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ON THE RESTRICTIONS OF THE TANGENT BUNDLE OF THE GRASSMANNIANS

EDOARDO BALLICO

Here we study extensions $0 \to S \to V \to Q \to 0$ of vector bundles over a projective variety X with trivial middle V (hence $V \to Q$ induces a map h from X to a Grassmannian G). For fixed X and Q and moving $V \to Q$ we study the induced local deformations of S. This gives morphisms h with suitable $h^*(TG)$.

Let X be a complete variety and Q a vector bundle on X; set $r := \operatorname{rank}(Q)$. Assume that Q is spanned (i.e. generated by its global sections). Hence there is a trivial vector bundle V on X and a surjection $q: V \to Q$; set $m := \operatorname{rank}(V)$ and $S := \operatorname{Ker}(q)$. Thus we have the following exact sequence on X.

$$0 \to S \to V \to Q \to 0. \tag{1}$$

The map q induces a morphism h from X to the Grassmannian $G := G(r, m)$. By the description of the tangent bundle TG on G we have $h^*(TG) \cong S^* \otimes Q$. Since the bundle $h^*(TG)$ reflects very much the geometry of h and X, it was an intensive object of study, in particular in the case $r = 1$, i.e. $G = \mathbf{P}^{m-1}$. Several questions are natural and their answer is known for certain X, m, r (e.g. $X = \mathbf{P}^1$, $r = 1$, see **[GIS]** and **[R]**). Fix X, m, and r; what are the possible $h^*(TG)$? Fix also h and $h^*(TG)$; what is the relation between the deformations of $h^*(TG)$ as abstract bundle on X and its deformations coming from deformations of h? For instance, if X is a curve, the set of nearby bundles can be stratified according to the numerical invariants of an Harder-Narasimhan filtration of the bundles ("Shatz stratification" in the sense of **[B2]**; see also **[He]**) (if $X = \mathbf{P}^1$ this is exactly the stratification according to isomorphism classes). Here we study a refined problem: fix X, m, r, and Q, and study the possible S obtained from different surjections $q: V \to Q$. The method is very simple: study the differential of the corresponding map of functors; under very strong cohomological conditions we will get a surjectivity of the differential map. To state our result we need to introduce a few notations.

Let $\mathbf{K}$ be an algebraically closed field; everything will be defined over $\mathbf{K}$. Fix the complete variety X, integers m, r, as above; when we will speak about Q and S we will assume that they are related by (1) (for some surjection $q: V \to Q$). Set $\boldsymbol{O} := \boldsymbol{O}_X$. Let $V(S)$ be a formal miniversal deformation space for S; if $\mathbf{K}$ is the complex number field $\mathbf{C}$, $V(S)$ will often mean a small representative of the germ at $[S]$ of $V(S)$ as complex space; we will work very often in cases (e.g. X a curve) in which $V(S)$ is smooth. We will consider only cases in which $V(S)$ is algebraizable (and consider a fixed representative of a germ of an algebraic scheme) or we will work over the complex number field with a representative of a germ of complex analytic space. As in the following, we will be rather vague about the general abstract set-up, just to allow as much flexibility as possible for the future; indeed we stress that the main reason of this note is the hope that somebody else will push much further the material considered here. We will use double quotation marks to denote some of these vague places. Let $H^0(V^* \otimes Q)'$ be the open subset of $H^0(\mathrm{Hom}(V, Q))$ formed by the surjections. On $H^0(V^* \otimes Q)'$ there is a universal family: its total space is $\{(x, q) \in V \times H^0(V^* \otimes Q)' : q(x) = 0\}$; we will often call $A(*, Q)$ this universal family (the integer m being fixed); of course, as schemes $A(*, Q) := H^0(V^* \otimes Q)'$. Hence each time we have a property P which may hold for a rank-$(m-r)$ vector bundle on X we have a subset of $H^0(V^* \otimes Q)'$ formed by the bundles with property P. Dually one can fix S and V and consider the rank-r quotient bundles of inclusions of constant rank $S \to V$ (i.e. consider the same concept for the dual of (1)); one have the right universal family $A(S, *)$, and so on (and call $A(*, Q)$ left universal family). Fix $(S, i) \in A(*, Q)$ (where i in the induced inclusion $S \to V$; in the following pages we will say only: fix $S \in A(*, Q)$ or fix $[S] \in A(*, Q)$). "There is a morphism $\rho_{S,i}$ from the germ (or completion) of $A(*, Q)$ at (S, i) to $V(S)$"; we will often write ρ_S instead of $\rho_{S,i}$. A property P for rank-$(m-r)$ bundles means essentially to have a *natural partition* (i.e. stable for pull-backs) on every family of rank-$(m-r)$ bundles on X. Fix a bundle S and a family of bundles on X parametrized by a scheme T with $o \in T$ corresponding to S; we will say that T is good with respect to a natural partition if the differential at o of "a induced map $T \to V(S)$" is surjective. These concepts are essentially contained in **[BH1]**, **[H1]** and **[H2]** (which of course were (and are!) a source of inspiration).

