write k_v for the completion of k with respect to the place v. Let S be a finite collection of places of k containing all infinite places. We write $\mathscr{O}_S$ for the ring of S-integers of k. For

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \in (k)^N,$$

we define the S-height

$$h_S(\vec{x}) = \prod_{v\in S} \max_{1\le n\le N}\{|x_n|_v\},$$

where $|\ |_v$ is normalized so as to satisfy the product formula. We define the field constant

$$\mathrm{const}_k = \left(\left(\frac{2}{\pi}\right)^s |\Delta_k|^{1/2}\right)^{1/d},$$

where s is the number of complex places of k and Δ_k is the discriminant of k.

We begin by proving the following generalization of Dirichlet's theorem.

THEOREM 1. *For each $v \in S$, let A_v be an $M\times N$ matrix over k_v and let X be a real number with $X > (\mathrm{const}_k)^{(M+N)/N}$. Then there exist $\vec{x} \in (\mathscr{O}_S)^N\backslash\{\vec{0}\}$ and $\vec{y} \in (\mathscr{O}_S)^M$ so that*

$$h_S(\vec{x}) \le X$$

and

$$\prod_{v\in S} |A_v\vec{x} - \vec{y}|_v \le (\mathrm{const}_k)^{(M+N)/M} X^{-N/M}.$$

Similarly, if we add an additional hypothesis we have:

THEOREM 2. *For each $v \in S$, let A_v be an $M\times N$ matrix over k_v. Assume that $A_v\vec{x} \notin (k)^M$ for all $v \in S$ and all $\vec{x} \in (\mathscr{O}_S)^N\backslash\{\vec{0}\}$. Then there exist infinitely many distinct pairs $(\vec{x}, \vec{y}) \in (\mathscr{O}_S)^N \times (\mathscr{O}_S)^M$ over projective space with $\vec{x} \ne \vec{0}$ satisfying*

$$(1.2)\quad h_S(\vec{x}, \vec{y})^N \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M \le (\mathrm{const}_k)^{(M+N)}(2N\mu(\{A_v\}_{v\in S}))^N,$$

where

$$\mu(\{A_v\}_{v\in S}) = \prod_{v\in S} \max\{1, |A_v|_v\}.$$

Next we show that (1.2) is sharp by demonstrating the existence of linear forms for which the fundamental quantity,

$$h_S(\vec{x}, \vec{y})^N \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M$$

cannot be made arbitrarily small. Let A_v be an $M \times N$ matrix over k_v for each $v \in S$. We shall say $\{A_v\}_{v\in S}$ is a *badly approximable S-system of linear forms* (*of dimension* $M \times N$) if there exists a constant $\tau = \tau(k, S, \{A_v\}_{v\in S}) > 0$ such that

$$\tau < h_S(\vec{x}, \vec{y})^N \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M$$

for every $\vec{x} \in (\mathscr{O}_S)^N \backslash \{\vec{0}\}$ and $\vec{y} \in (\mathscr{O}_S)^M$. We show that these S-systems always exist.

THEOREM 3. *Let k be any number field and S a finite collection of places of k containing all infinite places. Given any integers $M \geq 1$ and $N \geq 1$ there exists a badly approximable S-system of linear forms of dimension $M \times N$.*

Just as in the classical case, we may ask for the relationship between the S-systems $\{A_v\}_{v\in S}$ and $\{A_v^T\}_{v\in S}$. Toward this end in §4 we prove a transference theorem over number fields. As a consequence of our transference theorem we deduce:

THEOREM 4. *Let A_v be an $M \times N$ matrix over k_v for each $v \in S$. Then $\{A_v\}_{v\in S}$ is a badly approximable S-system of linear forms if and only if $\{A_v^T\}_{v\in S}$ is a badly approximable S-system of linear forms.*

We quantify this by proving the following generalized Khintchine's transference principle:

THEOREM 5. *Let ω be the supremum of all real numbers $\eta \geq 0$ such that there are infinitely many S-distinct pairs of vectors $(\vec{x}, \vec{y}) \in (\mathscr{O}_S)^N \times (\mathscr{O}_S)^M$ with $\vec{x} \neq \vec{0}$ satisfying*

$$h_S(\vec{x}, \vec{y})^{N(1+\eta)} \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M < 1.$$

Let ω^* be the supremum of all real numbers $\eta^* \geq 0$ so that there are infinitely many S-distinct pairs $(\vec{u}, \vec{w}) \in (\mathscr{O}_S)^M \times (\mathscr{O}_S)^N$ with $\vec{u} \neq \vec{0}$ satisfying

$$h_S(\vec{u}, \vec{w})^{M(1+\eta^*)} \prod_{v\in S} |A_v^T\vec{u} - \vec{w}|_v^N < 1.$$

Then

$$\omega^* \geq \frac{\omega}{(N-1)\omega + M + N - 1}$$

and

$$\omega \geq \frac{\omega^*}{(M-1)\omega^* + M + N - 1}.$$

Classically, theorems of this nature were proven via techniques in geometry of numbers, in particular, using Mahler's results on polar reciprocal bodies. We prove our generalized transference theorems by first developing the analog of the polar body in the setting of geometry of numbers over the adèles. We believe these results to be of independent interest with further applications outside the present work.

We organize our paper as follows:

Section 2: Notation and normalizations.

Section 3: The adelic polar body.

Section 4: A transference theorem over number fields.

Section 5: Dirichlet's theorem over number fields.

Section 6: Badly approximable S-systems of linear forms.

Acknowledgment. The author wishes to express his gratitude to Professor Jeffrey D. Vaaler for several useful comments and suggestions.

2. Notation and normalizations. In this section we define the basic terminology and notation that will be used throughout the remainder of this paper. We remark that our notation and normalizations are largely adopted from [1].

Let k be an algebraic number field of degree d over $\mathbb{Q}$. We write V_k for the collection of all nontrivial places of k. Suppose $v \in V_k$. If v is an archimedean place, we say v lies over infinity, denoted by $v|\infty$. If v is a nonarchimedean place then there exists a finite rational prime p such that v extends the place of p to V_k. In this case we say v lies over the finite rational prime p, written as $v \nmid \infty$ or $v|p$.

For each $v \in V_k$ we write k_v for the completion of k with respect to the place v. We define the local degree as

$$d_v = [k_v : \mathbb{Q}_v].$$

We now normalize two absolute values. For each place v of k, we normalize the absolute value $\| \ \|_v$ as follows:

(i) if $v|p$ then $\|p\|_v = p^{-1}$,

(ii) if $v|\infty$ then for $x \in k_v$, $\|x\|_v = |x|$ where $| \ |$ is the usual Euclidean absolute value on $\mathbb{R}$ or $\mathbb{C}$.

Thus $\| \ \|_v$ extends the usual p-adic absolute value if $v|p$ and the Euclidean absolute value if $v|\infty$. Our second normalized absolute value $| \ |_v$ is defined by

$$|x|_v = \|x\|_v^{d_v/d}.$$

This normalization gives rise to the *product formula*:

$$\prod_{v \in V_k} |x|_v = 1$$

for all $x \in k$, $x \neq 0$.

We extend our absolute values to vectors as follows. Let

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

denote a column vector in $(k_v)^N$. We define

$$|\vec{x}|_v = \max_{1 \leq n \leq N} \{|x_n|_v\}.$$

We extend the absolute value $\| \ \|_v$ in a similar manner.

Let us now consider extensions to matrices. Fix a place v of k and let $A = (a_{mn})$ be an $M \times N$ matrix over k_v. We define

$$|A|_v = \max_{\substack{1 \leq m \leq M \\ 1 \leq n \leq N}} \{|a_{mn}|_v\}.$$

Assume now that v is a finite place of k. We write $\mathscr{O}_v$ for the maximal compact (open) subring of k_v,

$$\mathscr{O}_v = \{x \in k_v : |x|_v \leq 1\}.$$

A subset R_v in $(k_v)^N$ is a *k_v-lattice* if it is a compact open $\mathscr{O}_v$-module in $(k_v)^N$. Clearly $(\mathscr{O}_v)^N$ is a k_v-lattice in $(k_v)^N$.

Let S be a finite collection of places of k containing all places lying over infinity. We define the *ring of S-integers* as

$$\mathscr{O}_S = \{x \in k : x \in \mathscr{O}_v \text{ for all } v \notin S\}.$$

We define the multiplicative *group of S-units* by

$$\mathscr{U}_S = \{x \in k : |x|_v = 1 \text{ for all } v \notin S\}.$$

For $\vec{x} \in (k)^N$ and $\vec{y} \in (k)^M$ we define the *S-height* of $\vec{x}$ and $\vec{y}$ as

$$h_S(\vec{x}, \vec{y}) = \prod_{v \in S} \max\{|\vec{x}|_v, |\vec{y}|_v\}.$$

Alternatively, we write $h_S(\vec{x})$ for the S-height of an individual vector, that is,

$$h_S(\vec{x}) = h_S(\vec{x}\,,\,\vec{0})$$

where $\vec{0}$ denotes the zero vector. Of course these are the same since

$$h_S\left(\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\right) = h_S(\vec{x}\,,\,\vec{y}),$$

where $\binom{\vec{x}}{\vec{y}} \in (k)^{M+N}$. In applications it will be convenient to use both forms of this S-height.

For each $v \in S$ let A_v be an $M \times N$ matrix over k_v. As we shall see, it will be useful to discuss all these matrices simultaneously. In view of this, we write $\{A_v\}_{v\in S}$, abbreviated as $\{A_v\}$, for the *collection of matrices* A_v *for* $v \in S$. As an example of this notation, we define the following function which is a measure of the size of the A_v's:

$$\mu(\{A_v\}_{v\in S}) = \mu(\{A_v\}) = \prod_{v\in S} \max\{1\,,\,|A_v|_v\}.$$

Let $k_{\mathbb{A}}$ denote the adèle ring of k. Elements of $k_{\mathbb{A}}$ shall be written as $x = (x_v)$ where x_v is the v-component of x for all $v \in V_k$. We write $(k_{\mathbb{A}})^N$ for the N-fold product of the adèles.

The additive group $k_{\mathbb{A}}$ is locally compact and thus there exists a Haar measure on $k_{\mathbb{A}}$ which is unique up to a multiplicative constant. We normalize this as follows.

(i) If $v|\infty$ and $k_v \cong \mathbb{R}$ we let β_v denote ordinary Lebesgue measure on $\mathbb{R}$.

(ii) If $v|\infty$ and $k_v \cong \mathbb{C}$ we let β_v denote Lebesgue measure on the complex plane multiplied by 2.

(iii) If $v|p$ we let β_v denote Haar measure on k_v normalized so that

$$\beta_v(\mathscr{O}_v) = |\mathscr{D}_v|_v^{d/2},$$

where $\mathscr{D}_v$ is the local different of k at v.

We now define a Haar measure β on $k_{\mathbb{A}}$ to be the product measure of the previously normalized local Haar measures:

$$\beta = \prod_{v\in V_k} \beta_v .$$

Technically, β determines a Haar measure on all open subgroups of the form

$$\prod_{v\in S} k_v \times \prod_{v\notin S} \mathscr{O}_v$$

where S is a finite collection of places of k containing all infinite places. Therefore the Haar measure on $k_{\mathbb{A}}$ is the unique measure which agrees with the product measure on these open subgroups. For each place v of k we let β_v^N denote the product measure on $(k_v)^N$. Similarly we define V to be the product measure β^N on $(k_{\mathbb{A}})^N$.

Let $\alpha \in k$ with $\alpha \neq 0$. By the product formula, $|\alpha|_v = 1$ for almost all v, that is $\alpha \in \mathscr{O}_v$ for almost all $v \in V_k$. Therefore $(\alpha, \alpha, \alpha, \dots) \in k_{\mathbb{A}}$ and so we may view $k \subseteq k_{\mathbb{A}}$ by the natural *diagonal map* $\eta : k \to k_{\mathbb{A}}$ defined by

$$\eta(\alpha) = (\alpha, \alpha, \alpha, \dots).$$

The set $\eta(k) \cong k$ is referred to as the set of *principal adèles.* In fact k is a discrete subset of $k_{\mathbb{A}}$ and under the natural quotient topology, $k_{\mathbb{A}}/k$ is a compact group having an induced Haar measure equal to 1. In the sequence which follows, a particular field constant will naturally arise. We define

$$\text{const}_k = \left(\left(\frac{2}{\pi} \right)^s |\Delta_k|^{1/2} \right)^{1/d},$$

where d is the degree of k over $\mathbb{Q}$, s is the number of complex places of k and Δ_k is the discriminant of k.

Let $x = (x_v)$ be an element of $k_{\mathbb{A}}$ and α be a positive real number. We define scalar multiplication, αx, to be the point $y = (y_v)$ in $k_{\mathbb{A}}$ determined by

$$y_v = \begin{cases} \alpha x_v & \text{if } v|\infty, \\ x_v & \text{if } v\nmid\infty. \end{cases}$$

We shall view elements of $(k_{\mathbb{A}})^N$ as column vectors $\vec{x}$ and extend our notion of scalar multiplication to vectors $\vec{x} \in (k_{\mathbb{A}})^N$ by

$$\alpha\vec{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_N \end{pmatrix}.$$

If $X \subseteq (k_{\mathbb{A}})^N$ then $\alpha X \subseteq (k_{\mathbb{A}})^N$ is obtained by applying scalar multiplication by α to each $\vec{x} \in X$.

We call a nonempty subset $R_v \subseteq (k_v)^N$ a *regular set* if it has the following form.

(i) If $v|\infty$ then R_v is a bounded, convex, closed, symmetric subset of $(k_v)^N$ with nonzero volume.

(ii) If $v\nmid\infty$ then R_v is a k_v-lattice in $(k_v)^N$.

