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ON SIEVED ORTHOGONAL POLYNOMIALS IX: ORTHOGONALITY ON THE UNIT CIRCLE

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We study sieved orthogonal polynomials on the unit circle and using a result of Szegõ we show that there is a one to one correspondence between a family of sieved orthogonal polynomials on the unit circle and two families of sieved orthogonal polynomials on the interval $[-1, 1]$, namely sieved polynomials of the first and second kinds. We find explicit representations of the sieved polynomials and the Herglotz transform of the measure with respect to which they are orthogonal.

1. Introduction. Since Al-Salam, Allaway and Askey [**1**] introduced sieved ultraspherical polynomials, the subject of sieved orthogonal polynomials on a finite interval have been studied extensively by several authors, [**3**], [**4**], [**7**], [**10**], [**14**]. In this work we consider the problem of sieved orthogonal polynomials on the unit circle.

We also explore the connection between symmetric sieved orthogonal polynomials on the interval $[-1, 1]$ and sieved orthogonal polynomials on the unit circle. This connection will shed new light on sieved random walk polynomials [**3**].

Let $d\mu$ be a finite positive Borel measure on the unit circle Γ and assume that $d\mu$ is normalized by

$$\frac{1}{2\pi}\int_0^{2\pi} d\mu(\theta) = 1, \qquad z = e^{i\theta} \in \Gamma.$$

We shall further assume that the support of $d\mu$ is infinite. Let $\{\Phi_n(z)\}_0^\infty$ be the sequence of monic polynomials orthogonal on Γ with respect to $d\mu$. The Φ_n's satisfy the recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{a}_n\Phi_n^*(z), \qquad n = 0, 1, \dots, \tag{1.1}$$

[**8**], [**9**], [**13**], where a_n is

$$a_n = -\overline{\Phi_{n+1}(0)}, \qquad n = 0, 1, \dots,$$

and the reciprocal polynomial of Φ_n is

$$\Phi_n^*(z) := z^n\overline{\Phi_n}(1/z).$$

It is well-known that given an infinite sequence of complex numbers $a_0, a_1, \ldots, a_n, \ldots$ with $|a_n| < 1$, $n = 0, 1, \ldots$, one can uniquely define a sequence of monic polynomials $\{\Phi_n(z)\}$, by demanding that $\Phi_0(z) = 1$ and $\Phi_n(z)$ satisfy (1.1). Furthermore the Φ_n's will be orthogonal on Γ [8], [13]. The a_n's are called the reflection coefficients.

In §2 we start with a measure $d\mu$, as above, or a sequence of monic polynomials $\{\Phi_n(z)\}$ orthogonal on the unit circle and a positive integer k, and generate a family of sieved orthogonal polynomials $\{S_n(z)\}$ by

$$(1.2)\quad S_0(z) = 1, \quad S_{n+1}(z) = zS_n(z) - \overline{\alpha_n} S_n^*(z), \qquad n = 0, 1, \ldots.$$

in (1.2), α_n is

$$(1.3)\qquad \alpha_n = \begin{cases} 0 & \text{if } k \nmid n+1, \\ a_{m-1} & \text{if } n+1 = mk. \end{cases}$$

Our first main result, Theorem 1.4 below, provides an explicit representation of the S_n's in terms of the Φ_n's and relates their orthogonality measures.

THEOREM 1.4. *Let* $\{S_n(z)\}$ *and* $\{\alpha_n\}$ *be defined as in* (1.2) *and* (1.3), *respectively. Then*

$$(1.5)\quad S_{nk+j}(z) = z^j \Phi_n(z^k), \qquad j = 0, 1, \ldots, k-1;\ n = 0, 1, \ldots.$$

Furthermore if $d\nu$ *is the measure with respect to which the* S_n*'s are orthogonal, then*

$$(1.6)\qquad d\nu(\theta) = k^{-1}\, d\mu(k\theta), \quad z = e^{i\theta}.$$

Theorem 1.4 will be proved in §2. One can think of Theorem 1.4 as a case when the polynomial mapping $T(z)$ of [2] is z^k and orthogonality is on the unit circle. The polynomial mapping approach does not seem to have been applied to orthogonal polynomials on the unit circle. In §2 we will also state and prove a result connecting the Herglotz transforms of the measures $d\mu$ and $d\nu$.