THEOREM 0.1. *Assume* $H^1(\boldsymbol{O}) = h^2(Q^* \otimes S) = 0$. *Then the morphism* $\rho_S: A(*, Q) \to V(S)$ *is smooth at* $[S]$; *hence* $A(*, Q)$ *is good at* $[S]$ *with respect to every natural partition.*

In the applications $V(S)$ will be (formally) smooth; hence 0.1 means exactly that for every property P the stratification of $A(*, Q)$ near S and of $V(S)$ will be "the same" up to a product with a completion at a point of a smooth variety with the expected dimension (or if you prefer an analytic disk of the expected dimension); in particular all the small deformations of $[S]$ as abstract vector bundle appear in $A(*, Q)$ with the right codimension, right incidence of closures of (families of) strata. We stress that 0.1 (in the case $X = \mathbf{P}^1$) will be a very good tool to obtain the existence. It seems to be powerful to obtain connectedness or irreducibility theorems for the strata of natural stratifications on $A(*, Q)$; for a very interesting case (without fixing Q, of course), see **[He]**.

The first section is devoted to the (very easy) proof of 0.1. Then in §2 we consider the case $X = \mathbf{P}^1$ (existence part (see 2.2), smoothness of the strata of the stratification of $A(*, Q)$ under isomorphism classes, closure of such strata, ...).

In §3 we consider several cases in which it seems possible to relate the geometry of the morphism $X \to G(r, n)$ to the choice of the surjection $q: V \to Q$.

In the last very short section we discuss an example for higher dimensional X. This example raises an interesting question (see the last two lines of §4).

1. This section is devoted to the proof of 0.1.

The following notations will be used in all the sections. Set $n := \dim(X)$. Fix a surjection $q: V \to Q$; hence q gives an exact sequence like (1). We will denote by h_q or h_V (if there is no danger of misunderstanding) the map $X \to G(r, m)$ induced by q; if $V = \boldsymbol{O} \otimes H^0(Q)$ and q is the natural map, set $h_Q := h_q$.

We need the following lemma whose cohomological proof is contained in its statement.

LEMMA 1.1. (a) *Assume* $h^1(\boldsymbol{O}) = 0$. *Then we have the surjectivity of the coboundary map* $\delta: H^0(V^* \otimes Q) \to H^1(S \otimes V^*)$ *of the exact sequence obtained tensoring* (1) *by* V^* (*in general* $\dim(\mathrm{Coker}(\delta)) \leq m^2 h^l(\boldsymbol{O})$).

(b) *Assume* $h^2(Q^* \otimes S^*) = 0$. *Then we have the surjectivity of the map* $H^1(S \otimes V^*) \to H^1(S \otimes S^*)$ *induced by the exact sequence obtained tensoring with* S *the dual of* (1).