For each $v \in V_k$ let R_v be a regular set in $(k_v)^N$. Assume that for almost all places v,

$$R_v = (\mathscr{O}_v)^N .$$

We now define

$$\mathscr{R} = \prod_{v \in V_k} R_v .$$

From our above assumption it is clear that $\mathscr{R} \subseteq (k_{\mathbb{A}})^N$. We shall call a subset $\mathscr{R}$ of $(k_{\mathbb{A}})^N$ *admissible* if it has the form described above. The set $\mathscr{R}$ is the adelic analog of the convex, symmetric set K in the classical geometry of numbers, and the rôle of the lattice $\mathbb{Z}^N$ in $\mathbb{R}^N$ is replaced by the discrete subgroup $(k)^N$ in $(k_{\mathbb{A}})^N$.

Let $\mathscr{R}$ be an admissible set in $(k_{\mathbb{A}})^N$. For each integer n, $1 \le n \le N$, we define the nth successive minimum λ_n of $\mathscr{R}$ with respect to $(k)^N$ by

$$\lambda_n = \inf\{\sigma > 0 : (\sigma\mathscr{R}) \cap (k)^N \text{ contains } n \text{ linearly independent vectors over } k\} .$$

By our assumptions on $\mathscr{R}$,

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N < \infty .$$

We now recall the adelic successive minima theorem which was proven independently by McFeat **[6]** and Bombieri and Vaaler (**[1]** Theorem 3).

THEOREM 2.1. *Let $\mathscr{R}$ be an admissible subset of $(k_{\mathbb{A}})^N$ and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the successive minima of $\mathscr{R}$ with respect to $(k)^N$. Then*

$$(\lambda_1\lambda_2\cdots\lambda_N)^d V(\mathscr{R}) \le 2^{dN} . \tag{2.1}$$

The following volume calculation will be quite useful. For each $v \in V_k$ let B_v be an $N \times N$ nonsingular matrix over k_v. Let

$$R_v = \{\vec{x} \in (k_v)^N : \|B_v\vec{x}\|_v \le 1\}$$

and assume that for almost all v, $R_v = (\mathscr{O}_v)^N$. Then $\mathscr{R} = \prod_{v \in V_k} R_v$ is an admissible subset of $(k_{\mathbb{A}})^N$. We now compute

$$\beta_v^N(R_v) = \begin{cases} 2^N |\det B_v|_v^{-d} & \text{if } v \text{ is real},\\ (2\pi)^N |\det B_v|_v^{-d} & \text{if } v \text{ is complex},\\ |\mathscr{D}_v|_v^{dN/2} |\det B_v|_v^{-d} & \text{if } v \nmid \infty . \end{cases}$$

Therefore

$$V(\mathscr{R}) = 2^{rN}(2\pi)^{sN}\left(\prod_{v\nmid\infty} |\mathscr{D}_v|_v^d\right)^{N/2} \prod_{v\in V_k} |\det B_v|_v^{-d}$$

where r is the number of real places of k. Hence

$$V(\mathscr{R}) = 2^{dN}\left(\frac{\pi}{2}\right)^{sN} |\Delta_k|^{-N/2} \prod_{v\in V_k} |\det B_v|_v^{-d} \tag{2.2}$$

follows from the identity

$$\prod_{v\nmid\infty} |\mathscr{D}_v|_v^d = |\Delta_k|^{-1}$$

along with the fact that $d = r + 2s$. Here Δ_k is the discriminant of k.

Alternatively we could report:

$$V(\mathscr{R}) = (2(\mathrm{const}_k)^{-1})^{dN} \prod_{v\in V_k} |\det B_v|_v^{-d}.$$

3. The adelic polar body. For each place v of k we define the bilinear form

$$\langle\cdot,\cdot\rangle : (k_v)^N \times (k_v)^N \to k_v \quad \text{by} \quad \langle\vec{x},\vec{y}\rangle = \sum_{n=1}^{N} x_n y_n$$

where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

are vectors in $(k_v)^N$. Now let $R_v \subseteq (k_v)^N$ be a regular set. For each v of k, we define the *local polar body* R_v^* by

$$R_v^* = \{\vec{x} \in (k_v)^N : |\langle\vec{x},\vec{y}\rangle|_v \le 1 \text{ for all } \vec{y} \in R_v\}.$$

LEMMA 3.1. *Let R_v be a regular set and R_v^* its polar body. Then*

(i) $R_v^* \subseteq (k_v)^N$.

(ii) R_v^* *is a regular set.*

(iii) $(R_v^*)^* = R_v$.

Proof. In case v is an infinite place, the lemma is a well-known result in linear analysis ([**2**] Chapter II.3, Corollary 3; II.4). Thus we need only prove the lemma when v is a finite place of k.

Part (i) is trivial and (ii) follows from the strong triangle inequality. We now prove (iii). Since R_v is a k_v-lattice in $(k_v)^N$, it follows that R_v is a finitely generated $\mathscr{O}_v$-module of degree N over $\mathscr{O}_v$ ([**8**] Chapter II.2, Theorem 1). Let

$$\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_N\} \subseteq (k_v)^N$$

be a basis for R_v over $\mathscr{O}_v$. Alternatively, if we let $W = (\vec{w}_1\vec{w}_2\cdots\vec{w}_N)$ be the $N \times N$ matrix with columns $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_N$, then

$$R_v = \{W\vec{\alpha} : \vec{\alpha} \in (\mathscr{O}_v)^N\}.$$

We now select linearly independent vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_N\} \subseteq (k_v)^N$ so that,

$$W^T U = \mathbf{1}_N$$

where $U = (\vec{u}_1\vec{u}_2\cdots\vec{u}_N)$ and $\mathbf{1}_N$ is the $N \times N$ identity matrix. We claim that

$$R_v^* = \{U\vec{\alpha} : \vec{\alpha} \in (\mathscr{O}_v)^N\}.$$

Let $\vec{\alpha}, \vec{\beta} \in (\mathscr{O}_v)^N$, then

$$|\langle U\vec{\alpha}, W\vec{\beta}\rangle|_v = |\vec{\beta}^T W^T U\vec{\alpha}|_v \leq \max_{1\leq n\leq N}\{|\beta_n\alpha_n|_v\} \leq 1.$$

So $R_v^* \supseteq \{U\vec{\alpha} : \vec{\alpha} \in (\mathscr{O}_v)^N\}$. Suppose now that $\vec{y} \in R_v^*$. There exists a vector $\vec{z} \in (k_v)^N$ so that $\vec{y} = U\vec{z}$. If $\vec{\alpha} \in (\mathscr{O}_v)^N$ then

$$|\langle W\vec{\alpha}, U\vec{z}\rangle|_v \leq 1. \tag{3.1}$$

Select $\vec{\alpha} = \vec{e}_n$, where $\vec{e}_n$ is the nth column of the $N \times N$ identity matrix. Inequality (3.1) reveals that

$$|\langle \vec{w}_n, U\vec{z}\rangle|_v = |\vec{e}_n^T\vec{z}|_v = |z_n|_v \leq 1 \quad \text{for } n = 1, 2, \dots, N.$$

Thus $\vec{z} \in (\mathscr{O}_v)^N$ and $R_v^* = \{U\vec{\alpha} : \vec{\alpha} \in (\mathscr{O}_v)^N\}$.

If $\vec{y} \in (R_v^*)^*$ then $\vec{y} = W\vec{\gamma}$ for some $\vec{\gamma} \in (k_v)^N$. Moreover, for all integers n, $1 \leq n \leq N$, we have

$$|\langle \vec{y}, \vec{u}_n\rangle|_v = |\vec{\gamma}^T W^T \vec{u}_n|_v = |\vec{\gamma}^T\vec{e}_n|_v = |\gamma_n|_v \leq 1,$$

so $\vec{\gamma} \in (\mathscr{O}_v)^N$ and thus $\vec{y} \in R_v$. We have now shown $(R_v^*)^* \subseteq R_v$. The reverse inclusion is trivial, so we have equality. □

If we select $W = U = \mathbf{1}_N$ in our above proof, we have:

COROLLARY 3.2. $((\mathscr{O}_v)^N)^* = (\mathscr{O}_v)^N$.

LEMMA 3.3. *Let v be any place of k and A be an $N \times N$ nonsingular matrix over k_v. Then for any regular set R_v we have*

$$(AR_v)^* = (A^T)^{-1} R_v^* .$$

Proof. Clearly $\langle \vec{x}, \vec{y} \rangle = \langle A\vec{x}, (A^T)^{-1}\vec{y} \rangle$ where $\vec{x}, \vec{y} \in (k_v)^N$. Thus $\vec{y} \in R_v^*$ if and only if $(A^T)^{-1}\vec{y} \in (AR_v)^*$. The lemma now follows. □

COROLLARY 3.4. *Let v be a finite place of k and R_v a k_v-lattice in $(k_v)^N$ with*

$$R_v = \{W\vec{\alpha} : \vec{\alpha} \in (\mathscr{O}_v)^N\}$$

where W is an $N \times N$, nonsingular matrix over k_v. Then

$$R_v^* = (W^T)^{-1} (\mathscr{O}_v)^N .$$

Proof. This follows immediately from Lemma 3.3 and Corollary 3.2. □

LEMMA 3.5. *Let R_v be a regular set in $(k_v)^N$.*

(i) *If $v|\infty$ and $k_v \cong \mathbb{R}$ then*

$$\frac{4^N}{(N!)^2} \le \beta_v^N(R_v)\beta_v^N(R_v^*) \le 4^N .$$

(ii) *If $v|\infty$ and $k_v \cong \mathbb{C}$ then*

$$\frac{4^{3N}}{\{(2N)!\}^2} \le \beta_v^N(R_v)\beta_v^N(R_v^*) \le 4^{3N} .$$

(iii) *If $v \nmid \infty$ then*

$$\beta_v^N(R_v)\beta_v^N(R_v^*) = |\mathscr{D}_v|_v^{dN}$$

where $\mathscr{D}_v$ is the local different at v.

Proof. Case (i) was proven by Mahler **[5]** in 1939. We remark that this inequality has recently been sharpened by a deep result of Bourgain and Milman **[3]**. Case (ii) follows immediately from (i) once we view R_v as a convex set inside $\mathbb{R}^{2N}$ $(\cong \mathbb{C}^N)$ and recall the extra factors of 2 due to our normalizations on β_v. We now prove (iii).

As before, let W be the $N \times N$, nonsingular matrix over k_v so that

$$R_v = \{W\vec{\alpha} : \vec{\alpha} \in (\mathscr{O}_v)^N\} .$$

Alternatively, we may express R_v as

$$R_v = \{\vec{x} \in (k_v)^N : \|W^{-1}\vec{x}\|_v \le 1\}.$$

Now from Corollary 3.4 we have

$$R_v^* = \{\vec{x} \in (k_v)^N : \|W^T\vec{x}\|_v \le 1\}.$$

Thus,

$$\beta_v^N(R_v) = \int_{(k_v)^N} \chi_{R_v}(\vec{x})\, d\beta_v^N(\vec{x}) = \int_{(k_v)^N} \chi_{(\mathscr{O}_v)^N}(W^{-1}\vec{x})\, d\beta_v^N(\vec{x}),$$

where $\chi_{R_v}(\vec{x})$ is the characteristic function of R_v. By making the change of variables $\vec{x} \to W\vec{x}$ and recalling the module in k_v ([**8**] Chapter 1, §2, Theorem 3, Corollary 3), we obtain

$$\begin{aligned}\beta_v^N(R_v) &= \operatorname{mod}_{k_v}(\det W) \int_{(k_v)^N} \chi_{(\mathscr{O}_v)^N}(\vec{x})\, d\beta_v^N(\vec{x}) \\ &= \|\det W^{-1}\|_v^{-d_v} \beta_v^N((\mathscr{O}_v)^N) \\ &= \|\det W^{-1}\|_v^{-d_v} \|\mathscr{D}_v\|_v^{d_v N/2}.\end{aligned}$$

Similarly we compute

$$\beta_v^N(R_v^*) = \|\det W^T\|_v^{-d_v} \|\mathscr{D}_v\|_v^{d_v N/2}.$$

The equality now follows. □

We now prove the adelic version of two theorems of Mahler [**5**]. Let $\mathscr{R} = \prod_v R_v$ be an admissible subset of $(k_{\mathbb{A}})^N$. We define the *adelic polar body* $\mathscr{R}^*$ by

$$\mathscr{R}^* = \prod_v R_v^*.$$

By Lemma 3.1 and Corollary 3.2 we see that $\mathscr{R}^*$ is another admissible subset of $(k_{\mathbb{A}})^N$ and $(\mathscr{R}^*)^* = \mathscr{R}$. Below we give bounds on the size of $\mathscr{R}^*$ relative to the size of $\mathscr{R}$.

THEOREM 3.6. *Let $\mathscr{R}$ be an admissible subset of $(k_{\mathbb{A}})^N$ and $\mathscr{R}^*$ its polar body. Then*

$$\frac{4^{(d+s)N}}{(N!)^{2r}[(2N)!]^{2s}|\Delta_k|^N} \le V(\mathscr{R})V(\mathscr{R}^*) \le \frac{4^{(d+s)N}}{|\Delta_k|^N}$$

where r and s are the number of real and complex places of k respectively and Δ_k is the discriminant of k.

Proof. The theorem follows from Lemma 3.5 after recalling that $d = r + 2s$ and

$$\prod_{v \nmid \infty} |\mathscr{D}_v|_v^{-d} = |\Delta_k|. \qquad \square$$

We now demonstrate the reciprocal relationship between the successive minima of $\mathscr{R}$ and $\mathscr{R}^*$.