Recall that a sequence of orthogonal polynomials $\{r_n(x)\}$ is a sequence of random walk polynomials if and only if there exists a sequence $\{d_n\}$, $0 < d_n < 1$, $n > 0$, $0 \le d_0 < 1$ such that

$$(1.7)\quad \begin{aligned} &r_0(x) = 1, \quad r_1(x) = x/(1-d_0), \\ &xr_n(x) = (1-d_n)r_{n+1}(x) + d_n r_{n-1}(x), \qquad n > 0. \end{aligned}$$

For example the spherical harmonics or ultraspherical polynomials [6] are random walk polynomials with $d_n = n/(2n+2\lambda)$. In §3 we point out that any sequence of symmetric polynomials $\{p_n(x)\}$ (i.e. $p_n(-x) = (-1)^n p_n(x)$) orthogonal on a subset of $[-1, 1]$ is actually a sequence of random walk polynomials. In **[3]** it was shown how a family of random walk polynomials (1.7) generates two families of sieved random walk polynomials. In §3 we show how a single sequence of sieved polynomials orthogonal on the unit circle, when projected on $[-1, 1]$, gives rise to the two aforementioned families. We also give a further generalization of **[3]**. In §4 we give a mild generalization of the sieving concept of §2. The results of §2 may be viewed as orthogonal polynomials on the Julia set associated with the polynomial mapping $T(z) = z^k$ while the results in §4 deal with Julia sets associated with iterations of different mappings.

2. Sieved polynomials on the circle. We first record a lemma needed in the proof of Theorem 1.4.

LEMMA 2.1 (**[11]**, **[12]**). *Let $\{\phi_n(z)\}$ be orthonormal polynomials with respect to a finite positive Borel measure $d\sigma$ on Γ. Then*

$$|\phi_n(e^{i\theta})|^{-2}\,d\theta \xrightarrow{W^*} d\theta\,, \quad \text{as } n \to \infty. \tag{2.2}$$

By (2.2) *we mean that the limiting relation*

$$\lim_{n\to\infty} \int_0^{2\pi} f(\theta)|\phi_n(e^{i\theta})|^{-2}\,d\theta = \int_0^{2\pi} f(\theta)\,d\sigma(\theta)\,,$$

holds for every function $f(\theta)$ in $C_{2\pi}$.

Proof of Theorem 1.4. We first use induction on n to show that

$$S_{nk}(z) = \Phi_n(z^k)\,, \qquad n = 0, 1, \ldots. \tag{2.3}$$

Clearly (2.3) holds when $n = 0$. If (2.3) holds for an n consider $S_{(n+1)k}(z)$. Since α_{nk+j} vanishes for $j = 0, 1, \ldots, k-2$, then (1.2) implies $S_{nk+j}(z) = z^j\Phi_n(z^k)$. Therefore

$$\begin{aligned} S_{(n+1)k}(z) &= zS_{nk+k-1}(z) - \overline{\alpha_{nk+k-1}}S^*_{nk+k-1}(z) \\ &= z^k\Phi_n(z^k) - \overline{a_n}\Phi^*_n(z^k) = \Phi_{n+1}(z^k)\,. \end{aligned}$$

In the above equality we used the fact

$$(S_{nk+k-1})^*(z) = (z^{k-1}\Phi_n(z^k))^* = \Phi^*_n(z^k)\,.$$

This completes the proof of (1.5). To prove the remaining part of Theorem 1.4 first note that for $n = 1, 2, \ldots$, we have, Geronimus

[8, p. 133],

$$\frac{1}{2\pi}\int_0^{2\pi}|\Phi_n(e^{i\theta})|^2\,d\mu(\theta)=\prod_{j=0}^{n-1}(1-|a_j|^2)=:N_n^2$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}|S_{nk}(e^{i\theta})|^2\,d\nu(\theta)=\prod_{j=0}^{nk-1}[1-|\alpha_j|^2]=\prod_{j=0}^{n-1}[1-|a_j|^2]=N_n^2;$$

hence the polynomials $\phi_n(z):=\Phi_n(z)/N_n$ and

$$s_{kn}(z):=S_{kn}(z)/N_n=\Phi_n(z^k)/N_n=\phi_n(z^k)$$

are orthonormal with respect to $d\mu$ and $d\nu$, respectively. Applying Lemma 2.1 and (2.3) we find that

$$\begin{aligned}\int_0^{2\pi}f(\theta)\,d\nu(\theta)&=\lim_{n\to\infty}\int_0^{2\pi}|s_{nk}(e^{i\theta})|^{-2}f(\theta)\,d\theta\\&=\lim_{n\to\infty}\int_0^{2\pi}|\phi_n(e^{ik\theta})|^{-2}f(\theta)\,d\theta\\&=\frac{1}{k}\lim_{n\to\infty}\int_0^{2k\pi}|\phi_n(e^{i\alpha})|^{-2}f(\alpha/k)\,d\alpha\\&=\frac{1}{k}\lim_{n\to\infty}\sum_{j=1}^{k}\int_{2(j-1)\pi}^{2j\pi}|\phi_n(e^{i\alpha})|^{-2}f(\alpha/k)\,d\alpha.\end{aligned}$$