Proof of 0.1. We need to interpret them $\gamma: H^0(V^* \otimes Q) \to H^1(S \otimes S^*)$ obtained composing the maps appearing in parts (a) and (b) of 1.1. The domain of γ is the tangent space to $H^0(V^* \otimes Q)'$ at the point q. The target of γ is the tangent space to the formal deformation space of S. Note that any family of deformations of q in $H^0(V^* \otimes Q)'$ gives (as kernel) a family of deformations of S; this correspondence is functorial; furthermore, it exists also for formal deformations. This gives a natural transformation τ of the corresponding functors. Evaluating τ on $\mathrm{Spec}(\mathbf{K}[\varepsilon])$ we get its differential $d\tau$. We see (perhaps up to sign) that $d\tau$ at the point corresponding to q is γ (after an identification of $H^1(S \otimes S^*)$ and $\mathrm{Ext}^1(S, S)$). Thus we get 0.1. □

2. In this section we assume $X = \mathbf{P}^1$. For simplicity we work over the complex number field. Let E be a vector bundle on X with $\mathrm{rank}(E) = m - r$, with E^* spanned. E is the direct sum of line bundles of degree a_j, $1 \leq j \leq m - r$, with $a_i \geq a_k$ if $i \geq k$ and $a_1 \leq 0$. We fix the degree x of the bundles considered. The diagram Δ associated to E is the ordered $(m-r)$-ple $(a_1, \dots, a_{m-r})$ of these $m-r$ integers. On the finite set of diagrams (with respect to $m-r$ and x) we consider the following partial order: if Δ' is associated to the bundle E' and Δ to E, set $\Delta' \leq \Delta$ if and only if $h^0(E'(t)) \geq h^0(E(t))$ for every integer t. By 0.1 we obtain that for every spanned Q the stratification of $A(*, Q)$ into isomorphism classes (i.e. diagrams) has, near any of its points $[S]$, all the good "natural" properties we can imagine; in particular it looks like a miniversal one. Let $B(\Delta)$ be the subset of $A(*, Q)$ parametrizing bundles with diagram Δ. By 0.1 and the description of a miniversal family due to Brieskorn ([**B1**]) we get that every $B(\Delta)$ is smooth and pure dimensional (if Δ corresponds to a bundle S, its codimension is $h^1(S \otimes S^*)$). For the same reasons for every Δ the closure $B'(\Delta)$ of the stratum $B(\Delta)$ is the union of all $B(\Delta')$ with $\Delta' \leq \Delta$. Fix Δ and $S \in B'(\Delta)$; let $A(\Delta, S)$ be the closure in $V(S)$ of the points corresponding to bundles with diagram Δ; we stress that the homological properties (dimension minus depth, ...) of $\boldsymbol{O}_{B'(\Delta), [S]}$ and $\boldsymbol{O}_{A(\Delta, S), [S]}$ (say for the reduced structure) are the same, i.e. the ones of $\boldsymbol{O}_{B'(\Delta), [S]}$ are "universal" and "as good as possible" (compare with the proofs in [**GIS**]).

For the existence part given in 2.2 we need the following well-known lemma.

LEMMA 2.1. *Assume* $X = \mathbf{P}^1$. *Fix integers* $s > r > 0$, a_i, $1 \leq i \leq s$, b_j, $1 \leq j \leq r$, *with* $a_1 \geq \cdots \geq a_s$, $b_1 \geq \cdots \geq b_r$, *and* *let* $E := \mathbf{O}(a_1) \oplus \cdots \oplus O(a_s)$, $F := \mathbf{O}(b_1) \oplus \cdots \oplus \mathbf{O}(b_r)$. *There is a surjection from* E *onto* F *if* $a_j \leq b_j$ *for all* j *with* $1 \leq j \leq r$. *If* $a_1 = a_s$, *this condition is necessary, too.*