THEOREM 3.7. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the successive minima associated with $\mathscr{R}$ and $\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*$ the successive minima associated with $\mathscr{R}^*$. For $n = 1, 2, \ldots, N$,*

$$1 \le (\lambda_n \lambda_{N+1-n}^*)^d \le \frac{(N!)^{2r}[(2N)!]^{2s}|\Delta_k|^N}{4^{sN}}.$$

Before proving this result we shall need to consider vectors associated with successive minima. Let $\vec{\varphi}_1, \vec{\varphi}_2, \ldots, \vec{\varphi}_N \in (k)^N$ be linearly independent vectors associated with the successive minima of $\mathscr{R}$. That is, for all integers n, $1 \le n \le N$, and any real number $\lambda > \lambda_n$, $\{\vec{\varphi}_1, \vec{\varphi}_2, \ldots, \vec{\varphi}_n\} \subseteq \lambda\mathscr{R}$. Let $\vec{\theta}_1, \vec{\theta}_2, \ldots, \vec{\theta}_N \in (k)^N$ be linearly independent vectors associated with the successive minima of $\mathscr{R}^*$.

LEMMA 3.8. *For any integers i, j with $1 \le i \le N$ and $1 \le j \le N$ we have*

$$\|\langle \vec{\varphi}_i, \vec{\theta}_j \rangle\|_v \le \begin{cases} \lambda_i \lambda_j^* & \text{for } v|\infty, \\ 1 & \text{for } v \nmid \infty. \end{cases}$$

Proof. For $v|\infty$ we let $\lambda > \lambda_i$ and $\lambda^* > \lambda_j^*$. Then

$$\lambda^{-1}\vec{\varphi}_i \in R_v \quad \text{and} \quad (\lambda^*)^{-1}\vec{\theta}_j \in R_v^*.$$

Thus by definition

$$\|\langle \lambda^{-1}\vec{\varphi}_i, (\lambda^*)^{-1}\vec{\theta}_j \rangle\|_v = (\lambda\lambda^*)^{-1}\|\langle \vec{\varphi}_i, \vec{\theta}_j \rangle\|_v \le 1.$$

Letting $\lambda \to \lambda_i$ and $\lambda^* \to \lambda_j^*$ establishes the first inequality. The second follows from the fact that when $v \nmid \infty$

$$\vec{\varphi}_i \in R_v \quad \text{and} \quad \vec{\theta}_j \in R_v^*. \qquad \square$$

Proof of Theorem 3.7. We first consider the lower bound. Fix an index n, $1 \le n \le N$. Define Φ, an $N \times n$ matrix over k by

$$\Phi = (\vec{\varphi}_1 \vec{\varphi}_2 \cdots \vec{\varphi}_n).$$

Clearly, by linear independence, Φ has full rank n. We now view Φ^T as a linear transformation:

$$\Phi^T : (k)^N \to (k)^n .$$

Let $\mathscr{K} \subseteq (k)^N$ denote the kernel of Φ^T, that is,

$$\mathscr{K} = \{\vec{x} \in (k)^N : \Phi^T \vec{x} = \vec{0}\} .$$

This forms an $N-n$ dimensional subspace of $(k)^N$. Since the $N+1-n$ vectors, $\vec{\theta}_1, \vec{\theta}_2, \dots, \vec{\theta}_{N+1-n}$, are linearly independent in $(k)^N$, there must exist an index j, with $1 \le j \le N+1-n$, so that $\vec{\theta}_j \notin \mathscr{K}$. That is, there must exist an index j such that

$$|\langle \vec{\varphi}_i , \vec{\theta}_j \rangle|_v \neq 0 \quad \text{for } 1 \le i \le n .$$

By Lemma 3.8 we have that if $v|\infty$ then

$$0 \neq |\langle \vec{\varphi}_i , \vec{\theta}_j \rangle|_v = (\|\langle \vec{\varphi}_i , \vec{\theta}_j \rangle\|_v)^{d_v/d} \le (\lambda_i \lambda_j^*)^{d_v/d} .$$

Hence, by the product formula and Lemma 3.8 we conclude that

$$\prod_v |\langle \vec{\varphi}_i , \vec{\theta}_j \rangle|_v = 1 \le \prod_{v|\infty} (\lambda_i \lambda_j^*)^{d_v/d} \cdot \prod_{v\nmid\infty} 1 .$$

Recalling that $1 \le i \le n$ and $1 \le j \le N+1-n$ we have

$$1 \le \lambda_i \lambda_j^* \le \lambda_n \lambda_{N+1-n}^* .$$

Since n was arbitrary, we have shown that for all $n = 1, 2, \dots, N$,

$$1 \le (\lambda_n \lambda_{N+1-n}^*)^d . \tag{3.2}$$

For the upper bound, we use the adelic successive minima theorem (2.1) to report that

$$(\lambda_1 \lambda_2 \cdots \lambda_N)^d V(\mathscr{R}) \le 2^{dN}$$

and

$$(\lambda_1^* \lambda_2^* \cdots \lambda_N^*)^d V(\mathscr{R}^*) \le 2^{dN} .$$

Multiplying these two inequalities and using Theorem 3.6 gives

$$\prod_{n=1}^{N} (\lambda_n \lambda_{N+1-n}^*)^d \le 4^{dN} \{V(\mathscr{R}) V(\mathscr{R}^*)\}^{-1} \le \frac{(N!)^{2r} [(2N)!]^{2s} |\Delta_k|^N}{4^{sN}} .$$

The result now follows from (3.2). □

As an application, we state and prove an adelic general transfer principle. Assume that $N > 1$. For each place v of k, let C_v be the v-adic cube:

$$C_v = \{\vec{x} \in (k_v)^N : \|\vec{x}\|_v \le 1\} .$$

We easily compute the polar body C_v^* to be

$$C_v^* = \begin{cases} \left\{\vec{x} \in (k_v)^n : \sum_{n=1}^{N} \|x_n\|_v \leq 1\right\} & \text{for } v|\infty, \\ \{\vec{x} \in (k_v)^N : \|\vec{x}\|_v \leq 1\} & \text{for } v\nmid\infty. \end{cases}$$

Let B_v be an $N \times N$ nonsingular matrix over k_v and define $R_v \subseteq (k_v)^N$ by

$$R_v = B_v^{-1} C_v .$$

Assume that for almost all v, $R_v = (\mathscr{O}_v)^N$. Define $T_v \subseteq (k_v)^N$ by

$$T_v = B_v^T C_v .$$

Let

$$\mathscr{R} = \prod_v R_v \quad \text{and} \quad \mathscr{T} = \prod_v T_v .$$

It is clear that $\mathscr{R}$ and $\mathscr{T}$ are admissible subsets of $(k_{\mathbb{A}})^N$.

THEOREM 3.9. *Let $\mathscr{R}$ and $\mathscr{T}$ be as above and λ_1, μ_1 the first successive minima of $\mathscr{R}$ and $\mathscr{T}$, respectively. Then*

$$\lambda_1 \leq c_k(N)\mu_1^{1/(N-1)} \left(\prod_v |\det B_v|_v\right)^{1/(N-1)}$$

and

$$\mu_1 \leq c_k(N)\lambda_1^{1/(N-1)} \left(\prod_v |\det B_v|_v\right)^{-1/(N-1)}$$

where

$$c_k(N) = \{N(\text{const}_k)^N\}^{1/(N-1)} .$$

Proof. From Lemma 3.3,

$$R_v^* = B_v^T C_v^* .$$

We claim that

$$\frac{1}{N}\mathscr{T} \subseteq \mathscr{R}^* \subseteq \mathscr{T} . \tag{3.3}$$

If $v\nmid\infty$ then $\vec{\alpha} \in T_v$ if and only if there exists $\vec{\beta} \in (\mathscr{O}_v)^N$ with $\vec{\alpha} = B_v^T \vec{\beta} \in R_v^*$. Thus

$$T_v = R_v^* \quad \text{for } v\nmid\infty .$$

Assume now that $v|\infty$. If $\vec{\alpha} \in \frac{1}{N} T_v$ then there exists a $\vec{\beta} \in (k_v)^N$ such that $\|\vec{\beta}\|_v \leq 1$ and $\vec{\alpha} = \frac{1}{N} B_v^T \vec{\beta}$. Since $\frac{1}{N}\sum_{n=1}^N \|\beta_n\|_v \leq \|\vec{\beta}\|_v \leq$

1, we have $\frac{1}{N}\vec{\beta} \in C_v^*$ and $\vec{\alpha} \in R_v^*$, thus $\frac{1}{N}T_v \subseteq R_v^*$, establishing our first containment. If $\sum_{n=1}^{N} \|\beta_n\|_v \leq 1$ then $\|\vec{\beta}\|_v \leq 1$. Thus if $\vec{\beta} \in C_v^*$ then $\vec{\beta} \in C_v$, establishing the second containment and proving the claim.

It follows from (3.3) that for each $n = 1, 2, \ldots, N$

$$\mu_n \leq \lambda_n^* \leq N\mu_n. \tag{3.4}$$

Again by the adelic successive minima theorem we have that

$$(\lambda_1\lambda_2 \cdots \lambda_N)^d V(\mathscr{R}) \leq 2^{dN},$$

so

$$\lambda_1^{d(N-1)} \leq 2^{dN}\{\lambda_N^d V(\mathscr{R})\}^{-1}.$$

From Theorem 3.7 we deduce

$$\lambda_1^{d(N-1)} \leq (\lambda_1^*)^d (2^{dN}/V(\mathscr{R})). \tag{3.5}$$

A simple calculation reveals that

$$V(\mathscr{R}) = 2^{dN} \left(\frac{\pi}{2}\right)^{sN} |\Delta_k|^{-N/2} \prod_v |\det B_v|_v^{-d}. \tag{3.6}$$

So from (3.6) along with (3.4) and (3.5) we conclude that

$$\lambda_1 \leq (N\mu_1)^{1/(N-1)} \left(\left(\frac{2}{\pi}\right)^{sN} |\Delta_k|^{N/2}\right)^{1/[d(N-1)]} \left(\prod_v |\det B_v|_v\right)^{1/(N-1)},$$

which is the first inequality. The second follows from symmetry. □

In some applications, we are given that a certain admissible set contains a non-zero point of $(k)^N$ and we wish to conclude that there exists a non-zero point of $(k)^N$ contained inside of a related admissible set. In light of this remark, the following corollary will be useful.

COROLLARY 3.10. *If there exists a* $\vec{\zeta} \in (k)^N \backslash \{\vec{0}\}$ *with* $\vec{\zeta} \in \mathscr{R}$, *then there exists an* $\vec{\xi} \in (k)^N \backslash \{\vec{0}\}$ *satisfying*:

$$\vec{\xi} \in c_k(N) \left\{\prod_v |\det B_v|_v\right\}^{-1/(N-1)} \mathscr{T}.$$

Proof. If $\vec{\zeta} \in \mathscr{R}$ then $\lambda_1 \leq 1$, so by Theorem 3.9,

$$\mu_1 \leq c_k(N) \left\{\prod_v |\det B_v|_v\right\}^{-1/(N-1)}.$$

The corollary follows by the definition of μ_1 and noting that each T_v is closed. □

4. A transference theorem over number fields. Transference theorems in Diophantine approximation are results in which information about a particular linear system implies information about the transposed linear system.

Let S be a finite collection of places of k containing all infinite places and let $\mathcal{O}_S$ be the ring of S-integers.

THEOREM 4.1. *For each $v \in S$, let A_v be an $M \times M$ matrix over k_v. Assume there exist vectors $\vec{x} \in (\mathcal{O}_S)^N$ and $\vec{y} \in (\mathcal{O}_S)^M$, not both identically zero, so that*

$$\|\vec{x}\|_v \le \|\delta_v\|_v \quad \text{and} \quad \|A_v\vec{x} - \vec{y}\|_v \le \|\varepsilon_v\|_v$$

where $\delta_v, \varepsilon_v \in k_v \backslash \{0\}$, for each $v \in S$. Let

$$\Lambda = \left(\prod_{v \in S} |\delta_v|_v^N |\varepsilon_v|_v^M \right)^{1/(M+N-1)} .$$

Then there exist vectors $\vec{u} \in (\mathcal{O}_S)^M$ and $\vec{w} \in (\mathcal{O}_S)^N$, not both identically zero, so that

$$\|\vec{u}\|_v \le \gamma_v \|\varepsilon_v\|_v^{-1} \quad \text{and} \quad \|A_v^T \vec{u} - \vec{w}\|_v \le \gamma_v \|\delta_v\|_v^{-1} \quad \text{for all } v \in S,$$

where

$$\gamma_v = \begin{cases} c_k(M+N)\Lambda & \text{if } v|\infty, \\ 1 & \text{if } v\nmid\infty. \end{cases} \tag{4.1}$$

Proof. For each place v of k, we define an $(M+N) \times (M+N)$ matrix B_v over k_v by the following rule: if $v \in S$ then

$$B_v = \left(\begin{array}{c|c} \delta_v^{-1} \mathbf{1}_N & 0 \\ \hline \varepsilon_v^{-1} A_v & \varepsilon_v^{-1} \mathbf{1}_M \end{array} \right),$$

if $v \notin S$ then $B_v = \mathbf{1}_{(M+N)}$ where $\mathbf{1}_N$ is the $N \times N$ identity matrix. We note that for $v \in S$,

$$(B_v^T)^{-1} = \left(\begin{array}{c|c} \delta_v \mathbf{1}_N & -\delta_v A_v^T \\ \hline 0 & \varepsilon_v \mathbf{1}_M \end{array} \right)$$

and for $v \notin S$, $(B_v^T)^{-1} = \mathbf{1}_{(M+N)}$. Now define R_v and $T_v \subseteq (k_v)^{M+N}$ by:

$$R_v = B_v^{-1} C_v \quad \text{and} \quad T_v = B_v^T C_v$$

where C_v is the $(M+N)$ dimensional v-adic cube as defined in §3. Let $\mathscr{R} = \prod_v R_v$ and $\mathscr{T} = \prod_v T_v$. It is clear that $\mathscr{R} \subseteq (k_{\mathbb{A}})^{M+N}$ and $\mathscr{T} \subseteq (k_{\mathbb{A}})^{M+N}$. Let

$$\vec{\zeta} = \begin{pmatrix} \vec{x} \\ -\vec{y} \end{pmatrix} \in (k)^{M+N} \backslash \{\vec{0}\}.$$

From our hypothesis we have that $\vec{\zeta} \in \mathscr{R}$. By Corollary 3.10 we conclude that there exists an $\vec{\xi} \in (k)^{M+N} \backslash \{\vec{0}\}$ satisfying:

$$\begin{aligned}\vec{\xi} &\in c_k(M+N) \left\{ \prod_v |\det B_v|_v \right\}^{-1/(M+N-1)} \mathscr{T} \\ &= c_k(M+N) \left\{ \prod_{v \in S} |\delta_v|_v^N |\varepsilon_v|_v^M \right\}^{1/(M+N-1)} \mathscr{T}.\end{aligned}$$

The theorem follows once $\vec{\xi}$ is viewed as

$$\vec{\xi} = \begin{pmatrix} \vec{w} \\ \vec{u} \end{pmatrix}. \qquad \square$$

REMARKS. 1. Upon first examination, it appears that we may always find a $\vec{u}$ and $\vec{w}$ in $(\mathscr{O}_S)^M$ and $(\mathscr{O}_S)^N$, respectively, that will satisfy the theorem by selecting $\vec{u} = \vec{0}$ and $\vec{w} \in (\mathscr{O}_S)^N$ to be v-adically small for all $v \in S$. However in practice we view $\prod_{v \in S} |\delta_v|_v$ as "large" and $\prod_{v \in S} |\varepsilon_v|_v$ as "small". In this situation, $\vec{u}$ being zero would force $\vec{w}$ to be zero, which is impossible. For if $\vec{u} = \vec{0}$ then we could report that

$$\|\vec{w}\|_v \leq \begin{cases} \gamma_v \|\delta_v\|_v^{-1} & \text{for } v \in S, \\ 1 & \text{for } v \notin S. \end{cases} \tag{4.2}$$

Thus by (4.1) and (4.2),

$$\begin{aligned}\prod_v |\vec{w}|_v &\leq \prod_{v \in S} (\gamma_v \|\delta_v\|_v^{-1})^{-d_v/d} = \prod_{v|\infty} (\gamma_v)^{d_v/d} \prod_{v \in S} |\delta_v|_v^{-1} \\ &= c_k(M+N) \left(\prod_{v \in S} |\delta_v|_v^N |\varepsilon_v|_v^M \right)^{1/(M+N-1)} \prod_{v \in S} |\delta_v|_v^{-1} \\ &= c_k(M+N) \prod_{v \in S} \left\{ |\delta_v|_v^{(1-M)/(M+N-1)} |\varepsilon_v|_v^{M/(M+N-1)} \right\}.\end{aligned} \tag{4.3}$$

Therefore, if $\prod_{v \in S} |\delta_v|_v$ is sufficiently large and $\prod_{v \in S} |\varepsilon_v|_v$ is sufficiently small then

$$\prod_v |\vec{w}|_v < 1.$$

But by the product formula this implies that $\vec{w} = \vec{0}$, which is a contradiction; thus $\vec{u} \neq \vec{0}$. Specifically, if

$$\prod_{v\in S} |\delta_v|_v \geq 1 \quad \text{and} \quad \prod_{v\in S} |\varepsilon_v|_v < c_k(M+N)^{-(M+N-1)/M}$$

then from (4.3) we may conclude that $\vec{u} \neq \vec{0}$.

2. Given that for each $v \in S$,

$$\|\vec{x}\|_v \leq \|\delta_v\|_v \quad \text{and} \quad \|A_v\vec{x} - \vec{y}\|_v \leq \|\varepsilon_v\|_v ,$$

we may deduce an upper bound for $h_S(\vec{x}, \vec{y})$ and $\prod_{v\in S} |A_v\vec{x} - \vec{y}|_v$.

For $v|\infty$ we have

$$\begin{aligned} \|\vec{y}\|_v &= \|A_v\vec{x} - \vec{y} - A_v\vec{x}\|_v \\ &\leq 2\max\{\|\varepsilon_v\|_v , \|A_v\vec{x}\|_v\} \\ &\leq 2\max\{\|\varepsilon_v\|_v , N\|A_v\|_v\|\vec{x}\|_v\} \\ &\leq 2N\max\{\|\varepsilon_v\|_v , \|A_v\|_v\|\delta_v\|_v\} \\ &\leq 2N\max\{1 , \|A_v\|_v\}\max\{\|\varepsilon_v\|_v , \|\delta_v\|_v\} . \end{aligned}$$

So $|\vec{y}|_v \leq (2N)^{d_v/d}\max\{1 , |A_v|_v\}\max\{|\varepsilon_v|_v , |\delta_v|_v\}$.

For $v \in S$, $v\nmid\infty$ we have:

$$|\vec{y}|_v \leq \max\{1 , |A_v|_v\}\max\{|\varepsilon_v|_v , |\delta_v|_v\} .$$

Hence

$$(4.4) \qquad \begin{cases} h_S(\vec{x}, \vec{y}) \leq 2N\mu(\{A_v\})\prod_{v\in S}\max\{|\varepsilon_v|_v , |\delta_v|_v\} \\ \text{and} \\ \prod_{v\in S}|A_v\vec{x} - \vec{y}|_v \leq \prod_{v\in S}|\varepsilon_v|_v , \end{cases}$$

where

$$\mu(\{A_v\}) = \prod_{v\in S}\max\{1 , |A_v|_v\} .$$

Similarly, from the conclusion of Theorem 4.1,

$$\|\vec{u}\|_v \leq \gamma_v\|\varepsilon_v\|_v^{-1} \quad \text{and} \quad \|A_v^T\vec{u} - \vec{w}\|_v \leq \gamma_v\|\delta_v\|_v^{-1} \quad \text{for each } v \in S,$$

we may report the following upper bounds.

For $v|\infty$:

$$\|\vec{w}\|_v \leq 2M\gamma_v\max\{1 , \|A_v\|_v\}\max\{\|\delta_v\|_v^{-1} , \|\varepsilon_v\|_v^{-1}\}$$

and for $v \in S$, $v\nmid\infty$:

$$\|\vec{w}\|_v \leq \gamma_v\max\{1 , \|A_v\|_v\}\max\{\|\delta_v\|_v^{-1} , \|\varepsilon_v\|_v^{-1}\} .$$

Therefore we conclude

$$(4.5)\quad \begin{cases} h_S(\vec{u}, \vec{w}) \\ \quad \leq 2Mc_k(M+N)\Lambda\mu(\{A_v\}) \prod_{v\in S} \max\{|\delta_v|_v^{-1}, |\varepsilon_v|_v^{-1}\} \\ \text{and} \\ \prod_{v\in S} |A_v^T\vec{u} - \vec{w}|_v \leq c_k(M+N)\Lambda \prod_{v\in S} |\delta_v|_v^{-1}. \end{cases}$$

5. Dirichlet's theorem over number fields.

LEMMA 5.1. *For each $v \in S$, let A_v be an $M \times N$ matrix over k_v and $\varepsilon_v \in k_v \backslash \{0\}$ with $|\varepsilon_v|_v < 1$. Select $\delta_v \in k_v \backslash \{0\}$ so that $|\delta_v|_v \geq 1$ for each $v \in S$ and*

$$(5.1)\qquad \prod_{v\in S} |\delta_v|_v^N |\varepsilon_v|_v^M = (\text{const}_k)^{(M+N)}.$$

Then there exist $\vec{x} \in (\mathscr{O}_S)^N \backslash \{\vec{0}\}$ and $\vec{y} \in (\mathscr{O}_S)^M$ satisfying

$$\|\vec{x}\|_v \leq \|\delta_v\|_v \quad \text{and} \quad \|A_v\vec{x} - \vec{y}\|_v \leq \|\varepsilon_v\|_v \quad \text{for all } v \in S.$$

REMARKS. We may always find $\delta_v \in k_v \backslash \{0\}$ for all $v \in S$ that satisfy (5.1) since S contains the infinite places of k. Again the constant const_k is given by

$$\text{const}_k = \left(\left(\frac{2}{\pi}\right)^s |\Delta_k|^{1/2}\right)^{1/d}.$$

Proof. For each place v of k we define B_v, an $(M+N)\times(M+N)$ matrix over k_v by:

$$B_v = \left(\begin{array}{c|c} \delta_v^{-1}\mathbf{1}_N & 0 \\ \hline \varepsilon_v^{-1}A_v & \varepsilon_v^{-1}\mathbf{1}_M \end{array}\right)$$

for all $v \in S$ and $B_v = \mathbf{1}_{(M+N)}$ for all $v \notin S$. Define $R_v \subseteq (k_v)^{M+N}$ by

$$R_v = \{\vec{z} \in (k_v)^{M+N} : \|B_v\vec{z}\|_v \leq 1\}$$

and let $\mathscr{R} = \prod_v R_v$. Clearly $\mathscr{R} \subseteq (k_{\mathbb{A}})^{M+N}$ and is admissible. We note that

$$\begin{aligned} V(\mathscr{R}) &= 2^{d(M+N)} \left(\frac{\pi}{2}\right)^{s(M+N)} |\Delta_k|^{-(M+N)/2} \left(\prod_v |\det B_v|_v\right)^{-d} \\ &= 2^{d(M+N)} \left(\left(\frac{\pi}{2}\right)^s |\Delta_k|^{-1/2}\right)^{(M+N)} \left(\prod_v |\delta_v|_v^N |\varepsilon_v|_v^M\right)^d \\ &= 2^{d(M+N)}. \end{aligned}$$

Theorem 2.1 reveals that

$$(\lambda_1\lambda_2\cdots\lambda_{(M+N)})^d V(\mathscr{R}) \le 2^{d(M+N)}.$$

Thus given the above volume calculation and the fact that the λ_n's are increasing, we have $\lambda_1 \le 1$. Hence there must exist a point $\binom{\vec{x}}{-\vec{y}} \in (k)^{M+N}\backslash\{\vec{0}\}$ so that

$$\begin{pmatrix} \vec{x} \\ -\vec{y} \end{pmatrix} \in \mathscr{R}.$$

By our definition of R_v for $v \notin S$ we see that

$$\begin{pmatrix} \vec{x} \\ -\vec{y} \end{pmatrix} \in (\mathscr{O}_S)^{M+N}.$$

In addition, for $v \in S$ we have

$$\|\vec{x}\|_v \le \|\delta_v\|_v \quad \text{and} \quad \|A_v\vec{x} - \vec{y}\|_v \le \|\varepsilon_v\|_v .$$

Now if $\vec{x} = \vec{0}$ then the previous inequality reduces to $\|\vec{y}\|_v \le \|\varepsilon_v\|_v < 1$ for all $v \in S$, so $h_S(\vec{y}) < 1$. Since $\vec{y} \in (\mathscr{O}_S)^M$, the product formula implies that $\vec{y} = \vec{0}$, which is impossible. Therefore we must have

$$\vec{x} \ne \vec{0}. \qquad \square$$

THEOREM 5.2. *For each $v \in S$, let A_v be an $M \times N$ matrix over k_v and let X be a real number with $X > (\text{const}_k)^{(M+N)/N}$. Then there exist $\vec{x} \in (\mathscr{O}_S)^N\backslash\{\vec{0}\}$ and $\vec{y} \in (\mathscr{O}_S)^M$ so that*

$$h_S(\vec{x}) \le X$$

and

$$\prod_{v\in S} |A_v\vec{x} - \vec{y}|_v \le (\text{const}_k)^{(M+N)/M} X^{-N/M}.$$

Proof. For each $v \in S$, select $\varepsilon_v \in k_v\backslash\{0\}$ so that $|\varepsilon_v|_v < 1$ and

$$\prod_{v\in S} |\varepsilon_v|_v^{-M} = (\text{const}_k)^{-(M+N)} X^N. \tag{5.2}$$

By Lemma 5.1, there exist $\vec{x} \in (\mathscr{O}_S)^N\backslash\{\vec{0}\}$ and $\vec{y} \in (\mathscr{O}_S)^M$ so that $|\vec{x}|_v \le |\delta_v|_v$ and $|A_v\vec{x} - \vec{y}|_v \le |\varepsilon_v|_v$ for each $v \in S$. Thus,

$$h_S(\vec{x}) \le \prod_{v\in S} |\delta_v|_v \quad \text{and} \quad \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v \le \prod_{v\in S} |\varepsilon_v|_v .$$

The theorem now follows from (5.1) and (5.2). $\square$

COROLLARY 5.3. *Let $\alpha_v \in k_v$ for each $v \in S$ and let $X > (\text{const}_k)^2$. Then there exist $x \in \mathscr{O}_S\backslash\{0\}$ and $y \in \mathscr{O}_S$ satisfying*

$$h_S(x) \le X$$

and

$$\prod_{v\in S}\left|\alpha_v - \frac{y}{x}\right|_v \le \frac{(\text{const}_k)^2}{X h_S(x)}.$$

Proof. This follows immediately from Theorem 5.2 with $M = N = 1$. □

The following two results are actually theorems about points in projective space over k. We shall phrase this in the following manner. Let $(\vec{x}_t, \vec{y}_t) \in (\mathscr{O}_S)^N \times (\mathscr{O}_S)^M$ with $\vec{x}_t \ne \vec{0}$ for $t = 1, 2$. We shall say that $(\vec{x}_1, \vec{y}_1)$ is *equivalent* to $(\vec{x}_2, \vec{y}_2)$, $(\vec{x}_1, \vec{y}_1) \sim (\vec{x}_2, \vec{y}_2)$, if there exists a constant $\alpha \in k$ such that $(\vec{x}_1, \vec{y}_1) = (\alpha\vec{x}_2, \alpha\vec{y}_2)$. If $(\vec{x}_1, \vec{y}_1)$ and $(\vec{x}_2, \vec{y}_2)$, are not equivalent we say they are *nonequivalent* or *relatively prime.* In addition, we shall say that $(\vec{x}_1, \vec{y}_1)$ is *S-equivalent* to $(\vec{x}_2, \vec{y}_2)$, $(\vec{x}_1, \vec{y}_1) \sim_S (\vec{x}_2, \vec{y}_2)$, if there exists a constant $u \in \mathscr{U}_S$ such that $(\vec{x}_1, \vec{y}_1) = (u\vec{x}_2, u\vec{y}_2)$. If two vectors are not S-equivalent we say that they are *non S-equivalent* or *S-distinct.* Clearly both $\sim$ and $\sim_S$ are equivalence relations on $(\mathscr{O}_S)^N \times (\mathscr{O}_S)^M$.