Proof. The last assertion is obvious by a lemma of Serre (see **[At]**, Th. 2). For the first assertion we may assume $s = r + 1$. Fix points $\{P_i\}$ on X. For every integer $1 \leq t \leq r$, set $E_t := \mathbf{O}(a_1) \oplus \cdots \oplus \mathbf{O}(a_t)$, $F_t := \mathbf{O}(b_1) \oplus \cdots \oplus \mathbf{O}(b_t)$. Start with a section of $\mathbf{O}(b_1 - a_1)$ with $(b_1 - a_1)P_1$ as divisor of poles and fix $t < r$; assume constructed an inclusion j (of sheaves) of E_t into F_t with generic rank t, rank at least $t - 1$ at each point of X, and with rank $t - 1$ at most at $P_1, \dots, P_t$. fix a section f of $\mathbf{O}(b_{t+1} - a_{t+1})$ with $(b_{t+1} - a_{t+1})P_{t+1}$ as divisor of poles; take a morphism $u: \mathbf{O}(b_{t+1}) \to F_t$ whose image at each point P_j on which j drops rank has image not contained in the image of j. The map $E_{t+1} \to F_{t+1}$ constructed with j, f, and u, has the same properties (for the integer $t + 1$) as t. Call $\mathbf{i}$ the map $E_r \to F$ obtained. We conclude taking a map $\mathbf{O}(a_{r+1}) \to F$ with the same property as the map u above. □

PROPOSITION 2.2. *Fix* $X = \mathbf{P}^1$, m, r, *and* Q. *A* rank-$(m - r)$ *bundle* S *on* X *fits in* (1) *if and only if* $\deg(\det(S)) = -\deg(\det(Q))$ *and every direct summand of* S *has degree at most* 0.

Proof. Fix a trivial vector bundle W with $\text{rank}(W) = r + 1$ and a surjection $w: W \to Q$; set $L := \text{Ker}(w)$. Hence $L \cong \mathbf{O}(-\deg(Q))$. Set $V := W \oplus W'$ with W' trivial and let $w': V \to Q$ be the surjection which agrees with w on the first factor and vanishes on the second factor. Thus w' induces (1) with $S \cong L \oplus W'$. By 0.1 it is sufficient to check that $L \oplus W'$ deform as abstract bundle to any other subbundle S' of a trivial bundle with $c_1(S') \cong c_1(S)$ and $\text{rank}(S') = m - r$. This is well-known (hint: use a sequence of "elementary moves" in which $\mathbf{O}(a) \oplus \mathbf{O}(b)$ with $a \geq b + 2$ deforms to $\mathbf{O}(a - 1) \oplus \mathbf{O}(b + 1)$). □

3. As in the classical case ($r = 1$), one can introduce the following notions about the extension (1) (or equivalently about the morphisms h_V and h_{V^*} induced by (1) and by its dual). The extension (1) is called right linearly normal (resp. right non-degenerate) if the induced map $V \to H^0(Q)$ is an isomorphism (resp. is injective); the right non-degeneracy means that $H^0(S) = 0$, i.e. that S has no trivial factor. The extension (1) is called left linearly normal (resp. left non-degenerate) if the dual of (1) is right linearly normal (resp. non-degenerate); thus left non-degeneracy is equivalent to the fact that Q has no trivial factor.