THEOREM 5.4. *For each $v \in S$, let A_v be an $M\times N$ matrix over k_v. Assume that $A_v\vec{x} \notin (k)^M$ for all $v \in S$ and all $\vec{x} \in (\mathscr{O}_S)^N\backslash\{\vec{0}\}$. Then there exist infinitely many relatively prime pairs $(\vec{x}, \vec{y}) \in (\mathscr{O}_S)^N\times(\mathscr{O}_S)^M$ with $\vec{x} \ne \vec{0}$ satisfying*

$$h_S(\vec{x}, \vec{y})^N \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M \le (\text{const}_k)^{(M+N)}(2N\mu(\{A_v\}))^N, \tag{5.3}$$

where

$$\mu(\{A_v\}) = \prod_{v\in S} \max\{1, |A_v|_v\}.$$

Proof. Assume that there are only finitely many nonequivalent pairs of vectors $(\vec{x}_1, \vec{y}_1), (\vec{x}_2, \vec{y}_2), \dots, (\vec{x}_T, \vec{y}_T)$ which satisfy the above conditions. For each integer t, $1 \le t \le T$, let $\vec{z}_t \in (\mathscr{O}_S)^{M+N}\backslash\{\vec{0}\}$ be defined by:

$$\vec{z}_t = \begin{pmatrix}\vec{x}_t\\ \vec{y}_t\end{pmatrix}.$$

For $\vec{z} \in (\mathscr{O}_S)^{M+N}\backslash\{\vec{0}\}$ define the set $E[\vec{z}] \subseteq k$ by setting

$$E[\vec{z}] = \{\alpha \in k\backslash\{0\} : \alpha\vec{z} \in (\mathscr{O}_S)^{M+N}\}.$$

We now claim that for each $1 \le t \le T$,

$$\inf_{\alpha \in E[\vec{z}_t]} \left\{ \prod_{v \in S} |A_v(\alpha \vec{x}_t) - (\alpha \vec{y}_t)|_v \right\} > 0 .$$

To prove this, let $\alpha \in E[\vec{z}_t]$. Clearly

$$\prod_{v \in S} |A_v(\alpha \vec{x}_t) - (\alpha \vec{y}_t)|_v = h_S(\alpha) \prod_{v \in S} |A_v \vec{x}_t - \vec{y}_t|_v$$

and by our hypothesis, $\prod_{v \in S} |A_v \vec{x}_t - \vec{y}_t|_v \neq 0$. Thus it is enough to show that

$$\inf_{\alpha \in E[\vec{z}_t]} \{h_S(\alpha)\} > 0 . \tag{5.4}$$

We observe that $\alpha \in E[\vec{z}_t]$ implies that

$$\prod_{v \notin S} |\alpha|_v |\vec{z}_t|_v \le 1$$

or

$$\prod_{v \notin S} |\alpha|_v \le \prod_{v \notin S} |\vec{z}_t|_v^{-1} .$$

By the product formula, since $\alpha \neq 0$

$$1 = \prod_{v} |\alpha|_v = \prod_{v \in S} |\alpha|_v \cdot \prod_{v \notin S} |\alpha|_v \le \prod_{v \in S} |\alpha|_v \cdot \prod_{v \notin S} |\vec{z}_t|_v^{-1} .$$

Hence

$$0 < \prod_{v \notin S} |\vec{z}_t|_v \le \prod_{v \in S} |\alpha|_v ,$$

so

$$0 < \prod_{v \notin S} |\vec{z}_t|_v \le \inf_{\alpha \in E[\vec{z}_t]} \{h_S(\alpha)\}$$

which establishes our claim.

Let

$$\delta = \min_{1 \le t \le T} \left\{ 1, \inf_{\alpha \in E[\vec{z}_t]} \left\{ \prod_{v \in S} |A_v(\alpha \vec{x}_t) - (\alpha \vec{y}_t)|_v \right\} \right\} .$$

From the claim above, $\delta > 0$. For each $v \in S$, select $\varepsilon_v \in k_v \backslash \{0\}$ so that $|\varepsilon_v|_v < 1$ and

$$0 < \prod_{v \in S} |\varepsilon_v|_v < \delta .$$

We apply Lemma 5.1 with this choice of $\{\varepsilon_v\}_{v \in S}$. Therefore there exist $\vec{x} \in (\mathscr{O}_S)^N \backslash \{\vec{0}\}$ and $\vec{y} \in (\mathscr{O}_S)^M$ satisfying

$$\|\vec{x}\|_v \le \|\delta_v\|_v \quad \text{and} \quad \|A_v \vec{x} - \vec{y}\|_v \le \|\varepsilon_v\|_v \quad \text{for each } v \in S .$$

By (4.4) we conclude that

$$h_S(\vec{x}, \vec{y}) \leq 2N\mu(\{A_v\}) \prod_{v\in S} |\delta_v|_v$$

and

$$\prod_{v\in S} |A_v\vec{x} - \vec{y}|_v \leq \prod_{v\in S} |\varepsilon_v|_v$$

since $|\varepsilon_v|_v < 1 \leq |\delta_v|_v$ for all $v \in S$. It follows from (5.1) that $(\vec{x}, \vec{y})$ satisfies (5.3), and thus must be equivalent to one of the pairs of vectors from our finite collection. That is, there exists an integer j, $1 \leq j \leq T$, so that $(\vec{x}, \vec{y}) \sim (\vec{x}_j, \vec{y}_j)$, hence there exists an $\alpha \in k$ such that

$$(\vec{x}, \vec{y}) = (\alpha\vec{x}_j, \alpha\vec{y}_j).$$

Clearly $\alpha \in E[\vec{z}_j]$ so

$$\prod_{v\in S} |A_v(\alpha\vec{x}_j) - (\alpha\vec{y}_j)|_v = \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v$$
$$\leq \prod_{v\in S} |\varepsilon_v|_v < \delta \leq \prod_{v\in S} |A_v(\alpha\vec{x}_j) - (\alpha\vec{y}_j)|_v$$

which is impossible. Thus $(\vec{x}, \vec{y})$ satisfies (5.3) but is not equivalent to any member of our finite set. Therefore there must be infinitely many relatively prime pairs of vectors in $(\mathscr{O}_S)^N \times (\mathscr{O}_S)^M$ which satisfy (5.3). □

COROLLARY 5.5. *For each* $v \in S$, *let* $\alpha_v \in k_v \backslash k$. *Then there exist infinitely many relatively prime pairs* $(x, y) \in (\mathscr{O}_S)^2$ *with* $x \neq 0$ *satisfying*

$$h_S(x, y) \prod_{v\in S} |\alpha_v x - y|_v \leq 2(\mathrm{const}_k)^2 \mu(\{\alpha_v\}).$$

Proof. This follows immediately from Theorem 5.4 with $M = N = 1$. □

REMARK. In the proof of Theorem 5.4 we saw that

$$\inf_{\alpha\in E[\vec{z}]} \{h_S(\alpha)\} > 0$$

(see inequality (5.4)). Here we remark that the infimum is actually attained, that is, there exists an $\alpha' \in E[\vec{z}]$ such that

$$0 < h_S(\alpha') = \inf_{\alpha\in E[\vec{z}]} \{h_S(\alpha)\}.$$

We prove this below.

Since $\vec{z} \in (\mathscr{O}_S)^{M+N}\backslash\{\vec{0}\}$, $|\vec{z}|_v = 1$ for almost all $v \in V_k$. Let $v_1, v_2, \ldots, v_J$ be the set of all nonarchimedean places of k such that $|\vec{z}|_v \neq 1$. For each v_j let z_j be the component of $\vec{z}$ such that

$$|z_j|_{v_j} = |\vec{z}|_{v_j}.$$

Let

$$\varepsilon = \min_{1\leq j\leq J}\{|z_j|_{v_j}^{-1}\}$$

and select $\tilde{v} \in S$ such that $\tilde{v}|\infty$. Clearly $\varepsilon > 0$ and thus by the strong approximation theorem, there exists a $\beta \in k\backslash\{0\}$ such that $|\beta|_v \leq 1$ for each $v \notin \{\tilde{v}, v_1, v_2, \ldots, v_J\}$ and

$$|z_j^{-1} - \beta|_{v_j} < \varepsilon \leq |z_j|_{v_j}^{-1} \tag{5.5}$$

for $j = 1, 2, \ldots, J$. Now select $\zeta \in k$ so that $|\zeta\beta|_v = 1$ for all $v \notin S \cup \{v_1, v_2, \ldots, v_J\}$ and $|\zeta|_{v_j} = 1$ for all $j = 1, 2, \ldots, J$. From (5.5) we have equality in the strong triangle inequality:

$$\begin{aligned}|\zeta\beta|_{v_j} &= |\zeta(z_j^{-1} - \beta) - \zeta z_j^{-1}|_{v_j}\\ &= \max\{|\zeta(z_j^{-1} - \beta)|_{v_j}, |\zeta z_j^{-1}|_{v_j}\}\\ &= |\zeta z_j^{-1}|_{v_j} = |z_j|_{v_j}^{-1}\end{aligned}$$

for all $j = 1, 2, \ldots, J$. We now claim that $\zeta\beta \in E[\vec{z}]$. Let $v \notin S$ and assume $v \notin \{v_1, v_2, \ldots, v_J\}$. Then $|\zeta\beta|_v = 1$ and $|\vec{z}|_v = 1$ so $(\zeta\beta)\vec{z} \in (\mathscr{O}_v)^{M+N}$. Assume now that $v = v_j$ for some $1 \leq j \leq J$. Then

$$|(\zeta\beta)\vec{z}|_v = |\zeta\beta|_{v_j}|\vec{z}|_{v_j} = |z_j|_{v_j}^{-1}|z_j|_{v_j} = 1,$$

so $(\zeta\beta)\vec{z} \in (\mathscr{O}_v)^{M+N}$. Thus, $(\zeta\beta)\vec{z} \in (\mathscr{O}_S)^{M+N}$ and therefore $\zeta\beta \in E[\vec{z}]$. We claim that

$$h_S(\zeta\beta) = \inf_{\alpha\in E[\vec{z}]}\{h_S(\alpha)\}.$$

If not, then there exists a $\gamma \in E[\vec{z}]$ such that

$$h_S(\gamma) < h_S(\zeta\beta).$$

Thus by the product formula there must exist a place $\tilde{v} \notin S$ so that $|\gamma|_{\tilde{v}} > |\zeta\beta|_{\tilde{v}}$. Hence

$$|\gamma\vec{z}|_{\tilde{v}} > |(\zeta\beta)\vec{z}|_{\tilde{v}} = 1$$

and so $\gamma \notin E[\vec{z}]$, a contradiction. Thus,

$$h_S(\zeta\beta) = \inf_{\alpha\in E[\vec{z}]}\{h_S(\alpha)\}. \qquad \square$$

6. Badly approximable S-systems of linear forms. We now show that Theorems 5.4 and 5.5 are sharp by demonstrating the existence of linear forms for which the fundamental quantity

$$h_S(\vec{x}, \vec{y})^N \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M$$

cannot be made arbitrarily small. In this section we define such forms and use Theorem 4.1 to prove some transference results.

Before we develop the theory of badly approximable S-systems, we consider the classical situation. Let A be an $M \times N$ matrix over $\mathbb{R}$. The linear system A is said to be badly approximable if there exists a constant $\tau = \tau(A, M, N) > 0$ so that

$$\tau < |\vec{x}|^N |A\vec{x} - \vec{y}|^M \tag{6.1}$$

for all $\vec{x} \in \mathbb{Z}^N \backslash \{\vec{0}\}$ and all $\vec{y} \in \mathbb{Z}^M$. Alternatively, it is easy to show that the linear system A is badly approximable if and only if there exists a constant $\tau' = \tau'(A, M, N) > 0$ so that

$$\tau' < \max\{|\vec{x}|, |\vec{y}|\}^N |A\vec{x} - \vec{y}|^M \tag{6.2}$$

for all $\vec{x} \in \mathbb{Z}^N \backslash \{\vec{0}\}$ and $\vec{y} \in \mathbb{Z}^M$.

This leads us to the following definition. Let A_v be an $M \times N$ matrix over k_v for each $v \in S$. We say $\{A_v\}_{v\in S}$ is a *badly approximable S-system of linear forms* (*of dimension* $M \times N$) if there exists a constant $\tau = \tau(k, S, \{A_v\}) > 0$ such that

$$\tau < h_S(\vec{x}, \vec{y})^N \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M$$

for every $\vec{x} \in (\mathscr{O}_S)^N \backslash \{\vec{0}\}$ and $\vec{y} \in (\mathscr{O}_S)^M$.

In view of our transference theorem of §4 we prove the following.

THEOREM 6.2. *Let A_v be an $M \times N$ matrix over k_v for each $v \in S$. Then $\{A_v\}_{v\in S}$ is a badly approximable S-system of linear forms if and only if $\{A_v^T\}_{v\in S}$ is a badly approximable S-system of linear forms.*

Proof. Assume that $\{A_v^T\}_{v\in S}$ is a badly approximable S-system of linear forms. Let $\tau' > 0$ be a constant so that

$$h_S(\vec{u}, \vec{w})^M \prod_{v\in S} |A_v^T\vec{u} - \vec{w}|_v^N > \tau' \tag{6.3}$$

for all $\vec{u} \in (\mathscr{O}_S)^M \backslash \{\vec{0}\}$ and $\vec{w} \in (\mathscr{O}_S)^N$. Let $\vec{x} \in (\mathscr{O}_S)^N \backslash \{\vec{0}\}$ and $\vec{y} \in (\mathscr{O}_S)^M$. We claim that, for all $v \in S$, $A_v\vec{x} - \vec{y} \neq \vec{0}$. If there were to exist a $\tilde{v}$ in S so that $A_{\tilde{v}}\vec{x} - \vec{y} = \vec{0}$ then we could select an $\varepsilon_{\tilde{v}} \in k_{\tilde{v}}\backslash\{0\}$ so $\tilde{v}$-adically small that Theorem 4.1 along with (4.5) would contradict (6.3).

We wish to find a $\tau > 0$ so that

$$h_S(\vec{x}, \vec{y})^N \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M > \tau . \tag{6.4}$$

Select δ_v, $\varepsilon_v \in k_v$ so that

$$\|\delta_v\|_v = \max\{\|\vec{x}\|_v, \|\vec{y}\|_v\} \quad \text{and} \quad \|\varepsilon_v\|_v = \|A_v\vec{x} - \vec{y}\|_v \tag{6.5}$$

for each $v \in S$. From our above claim, $\|\varepsilon_v\|_v > 0$ for all $v \in S$. Since $\vec{x} \neq \vec{0}$, $h_S(\vec{x}, \vec{y}) \geq 1$ and so we may assume that

$$\prod_{v\in S} |\varepsilon_v|_v^M < c_k(M+N)^{-(M+N-1)} \tag{6.6}$$

otherwise (6.4) is trivial: select $\tau = c_k(M+N)^{-(M+N-1)}$. We apply Theorem 4.1 with $\vec{x}$, $\vec{y}$, $\{\varepsilon_v\}$, $\{\delta_v\}$ as above and conclude that there exist $\vec{u} \in (\mathscr{O}_S)^M\backslash\{\vec{0}\}$ and $\vec{w} \in (\mathscr{O}_S)^N$ satisfying

$$\|\vec{u}\|_v \leq \gamma_v\|\varepsilon_v\|_v^{-1} \quad \text{and} \quad \|A_v^T\vec{u} - \vec{w}\|_v \leq \gamma_v\|\delta_v\|_v^{-1} \tag{6.7}$$

for all $v \in S$. The fact that $\vec{u} \neq \vec{0}$ follows from (6.6) and the first remark after Theorem 4.1. We now compute upper bounds for $|\vec{w}|_v$.

For $v|\infty$:

$$\begin{aligned}\|\vec{w}\|_v &= \|A_v^T\vec{u} - \vec{w} - A_v^T\vec{u}\|_v \\ &\leq 2M\max\{\gamma_v\|\delta_v\|_v^{-1}, \gamma_v\|A_v^T\|_v\|\varepsilon_v\|_v^{-1}\}.\end{aligned}$$

However,

$$\begin{aligned}\|\varepsilon_v\|_v &= \|A_v\vec{x} - \vec{y}\|_v \leq 2\max\{\|A_v\vec{x}\|_v, \|\vec{y}\|_v\} \\ &\leq 2N\max\{1, \|A_v\|_v\}\max\{\|\vec{x}\|_v, \|\vec{y}\|_v\} \\ &= 2N\max\{1, \|A_v\|_v\}\|\delta_v\|_v .\end{aligned}$$

Therefore putting these two inequalities together we conclude that

$$\begin{aligned}\|\vec{w}\|_v &\leq 2M\gamma_v \max\left\{2N\max\{1, \|A_v\|_v\}, \|A_v^T\|_v\right\}\|\varepsilon_v\|_v^{-1} \\ &\leq 4MN\gamma_v\max\{1, \|A_v\|_v\}\|\varepsilon_v\|_v^{-1} .\end{aligned}$$

So for $v|\infty$, $|\vec{w}|_v \leq (4MN\gamma_v)^{d_v/d}\max\{1, |A_v|_v\}|\varepsilon_v|_v^{-1}$. Similarly, for $v\nmid\infty$, $v \in S$:

$$|\vec{w}|_v \leq (\gamma_v)^{d_v/d}\max\{1, |A_v|_v\}|\varepsilon_v|_v^{-1} .$$

Hence from (6.7) we have

$$h_S(\vec{u}, \vec{w}) \le 4MNc_k(M+B)\Lambda\mu(\{A_v\}) \prod_{v\in S} |\varepsilon_v|_v^{-1}$$

and

$$\prod_{v\in S} |A_v^T\vec{u} - \vec{w}|_v \le c_k(M+N)\Lambda \prod_{v\in S} |\delta_v|_v^{-1},$$

where we recall $\Lambda = (\prod_{v\in S} |\delta_v|_v^N |\varepsilon_v|_v^M)^{1/(M+N-1)}$. Thus,

$$\begin{aligned} h_S(\vec{u}, \vec{w})^M &\prod_{v\in S} |A_v^T\vec{u} - \vec{w}|_v^N \\ &\le (4MN\mu(\{A_v\}))^M c_k(M+N)^{(M+N)}\Lambda^{(M+N)} \left(\prod_{v\in S} |\delta_v|_v^N |\varepsilon_v|_v^M\right)^{-1} \\ &= (4MN\mu(\{A_v\}))^M c_k(M+N)^{(M+N)}\Lambda. \end{aligned}$$

Since $\vec{u} \in (\mathscr{O}_S)^M \backslash \{\vec{0}\}$ and $\vec{w} \in (\mathscr{O}_S)^N$, (6.3) is satisfied, and so

$$\left\{\tau'(4MN\mu(\{A_v\}))^{-M} c_k(M+N)^{-(M+N)}\right\}^{(M+N-1)} \le \Lambda^{(M+N-1)}.$$

Therefore from (6.5) we conclude that

$$h_S(\vec{x}, \vec{y})^N \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M > \tau,$$

where

$$\tau = \min\left\{\{\tau'(4MN\mu(\{A_v\}))^{-M} c_k(M+N)^{-(M+N)}\}^{(M+N-1)}, \; c_k(M+N)^{-(M+N-1)}\right\}.$$

Since $\vec{x}$ and $\vec{y}$ were arbitrary, we have that $\{A_v\}_{v\in S}$ is a badly approximable S-system of linear forms. The reverse implication follows from symmetry. □

We now give a quantitative version of Theorem 6.2 by proving the generalized "Khintchine's transference principle" (see [7] Chapter IV, §5).

THEOREM 6.3. *Let ω be the supremum of all real numbers $\eta \ge 0$ such that there are infinitely many S-distinct pairs of vectors $(\vec{x}, \vec{y}) \in (\mathscr{O}_S)^N \times (\mathscr{O}_S)^M$ with $\vec{x} \ne \vec{0}$ satisfying*

$$h_S(\vec{x}, \vec{y})^{N(1+\eta)} \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M < 1. \tag{6.8}$$

Let ω^* be the supremum of all real numbers $\eta^* \geq 0$ so that there are infinitely many S-distinct pairs $(\vec{u}, \vec{w}) \in (\mathcal{O}_S)^M \times (\mathcal{O}_S)^N$ with $\vec{u} \neq \vec{0}$ satisfying

$$h_S(\vec{u}, \vec{w})^{M(1+\eta^*)} \prod_{v \in S} |A_v^T \vec{u} - \vec{w}|_v^N < 1. \tag{6.9}$$

Then

$$\omega^* \geq \frac{\omega}{(N-1)\omega + M + N - 1} \tag{6.10}$$

and

$$\omega \geq \frac{\omega}{(M-1)\omega^* + M + N - 1}. \tag{6.11}$$

Proof. We shall first prove inequality (6.10). Clearly if $\omega = 0$ then (6.10) is satisfied. We now assume that $\omega > 0$. We further assume that if $N > 1$ then $\omega^* < \frac{1}{N-1}$ (otherwise (6.10) follows immediately). Suppose that $0 < \eta < \omega$, $\eta^* > \omega^*$ and similarly if $N > 1$ then $\eta^* < \frac{1}{N-1}$.

Since $\eta < \omega$, (6.8) is satisfied for infinitely many S-distinct pairs $(\vec{x}, \vec{y}) \in (\mathcal{O}_S)^N \times (\mathcal{O}_S)^M$ with $\vec{x} \neq \vec{0}$. For each such pair $(\vec{x}, \vec{y})$ and $v \in S$, select $\gamma_v = \gamma_v(\vec{x}, \vec{y}) \in k_v$ so that if

$$\begin{cases} \|\delta_v\|_v = \max\{\|\vec{x}\|_v, \|\vec{y}\|_v\} \\ \text{and} \\ \|\varepsilon_v\|_v = \max\{\|A_v\vec{x} - \vec{y}\|_v, \|\gamma_v\|_v\} \end{cases} \tag{6.12}$$

then $\|\varepsilon_v\|_v > 0$ and

$$\prod_{v \in S} (|\delta_v|_v^{N(1+\eta)} |\varepsilon_v|_v^M) < 1. \tag{6.13}$$

Since (6.8) is satisfied for infinitely many S-distinct $(\vec{x}, \vec{y})$, we may force $\prod_{v\in S} |\varepsilon_v|_v$ to be small by making $\prod_{v \in S} |\delta_v|_v$ sufficiently large. By Theorem 4.1 and the remarks which follow it, we may find $\vec{u} \in (\mathcal{O}_S)^M$ and $\vec{w} \in (\mathcal{O}_S)^N$ with $\vec{u} \neq \vec{0}$ satisfying

$$\begin{cases} h_S(\vec{u}, \vec{w}) \leq 4MNc_k(M+N)\Lambda\mu(\{A_v\}) \prod_{v\in S} |\varepsilon_v|_v^{-1} \\ \text{and} \\ \prod_{v \in S} |A_v^T\vec{u} - \vec{w}|_v \leq c_k(M+N)\Lambda \prod_{v\in S} |\delta_v|_v^{-1}, \end{cases} \tag{6.14}$$

where

$$\Lambda = \left(\prod_{v \in S} |\delta_v|_v^N |\varepsilon_v|_v^M\right)^{1/(M+n-1)}.$$

Now since $\eta^* > \omega^*$, (6.9) is satisfied for only finitely many $(\vec{u}, \vec{w}) \in (\mathscr{O}_S)^M \times (\mathscr{O}_S)^N$ with $\vec{u} \neq \vec{0}$, say $(\vec{u}_1, \vec{w}_1), (\vec{u}_2, \vec{w}_2), \dots, (\vec{u}_J, \vec{w}_J)$. Let

$$\Gamma = \min_{1 \le j \le J} \left\{ \prod_{v \in S} |A_v^T \vec{u}_j - \vec{w}_j|_v \right\}.$$

We claim that $\Gamma > 0$. If $\Gamma = 0$ then there would exist an integer l, $1 \le l \le J$, and a place $\tilde{v} \in S$ such that $A_{\tilde{v}}^T \vec{u}_l - \vec{w}_l = \vec{0}$. Select an infinite sequence of S-distinct points $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathscr{O}_S$. Then $\{(\alpha_n \vec{u}_l, \alpha_n \vec{w}_l)\}_{n=1}^{\infty}$ is an infinite collection of S-distinct vectors in $(\mathscr{O}_S)^N \times (\mathscr{O}_S)^M$, all of which satisfy

$$\prod_{v \in S} |A_v^T(\alpha_n \vec{u}_l) - (\alpha_n \vec{w}_l)|_v = 0$$

and thus trivially satisfy (6.9), a contradiction. Therefore $\Gamma > 0$. We now make $\prod_{v \in S} |\delta_v|_v$ so large that

$$c_k(M+N)\Lambda \prod_{v \in S} |\delta_v|_v^{-1} < \Gamma.$$

Thus since Γ was the minimum, the $(\vec{u}, \vec{w})$ which satisfies (6.14) is not S-equivalent to any pair from our finite collection. Therefore it does not satisfy (6.9) so

$$\left(\prod_{v \in S} |A_v^T \vec{u} - \vec{w}|_v \right) h_S(\vec{u}, \vec{w})^{\frac{M}{N}(1+\eta^*)} \ge 1.$$

By the upper bounds of (6.14), the above inequality reveals that

$$c_k(M+N)\Lambda \prod_{v \in S} |\delta_v|_v^{-1} \cdot \left(4MNc_k(M+N)\Lambda\mu(\{A_v\}) \prod_{v \in S} |\varepsilon_v|_v^{-1} \right)^{\frac{M}{N}(1+\eta^*)} \ge 1.$$

Raising both sides to the $(N(M+N-1))$th power and recalling the definition of Λ yields

$$Q \left(\prod_{v \in S} |\delta_v|_v \right)^{N+MN\eta^*} \left(\prod_{v \in S} |\varepsilon_v|_v \right)^{M-M(N-1)\eta^*} \ge 1,$$

where

$$Q = c_k(M+N)^{(M+N-1)(N+M(1+\eta^*))} (4MN\mu(\{A_v\}))^{(M+N-1)(1+\eta^*)M}.$$

Since $\eta^* < \frac{1}{N-1}$, $M - M(N-1)\eta^* > 0$. Taking $(M - M(N-1)\eta^*)$th roots of both sides reveals that

$$Q' \prod_{v \in S} |\varepsilon_v|_v \left\{ \prod_{v \in S} |\delta_v|_v \right\}^{(N/M)((1+M\eta^*)/(1-(N-1)\eta^*))} \geq 1,$$

where

$$Q' = Q^{1/(M-M(N-1)\eta^*)}.$$

Since we may choose $\prod_{v \in S} |\delta_v|_v$ to be large, the previous inequality along with (6.13) leads us to conclude that

$$1 + \eta \leq \frac{1 + M\eta^*}{1 - (N-1)\eta^*}$$

or

$$\eta^* \geq \frac{\eta}{(N-1)\eta + M + N - 1}.$$

Letting $\eta^* \to \omega^*$ from above and $\eta \to \omega$ from below gives (6.10). Inequality (6.11) follows from symmetry. □

Below we demonstrate the existence of badly approximable S-system of linear forms of arbitrary dimension. We begin with the following

THEOREM 6.4. *Suppose* $\{1, \alpha_1, \alpha_2, \dots, \alpha_N\}$ *is a basis for the algebraic number field* K *of degree* $N+1$ *over* k. *Let us further assume that* K *may be embedded into* k_v *for each* $v \in S$. *Let* $\vec{\alpha}^T = (\alpha_1 \alpha_2 \cdots \alpha_N)$, *so* $\vec{\alpha}$ *may be viewed as a vector in* $(k_v)^N$ *for each* $v \in S$. *Then* $\{\vec{\alpha}^T\}_{v \in S}$ *is a badly approximable* S*-system of linear forms of dimension* $1 \times N$.