(3.1) Fix a spanned vector bundle E on a complete variety X; set $n := \dim(X)$ and $r := \mathrm{rank}(E)$. We assume that $H^0(E)$ gives an embedding of X; we want to find $W \subseteq H^0(E)$ with $\dim(W)$ as low as possible, W spanning E (this is again **[At]**, Th. 2) and such that the induced morphism h_W is an embedding. As in the classical case $(r = 1)$ in general one cannot do better than a dimensional count. First we will consider the injectivity part of h_W and then (assuming for simplicity X smooth) the differential of h_W. We need to define several invariants of (X, E) (with E spanned); set $V := H^0(E)$. For every $P \in X$, by assumption the Kernel $V(P)$ of the map $V \to E|_{\{P\}}$ has codimension r in V; if $\{x, y\} \subset X$, with $x \neq y$, set $V(x, y) := \mathrm{Ker}(V \to E|_{\{x,y\}})$ and $d(x, y) := \dim(V(x, y))$; if $\mathbf{v} \in T_xX$ (Zariski tangent space), set $V(\mathbf{v}) := \mathrm{Ker}(V \to E|_{\mathbf{v}})$ and $d(\mathbf{v}) := \dim(V(\mathbf{v}))$; for $0 \leq i \leq 2n$, set $r_i := \min\{t\colon$ there is an i-dimensional family of pairs $(x, y) \in X \times X \backslash \Delta$ with $d(x, y) = r_i\}$; set $k_i = \min\{t\colon$ there is an i-dimensional family of pairs $(x, \mathbf{v})$ with $x \in X$, $\mathbf{v} \in T_xX$ and $d(\mathbf{v}) = r_i\}$; $k_i' = \min\{t\colon$ there is an i-dimensional family of pairs $(x, \mathbf{v})$ with $x \in X_{\mathrm{reg}}$, $\mathbf{v} \in T_xX$ and $d(\mathbf{v}) = r_i\}$; set $k(E) := \max\{i + k_i\}$ and $k'(E) := \max\{i + k_i', 1 \leq i < 2n\}$. By semicontinuity we have $r_i \geq r_j$ if $i \leq j$; the assumption "h_E an embedding" is equivalent to $r_0 > 0$ (by the definition of h_E).

PROPOSITION 3.2. (a) (*injectivity part*) *Fix a general* $W \subseteq H^0(E)$ *with* $\dim(W) \geq \max(i + r_i, 0 \leq i \leq 2n\}$. *Then* h_W *is injective.*

(b) *Fix a general* $W \subseteq H^0(E)$ *with* $\dim(W) \geq k(E)$ (*resp.* $k'(E)$). *Then* h_W *is an embedding* (*resp.* h_W *embeds* X_{reg}).

Proof. (a) counts the set of triples (W, x, y) for which $\dim(W \cap V(x, y)) \leq r$. For part (b) the proof is again a dimensional count; we stress that one needs all $\mathbf{v}$ in the Zariski tangent space, not only in the tangent cone of X at x.

(3.3) Of course, Lemma 2.1 gives a sufficient condition for the existence of morphisms from $\mathbf{P}^1$ into a partial flag variety with given quotient bundles; but it is an easy exercise to do better. Proposition 3.2 gives a way to pass from "morphisms" to "embeddings" or "injective morphisms". A similar dimensional count (easier since now $n = 1$) allows one to pass from "morphisms" to "birational morphisms with image with given (very small) number of nodes and cusps".

(3.4) Fix a singular curve C and let $f: D \to C$ be its normalization. Fix a vector bundle E on C; set $r := \text{rank}(E)$, $N := h^0(E)$. Suppose you know $f^*(E)$; what other data are necessary to reconstruct E? This is classical for line bundles and certain well-known in general. This problem is easier in the case of singularities with a modulo in the sense of Rosenlicht (see **[Se]**), i.e. singularities obtained from a smooth curve glueing a positive divisor; we will consider as examples only the case of ordinary nodes and ordinary cusps.

Assume P is an ordinary node of C and let $\{P', P''\} := f^{-1}(P)$; let M' and M'' be the fibers of $f^*(E)$ over P' and P''. The existence of E induces an identification of M' with M''; vice versa, if you have an isomorphism of M' and M'' you can descend $f^*(E)$ to a bundle on $\mathbf{C}$; hence, given $f^*(E)$, all possible bundles E's are parameterized by $\text{GL}(r, \mathbf{K})$ (of course if $\text{Sing}(C) = \{P\}$, but since $\text{Sing}(C)$ is finite and the descent problem local, it is sufficient to consider one singular point at each step).