Proof. For any place v of k, let

$$V_{K,v} = \{w \in V_K : w | v\}.$$

By our hypothesis, for each $v \in S$, K may be embedded into k_v, therefore there exists a $\tilde{w} = \tilde{w}(v) \in V_{K,v}$, so that $[K_{\tilde{w}} : k_v] = 1$. Thus $\| \, \|_{\tilde{w}} = \| \, \|_v$ and so

$$(6.15) \qquad | \, |_{\tilde{w}} = \| \, \|_{\tilde{w}}^{[K_{\tilde{w}} : k_v][k_v : \mathbb{Q}_v]/[K : k][k : \mathbb{Q}]} = | \, |_v^{1/[K:k]}.$$

Let $\vec{x} \in (\mathcal{O}_S)^N \backslash \{\vec{0}\}$ and $y \in \mathcal{O}_S$. Let $\beta = \sum_{n=1}^N \alpha_n x_n - y \in K$. From (6.15) we conclude

$$(6.16) \qquad \prod_{v \in S} |\beta|_v = \prod_{\substack{\tilde{w} \in V_{K,v} \\ v \in S}} |\beta|_{\tilde{w}}^{N+1}.$$

By linear independence, $\beta \neq 0$, so by the product formula

$$1 = \prod_{v \in V_K} |\beta|_w = \prod_{v \in V_K} \left\{ \prod_{w \in V_{K,v}} |\beta|_w \right\}^{N+1} .$$

We now conclude

$$(6.17) \quad 1 = \prod_{v \in S} \left\{ \prod_{w \in V_{K,v}} |\beta|_w \right\}^{N+1} \prod_{v \notin S} \left\{ \prod_{w \in V_{K,v}} |\beta|_w \right\}^{N+1}$$

$$= \prod_{\substack{\tilde{w} \in V_{K,v} \\ v \in S}} |\beta|_{\tilde{w}}^{N+1} \prod_{\substack{w \in V_{K,v} \\ v \in S \\ w \neq \tilde{w}}} |\beta|_w^{N+1} \prod_{v \notin S} \left\{ \prod_{w \in V_{K,v}} |\beta|_w \right\}^{N+1} .$$

Since $\vec{x} \in (\mathscr{O}_S)^N$ and $y \in \mathscr{O}_S$ we easily have:

$$\prod_{v \notin S} \left\{ \prod_{w \in V_{K,v}} |\beta|_w \right\}^{N+1} \leq \prod_{v \notin S} \left\{ \prod_{w \in V_{K,v}} \max\{1, |\vec{\alpha}|_w\} \right\}^{N+1} .$$

Define the constant $B_1 = B_1(K, k, S, \vec{\alpha})$ as

$$B_1 = \prod_{v \notin S} \left\{ \prod_{w \in V_{K,v}} \max\{1, |\vec{\alpha}|_w\} \right\}^{N+1} .$$

Therefore from (6.16), (6.17) and the previous bound:

$$(6.18) \qquad 1 \leq \prod_{v \in S} |\beta|_v \cdot \prod_{\substack{w \in V_{K,v} \\ v \in S \\ w \neq \tilde{w}}} |\beta|_w^{N+1} \cdot B_1 .$$

It is easy to check that

$$\prod_{\substack{w \in V_{K,v} \\ v \in S \\ w \neq \tilde{w}}} |\beta|_w^{N+1} \leq B_2 \left\{ \prod_{\substack{w \in V_{K,v} \\ v \in S \\ w \neq \tilde{w}}} \max\{|\vec{x}|_w, |y|_w\} \right\}^{N+1} ,$$

where

$$B_2 = B_2(K, k, S, \vec{\alpha}) = \left\{ \prod_{\substack{w \in V_{K,v} \\ v \in S}} 2N \max\{1, |\vec{\alpha}|_w\} \right\}^{N+1} .$$

So from (6.18) and the above inequality we conclude

$$1 \le \prod_{v\in S} |\beta|_v \left\{ \prod_{\substack{w\in V_{K,v} \\ v\in S \\ w\neq\tilde{w}}} \max\{|\vec{x}|_w\,,\,|y|_w\} \right\}^{N+1} B_1 B_2 . \tag{6.19}$$

We note that for $z \in k$ and $w|v$,

$$\|z\|_w = \|z\|_v$$

and by (6.15)

$$|z|_w = |z|_v^{[K_w\,:\,k_v]/[K\,:\,k]} .$$

Since $\vec{x} \in (k)^N$ and $y \in k$, (6.19) reveals

$$\begin{aligned}
1 &\le \prod_{v\in S} |\beta|_v \left\{ \prod_{\substack{w\in V_{K,v} \\ v\in S \\ w\neq\tilde{w}}} \max\{|\vec{x}|_v\,,\,|y|_v\}^{[K_w\,:\,k_v]} \right\}^{\frac{N+1}{N+1}} \cdot B_1 B_2 \\
&= \prod_{v\in S} |\beta|_v \left\{ \prod_{v\in S} \max\{|\vec{x}|_v\,,\,|y|_v\}^{\sum_{w|v}[K_w\,:\,k_v]-[K_{\tilde{w}}\,:\,k_v]} \right\} B_1 B_2 \\
&= \prod_{v\in S} |\beta|_v \prod_{v\in S} \max\{|\vec{x}|_v\,,\,|y|_v\}^{[K\,:\,k]-1} \cdot B_1 B_2 \\
&= \prod_{v\in S} |\beta|_v \prod_{v\in S} \max\{|\vec{x}|_v\,,\,|y|_v\}^{N} \cdot B_1 B_2 \\
&= \prod_{v\in S} \left| \sum_{n=1}^{N} \alpha_n x_n - y \right|_v \cdot h_S(\vec{x}\,,\,y)^N B_1 B_2 .
\end{aligned}$$

Hence

$$h_S(\vec{x}\,,\,y)^N \prod_{v\in S} \left| \sum_{n=1}^{N} \alpha_n x_n - y \right|_v > \frac{1}{2B_1B_2}$$

and therefore $\{\vec{\alpha}^T\}_{v\in S}$ is a badly approximable S-system of linear forms. □

We proceed now to show the existence of badly approximable S-systems of linear forms of dimension $M \times N$.

Let $\{a_1\,,\,a_2\,,\,\dots\,,\,a_L\}$ be a complete set of distinct conjugates contained in some splitting field of k. Let

$$f(x) = \sum_{l=0}^{L} c_l x^l \in k[x]$$

be the monic irreducible polynomial associated with $\{a_l\}_{l=1}^{L}$. That is,

$$f(x) = \prod_{l=1}^{L}(x - a_l).$$

We now recall that the coefficients $c_l \in k$ are the elementary symmetric polynomials of $a_1, a_2, \dots, a_L$. Write

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{pmatrix}$$

and let

$$P(y) = \sum_{l=0}^{L-1} x_{l+1} y^l.$$

Define the polynomial $F(\vec{x}) = \prod_{l=1}^{L} P(a_l)$. Trivially the coefficients of F are symmetric polynomials of $a_1, a_2, \dots, a_L$. By the Fundamental Theorem on Symmetric Functions, every symmetric polynomial of $a_1, a_2, \dots, a_L$ is a polynomial in $c_0, c_1, \dots, c_{L-1}$. We have already noted that the elementary symmetric polynomials of $a_1, a_2, \dots, a_L$ are elements of k. Therefore the coefficients of $F(\vec{x})$ are also in k. Thus we have just discovered that

$$F(\vec{x}) \in k[x_1, x_2, \dots, x_L].$$

Alternatively, we may describe $F(\vec{x})$ in the following manner. For each integer l, $l = 1, 2, \dots, L$, we define the L dimension column vector

$$\vec{a}_l = \begin{pmatrix} 1 \\ a_l \\ a_l^2 \\ \vdots \\ a_l^{L-1} \end{pmatrix}.$$

Hence

$$F(\vec{x}) = \prod_{l=1}^{L}(\vec{a}_l^T \vec{x}) \in k(a_1, a_2, \dots, a_L)[x_1, x_2, \dots, x_L].$$

We remark that $F(\vec{x})$ is a norm form. From our previous observations we have seen that

$$\prod_{l=1}^{L}(\vec{a}_l^T \vec{x}) \in k[x_1, x_2, \dots, x_L].$$

It now follows that we may always find an element $\gamma \in k\backslash\{0\}$ so that if we consider the complete set of distinct conjugates

$$\alpha_1, \alpha_2, \ldots, \alpha_L,$$

where $\alpha_l = \gamma a_l$ for each $l = 1, 2, \ldots, L$, then

$$G(\vec{x}) = \prod_{l=1}^{L} (\vec{\alpha}_l^T \vec{x}) \in \mathcal{O}_S[x_1, x_2, \ldots, x_L].$$

This inspires the definition below.

A set $\{\alpha_1, \alpha_2, \ldots, \alpha_L\}$ contained in some splitting field of k is called a *set of S-algebraic integers* if the following three conditions hold.

(i) $\{\alpha_1, \alpha_2, \ldots, \alpha_L\}$ is a complete set of distinct conjugates.

(ii) The field $k(\alpha_1, \alpha_2, \ldots, \alpha_L)$ may be embedded into k_v for each $v \in S$.

(iii) The polynomial $G(\vec{x}) = \prod_{l=1}^{L} (\vec{\alpha}_l^T \vec{x})$ is an element of $\mathcal{O}_S[x_1, x_2, \ldots, x_L]$.

From our above remarks we conclude that any set $\{\alpha_1, \alpha_2, \ldots, \alpha_L\}$ which satisfies conditions (i) and (ii) may be multiplied by a suitable constant of k so as to produce a set of S-algebraic integers.

Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_{M+N}\}$ is a set of S-algebraic integers. We define the following two matrices:

$$\Phi_{11} = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{M-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{M-1} \\ \vdots & \vdots & & & \vdots \\ 1 & \alpha_M & \alpha_M^2 & \cdots & \alpha_M^{M-1} \end{pmatrix}$$

and

$$\Phi_{12} = \begin{pmatrix} \alpha_1^M & \alpha_1^{M+1} & \cdots & \alpha_1^{M+N-1} \\ \alpha_2^M & \alpha_2^{M+1} & \cdots & \alpha_2^{M+N-1} \\ \vdots & \vdots & & \vdots \\ \alpha_M^M & \alpha_M^{M+1} & \cdots & \alpha_M^{M+N-1} \end{pmatrix}.$$

Clearly Φ_{11} is a nonsingular $M \times M$ matrix. For each $v \in S$ we define the $M \times N$ matrix $A_v(\alpha_1, \alpha_2, \ldots, \alpha_M)$ over k_v as

$$A_v(\alpha_1, \alpha_2, \ldots, \alpha_M) = -(\Phi_{11})^{-1}(\Phi_{12}).$$

We now prove that badly approximable S-systems of linear forms of dimension $M \times N$ exist.

THEOREM 6.5. *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{M+N}\}$ be a set of S-algebraic integers. For each $v \in S$ let A_v be the $M \times N$ matrix over k_v defined by*

$$A_v = A_v(\alpha_1, \alpha_2, \ldots, \alpha_M).$$

Then $\{A_v\}_{v\in S}$ is a badly approximable S-system of linear forms of dimension $M \times N$.

Proof. Since there is an embedding of $k(\alpha_1, \alpha_2, \ldots, \alpha_{M+N})$ into k_v, we may view

$$\vec{\alpha}_l = \begin{pmatrix} 1 \\ \alpha_l \\ \alpha_l^2 \\ \vdots \\ \alpha_l^{M+N-1} \end{pmatrix}$$

as a vector in $(k_v)^{M+N}$ for each integer $l = 1, 2, \ldots, M+N$ and each $v \in S$.

Suppose $K = k(\alpha_1) = \{\vec{\alpha}_1^T \vec{u} : \vec{u} \in (k)^{M+N}\}$. Then clearly by condition (iii),

$$\mathrm{Norm}_{K/k}(\vec{\alpha}_1^T \vec{u}) = \prod_{l=1}^{M+N} \vec{\alpha}_l^T \vec{u} \in \mathscr{O}_S[u_1, u_2, \ldots, u_{M+N}].$$

Thus for all $\vec{u} \in (\mathscr{O}_S)^{M+N} \backslash \{\vec{0}\}$,

$$1 \le \prod_{v\in S} \left| \prod_{l=1}^{M+N} \vec{\alpha}_l^T \vec{u} \right|_v . \tag{6.20}$$

We now define the $(M+N) \times (M+N)$ Vandermonde matrix Φ by:

$$\Phi = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{M-1} & \alpha_1^M & \cdots & \alpha_1^{M+N-1} \\ \vdots & & & & \vdots & \vdots & & \vdots \\ 1 & \alpha_M & \alpha_M^2 & \cdots & \alpha_M^{M-1} & \alpha_M^M & \cdots & \alpha_M^{M+N-1} \\ 1 & \alpha_{M+1} & \alpha_{M+1}^2 & \cdots & \alpha_{M+1}^{M-1} & \alpha_{M+1}^M & \cdots & \alpha_{M+1}^{M+N-1} \\ \vdots & & & & \vdots & \vdots & & \vdots \\ 1 & \alpha_{M+N} & \alpha_{M+N}^2 & \cdots & \alpha_{M+N}^{M-1} & \alpha_{M+N}^M & \cdots & \alpha_{M+N}^{M+N-1} \end{pmatrix}.$$

From our above notation, we could write

$$\Phi = \begin{pmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_{M+N}^T \end{pmatrix}.$$

In addition, it will be useful to partition Φ in the following manner:

$$\Phi = \left(\begin{array}{c|c} \Phi_{11} & \Phi_{12} \\ \hline \Phi_{21} & \Phi_{22} \end{array} \right),$$

where Φ_{11} is $M \times M$, Φ_{12} is $M \times N$, Φ_{21} is $N \times M$ and Φ_{22} is $N \times N$.

It shall also be convenient to define the function $\varepsilon : V_k \to \mathbb{Q}^+$ by

$$\varepsilon(v) = \begin{cases} d_v/d & \text{if } v|\infty, \\ 0 & \text{if } v \nmid \infty. \end{cases}$$

We are now in a position to show that $\{A_v\}_{v\in S} = \{-(\Phi_{11})^{-1}(\Phi_{12})\}_{v\in S}$ is a badly approximable S-system of linear forms of dimension $M \times N$. Suppose that $\vec{x} \in (\mathcal{O}_S)^N \backslash \{\vec{0}\}$ and $\vec{y} \in (\mathcal{O}_S)^M$. We argue that (6.20) and our previous notation give

$$\begin{aligned}
1 &\le \prod_{v\in S} \left| \prod_{l=1}^{M+N} \vec{\alpha}_l^T \binom{\vec{y}}{\vec{x}} \right|_v \\
&= \prod_{v\in S} \left\{ \left| \prod_{m=1}^{M} \vec{\alpha}_m^T \binom{\vec{y}}{\vec{x}} \right|_v \left| \prod_{n=M+1}^{M+N} \vec{\alpha}_n^T \binom{\vec{y}}{\vec{x}} \right|_v \right\} \\
&\le \prod_{v\in S} \left\{ |\Phi_{11}\vec{y} + \Phi_{12}\vec{x}|_v^M |\Phi_{21}\vec{y} + \Phi_{22}\vec{x}|_v^N \right\} \\
&= \prod_{v\in S} \left\{ |\Phi_{11}(A_v\vec{x} - \vec{y})|_v^M |\Phi_{21}\vec{y} + \Phi_{22}\vec{x}|_v^N \right\} \\
&\le \prod_{v\in S} \Big\{ \max\{M, N\}^{(M+N)\varepsilon(v)} \max\{1, |\Phi_{11}|_v\}^M \\
&\qquad \cdot |A_v\vec{x} - \vec{y}|_v^M \max\{1, |\Phi_{21}|_v\}^N \max\{1, |\Phi_{22}|_v\}^N \\
&\qquad\qquad \cdot \max\{|\vec{x}|_v, |\vec{y}|_v\}^N \Big\} \\
&= \max\{M, N\}^{(M+N)} \mu(\{\Phi_{11}\})^M \mu(\{\Phi_{21}\})^N \mu(\{\Phi_{22}\})^N \\
&\qquad \cdot \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M h_S(\vec{x}, \vec{y})^N,
\end{aligned}$$

where

$$\mu(\{B_v\}) = \prod_{v\in S} \max\{1, |B_v|_v\}.$$

Therefore

$$\tau < h_S(\vec{x}, \vec{y})^N \prod_{v\in S} |A_v\vec{x} - \vec{y}|_v^M,$$

where

$$\tau = \left\{2\max\{M, N\}\mu(\{\Phi_{11}\})\mu(\{\Phi_{21}\})\mu(\{\Phi_{22}\})\right\}^{-(M+N)}.$$

Hence $\{A_v\}_{v\in S}$ is a badly approximable S-system of linear forms of dimension $M \times N$. □

We end this section with two examples. Towards this end we shall need two auxiliary polynomials. We describe our first polynomial $f(x) \in k[x]$ below. Fix a place $\tilde{v} \notin S$ such that there exists a point $b \in k$ so that

$$\|b\|_{\tilde{v}} = \|\pi_{\tilde{v}}\|_{\tilde{v}},$$

where $\pi_{\tilde{v}}$ is the generator of the maximal ideal of $\mathscr{O}_{\tilde{v}}$, and

$$\|b\|_v = 1 \quad \text{for all } v \in S,\ v \nmid \infty.$$

Choose an $a \in k$ so that

$$\|a\|_{\tilde{v}} \le 1$$

and

$$\|a\|_v > \begin{cases} \max\{1, \|N\|_v\} & \text{for } v \in S,\ v\nmid\infty, \\ 1 + \frac{1}{\|b\|_v} & \text{for } v|\infty. \end{cases} \tag{6.21}$$

We define $f(x) = x^N - abx + b$.

Lemma 6.6. (i) *The polynomial $f(x)$ is irreducible over k.*
(ii) *For each $v \in S$ there exists an $\alpha_v \in k_v$ such that $f(\alpha_v) = 0$.*

Proof. The irreducibility of $f(x)$ follows immediately from the p-adic version of Eisenstein's theorem with the specified place being $\tilde{v}$ (see [4], Chapter 6).

It thus remains for us to demonstrate the existence of a root of $f(x)$ in k_v for each $v \in S$. First we consider the case when v is a finite place in S. Let

$$h(x) = \frac{1}{a}f(x) = \frac{1}{a}x^N - bx + \frac{b}{a} \in \mathscr{O}_v[x].$$

We immediately compute

$$h\left(\frac{1}{a}\right) = \frac{1}{a^{N+1}} \quad \text{and} \quad h'\left(\frac{1}{a}\right) = \frac{N}{a^N} - b.$$

From (6.21), $|\frac{N}{a^N}|_v < 1$, thus we have equality in the strong triangle inequality:

$$\left|h'\left(\frac{1}{a}\right)\right|_v = |b|_v = 1.$$

To summarize, we have just found an element $\frac{1}{a} \in \mathscr{O}_v$ which satisfies

$$\left|h\left(\frac{1}{a}\right)\right|_v < 1 = \left|h'\left(\frac{1}{a}\right)\right|_v^2 .$$

Therefore by Hensel's lemma there exists an $\alpha_v \in k_v$ such that $h(\alpha_v) = 0$. Clearly $f(\alpha_v) = 0$.

Now suppose $v|\infty$. If $k_v \cong \mathbb{C}$ then it is clear that there is a root α_v of $f(x)$ in k_v. If $k_v \cong \mathbb{R}$ then we may view a and b as real nonzero numbers. If $a > 0$ then (6.21) reveals

$$f(0)f(1) = (b)(1 - ab + b) < 0,$$

so by the intermediate value theorem there exists an $\alpha_v \in k_v$ with $f(\alpha_v) = 0$. If $a < 0$ then by (6.21)

$$f(0)f(-1) = (b)((-1)^N + ab + b) < 0,$$

and thus there exists a root α_v of $f(x)$ in k_v. □

We now describe our second, slightly more complicated, auxiliary polynomial. Select N distinct elements $a_1, a_2, \ldots, a_N$ of k satisfying

$$\|a_n\|_v < 1 \quad \text{for each } n = 1, 2, \ldots, N \text{ and each } v \in S,\ v \nmid \infty \text{ and for } v = \tilde{v},$$

and

$$1 < \min_{1 \le m < n \le N} \{\|a_m - a_n\|_v\} \quad \text{for all } v|\infty. \tag{6.22}$$

Select $\beta \in k$ so that

$$\|\beta\|_{\tilde{v}} = \|\pi_{\tilde{v}}\|_{\tilde{v}}$$

and

$$\|\beta\|_v < \begin{cases} \min\limits_{1 \le m \le N} \left\{ \prod\limits_{\substack{n=1 \\ n \ne m}}^{N} \|a_m - a_n\|_v \right\}^2 & \text{for } v \in S,\ v \nmid \infty, \\ \min\limits_{1 \le m \le N} \left\{ \prod\limits_{n=1}^{N} \left\|a_m - \frac{1}{2} - a_n\right\|_v , \right. & \\ \qquad \left. \prod\limits_{n=1}^{N} \left\|a_m + \frac{1}{2} - a_n\right\|_v \right\} & \text{for } v|\infty. \end{cases} \tag{6.23}$$

We define $g(x) \in k[x]$ by

$$g(x) = \prod_{n=1}^{N} (x - a_n) + \beta .$$

LEMMA 6.7. (i) *The polynomial* $g(x)$ *is irreducible over* k.
(ii) *The polynomial* $g(x)$ *splits completely in* k_v *for each* $v \in S$.

Proof. Write $g(x) = \sum_{n=0}^{N} c_n x^n$. Thus $c_N = 1$, $\|c_n\|_{\tilde{v}} < 1$ for all $n = 1, 2, \ldots, N-1$ and since we have equality in the strong triangle inequality if $N > 1$,

$$\|c_0\|_{\tilde{v}} = \left\| (-1)^N \prod_{n=1}^{N} a_n + \beta \right\|_{\tilde{v}} = \|\beta\|_{\tilde{v}} = \|\pi_{\tilde{v}}\|_{\tilde{v}}.$$

Thus by Eisenstein's theorem, $g(x)$ is irreducible over k.

Suppose now that v is a finite place in S. For any a_m we see that $a_m \in \mathscr{O}_v$ satisfies

$$g(a_m) = \beta \quad \text{and} \quad g'(a_m) = \prod_{\substack{n=1 \\ n \neq m}}^{N} (a_m - a_n).$$

From (6.23):

$$|g(a_m)|_v = |\beta|_v < \min_{1 \le m \le N} \left\{ \prod_{\substack{n=1 \\ n \neq m}}^{N} |a_m - a_n|_v \right\}^2 \le |g'(a_m)|_v^2.$$

Therefore Hensel's lemma gives the existence of a unique $\alpha_m \in k_v$ such that $g(\alpha_m) = 0$ and

$$|a_m - \alpha_m|_v \le \frac{|g(a_m)|_v}{|g'(a_m)|_v}.$$

Thus $\alpha_1, \alpha_2, \ldots, \alpha_N$ are N distinct zeros of $g(x)$. Since the degree of $g(x)$ is N, this must be all the zeros and hence $g(x)$ splits completely in k_v.

Now suppose $v|\infty$. If $k_v \cong \mathbb{C}$ then trivially $g(x)$ splits completely in k_v. If $k_v \cong \mathbb{R}$ then by (6.22) and (6.23) it is easy to verify that

$$(-1)^0 g(a_N + \tfrac{1}{2}) > 0; \quad (-1) g(a_{N-1} + \tfrac{1}{2}) > 0;$$
$$(-1)^2 g(a_{N-2} + \tfrac{1}{2}) > 0; \cdots ; (-1)^{N-1} g(a_1 + \tfrac{1}{2}) > 0;$$
$$(-1)^N g(a_1 - \tfrac{1}{2}) > 0.$$

So $g(x)$ has N distinct real zeros, thus $g(x)$ splits completely in k_v. □

Our first example may be phrased as follows.

THEOREM 6.8. *Let k be any algebraic number field and S any finite collection of places of k containing all infinite places. Then for any integer $N \geq 1$, there exists a badly approximable S-system of linear forms of dimension $1 \times N$.*

Proof. Let $f(x) = x^{N+1} - abx + b$ be the irreducible polynomial of Lemma 6.6. Factor $f(x)$ over some algebraic closure:

$$f(x) = \prod_{n=1}^{N+1} (x - \alpha_n)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{N+1}$ are the distinct roots of $f(x)$. Clearly

$$k(\alpha_1) \cong k(\alpha_2) \cong \cdots \cong k(\alpha_{N+1})$$

and $[k(\alpha_n) : k] = N + 1$ for each $n = 1, 2, \ldots, N + 1$. Let $\alpha = \alpha_1$ and define $K = k(\alpha)$. Of course $[K : k] = N + 1$ and $\{1, \alpha, \alpha^2, \ldots, \alpha^N\}$ form a basis for K over k. For each $v \in S$, there exists a root, say α_v, of $f(x)$ such that $\alpha_v \in k_v$. That is, $k(\alpha_v) \subseteq k_v$. Since

$$K = k(\alpha) \cong k(\alpha_v) \subseteq k_v,$$

K may be embedded into k_v for each $v \in S$. Let

$$\vec{\alpha} = \begin{pmatrix} \alpha \\ \alpha^2 \\ \vdots \\ \alpha^N \end{pmatrix}.$$

Therefore by Theorem 6.4 we conclude that $\{\vec{\alpha}^T\}_{v \in S}$ is a badly approximable S-system of linear forms of dimension $1 \times N$. □

We now construct an example in the most general setting.

THEOREM 6.9. *Let k be any algebraic number field and S any finite collection of places of k containing all infinite places. Then for any integers $M \geq 1$ and $N \geq 1$, there exists a badly approximable S-system of linear forms of dimension $M \times N$.*

Proof. Let $g(x) = \prod_{n=1}^{M+N}(x - a_n) + \beta$ be the irreducible polynomial of Lemma 6.7. Let $\alpha_1, \alpha_2, \ldots, \alpha_{M+N}$ be the distinct zeros of $g(x)$ over some algebraic closure of k. From Lemma 6.7 we have

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{M+N}\} \subseteq k_v$$

for each $v \in S$. Thus, multiplying the above set by a suitable constant of k if necessary, we may assume that $\{\alpha_1, \alpha_2, \ldots, \alpha_{M+N}\}$ is a set of S-algebraic integers. Therefore by Theorem 6.5, there exists a badly approximable S-system of linear forms of dimension $M \times N$. □

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Received March 23, 1990.

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