Now assume that P is an ordinary cusp of C; let $A \in D$ with $f(A) = P$. Let M (resp. M') be the fiber of $f^*(E)$ over A (resp. over the length two scheme corresponding to the positive divisor $2A$). By restriction of M' to A, we have a surjection $s: M' \to M$; the existence of E is equivalent to a choice of a splitting of s; thus, after a choice of bases, the possible bundles E's are parameterized by $M(r \times r; \mathbf{K})$, the matrix B corresponding to a surjection with matrix $(\text{Id}_r, B)^t$.

Of course, we are interested in the case E spanned, hence giving a morphism to an appropriate Grassmannian. We are even more interested in the opposite problem: given C and, perhaps, $f^*(E)$, find the possible spanned E and the corresponding restricted tangent bundles (restricted to C, not only their pull-backs to D).

(3.5) As in the classical case there are the standard enumerative formulas (genus formula, double point formula (**[Fu]**, 9.3 and 9.3.1), ...) when the conditions of 2.2 are not satisfied. For instance if $n = r = 2$, $m = 4$ and X is smooth one can get the double point formula; just to obtain the same formula up to a non-zero constant (i.e. just to get the numerical obstruction to the fact that h_q is an embedding) one can simply use exactly the calculations in **[Ha]**, p. 434, (which is the case of smooth surfaces in $\mathbf{P}^4$ instead of the case of smooth surfaces in the smooth quadric $G(2, 4) \subset \mathbf{P}^5$). Furthermore, as in 3.2, one can find (just counting dimensions) sufficient condition for the absence, for general W, of higher order degeneracy loci (e.g. triple points).

4. If $\dim(X) = 2$ the hypotheses of 0.1 are almost never satisfied. If $\dim(X) > 2$ the hypotheses of 0.1 are satisfied (for sufficiently positive Q) essentially only if S is infinitesimally rigid, i.e. $H^1(S^* \otimes S) = 0$. However in this case (as we will see in an example) the elementary approach of this paper is very useful to construct examples of maps to Grassmannians with certain restricted tangent bundle; for instance in the example 4.1 with $X = \mathbf{P}^3$, S will be stable and $S^* \otimes Q$ will be a direct sum of two copies of a stable vector bundle.

EXAMPLE 4.1. Set $X = \mathbf{P}^3$, $m = 5$, $r = 2$; fix an integer $t > 0$ and take $Q = \boldsymbol{O}(6t)^{\oplus 2}$. Fix any surjection $j\colon V \to Q$, and set $S := \operatorname{Ker}(j)$. We want to check when S is stable. Since $\operatorname{rank}(S) = 3$ (this is important!) S is stable if and only if $h^0(S(4t)) = h^0(S^*(-4t)) = 0$. By the dual of (1) we see that $h^0(S^*(-1)) = 0$ (so from this side S has a high order of stability for every surjection j). The vanishing of $H^0(S(4t))$ is not true for every j. We will check that it is true for general j, i.e. that for general j the induced map $j_*\colon H^0(\boldsymbol{O}(4t))^{\oplus 5} \to H^0(\boldsymbol{O}(10t))$ is injective. By semicontinuity it is sufficient to check this for just one map j (even not surjective). Fix $f_i \in H^0(\boldsymbol{O}(6t))$, $1 \le i \le 5$, with $\dim(\{f_1 = f_2 = f_3 = 0\}) = 0$ and $\dim(\{f_4 = f_5 = 0\}) = 1$. Map V into the first factor of Q using $(f_1, f_2, f_3, 0, 0)$ and on the second factor using $(0, 0, 0, f_4, f_5)$. Fix $(g_1, \dots, g_5) \in \operatorname{Ker}(j_*)$. The regularity of the sequence $\{f_1, f_2, f_3\}$ gives $g_1 = g_2 = g_3 = 0$. Similarly we get $g_4 = g_5 = 0$. □

There are obvious variations of 4.1 ($m = 6$, $r = 3$, or X a complete intersection, or...). We want to stress only that the vanishing of $H^0(S(4t))$ was equivalent to a very weak form of a maximal rank problem which seems to be interesting.

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UNIVERSITY OF TRENTO
38050 POVO (TN), ITALY