If we restrict ourselves to functions f with period $2\pi/k$, then the extreme right-hand side of the above equalities is

$$\int_0^{2\pi}f(\theta/k)\,d\mu(\theta).$$

Finally the uniqueness of a Borel measure representing a continuous linear functional on $C_{2\pi}$ establishes (1.6). □

The Herglotz transform of a finite Borel measure $d\sigma(\theta)$ supported on $[0, 2\pi)$ is defined by

$$F(\sigma;z):=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\sigma(\theta),\qquad |z|<1. \tag{2.4}$$

In the open unit disc $F(\sigma;z)$ is an analytic function and has positive real part [8, p. 23]. Furthermore the following inversion formula holds

$$\frac{1}{2}[\sigma(\theta+0)+\sigma(\theta-0)]=\text{Constant}+\lim_{r\to1^-}\int_0^{\theta}\operatorname{Re}\{F(\sigma;re^{i\phi})\}\,d\phi. \tag{2.5}$$

THEOREM 2.6. *Let $d\nu$ be as in Theorem* 1.4; *then*

$$F(\nu\,;\,z) = F(\mu\,;\,z^k), \qquad |z| < 1. \tag{2.7}$$

Before proving Theorem 2.6 we recall some basic facts. The monic orthogonal polynomials of the second kind associated with $\{\Phi_n(z)\}$ are [**8**, p. 5]

$$\Psi_n(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}[\Phi_n(e^{i\theta}) - \Phi_n(z)]\,d\mu(\theta), \qquad n > 0, \quad \Psi_0(z) = 1. \tag{2.8}$$

Note that Geronimus [**8**, (1.13), p. 5] erroneously defines $\Psi_0(z)$ by letting $n = 0$ in the integral in (2.8). It is not difficult to use (1.2) and (1.2′) on page 2 in [**8**] to show that $\{\Psi_n(z)\}$ can be generated by

$$\Psi_0(z) = 1\,, \quad \Psi_{n+1}(z) = z\Psi_n(z) + \overline{a_n}\,\Psi_n^*(z), \qquad n = 0, 1, 2\dots. \tag{2.9}$$

The reflection coefficients in (2.9) have the opposite sign of the reflection coefficients in (1.1). Moreover [**8**, (1.16), p. 6]

$$F(\mu\,;\,z) - \frac{\Psi_n^*(z)}{\Phi_n^*(z)} = O(z^{n+1}), \qquad |z| < 1, \quad \text{as } n \to \infty. \tag{2.10}$$

Proof of Theorem (2.6). Let $\{\Psi_n(z)\}$ and $\{B_n(z)\}$ be the monic polynomials of the second kind associated with the polynomials orthogonal with respect to $d\mu$ and $d\nu$, respectively. Now (2.9) implies $B_n(z) = \Psi_n(z^k)$. Therefore

$$\begin{aligned} F(\nu\,;\,z) &= \lim_{n\to\infty} B_{nk}^*(z)/S_{nk}^*(z) \\ &= \lim_{n\to\infty} \Psi_n^*(z^k)/\Phi_n^*(z^k) = F(\mu\,;\,z^k), \qquad |z| < 1, \end{aligned}$$

and the theorem follows. □

3. Random walk polynomials. Let $\{p_n(x)\}$ be a sequence of monic symmetric polynomials orthogonal on a subset of $[-1, 1]$. To see that they must be a sequence of monic random walk polynomials assume $\{p_n(x)\}$ to be monic and generated by

$$p_0(x) = 1, \quad p_1(x) = x, \quad xp_n(x) = p_{n+1}(x) + b_n p_{n-1}(x), \qquad n > 0.$$

It follows from [**5**, §4.2, p. 108] that the p_n 's are orthogonal on a subset of $[-1, 1]$ if and only if $\{b_n : 0 < n < \infty\}$ is a chain sequence, i.e., there is a parameter sequence $\{g_n : 0 \le n < \infty\}$ such that b_n admits the factorization $b_n = g_n(1 - g_{n-1})$, $n > 0$, with $0 \le g_0 < 1$ and

$0 < g_n < 1$, $n > 0$. If $\{b_n\}$ is a chain sequence one can always choose $\{g_n\}$ to be a minimal parameter sequence, i.e. $g_0 = 0$, [**5**, Theorem 5.3, p. 94]. Therefore, $\{p_n(x)\}$ is a sequence of constant multiples of random walk polynomials.

Next we start with a monic polynomial set $\{\Phi_n(z)\}$ orthogonal on the unit circle with respect to $d\mu(\theta)$, $z = e^{i\theta}$. Let $d\rho$ be the positive Borel measure defined by

$$d\mu(\theta) = |\sin\theta|\, d\rho(\cos\theta). \tag{3.1}$$

Further, let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be the monic polynomials orthogonal on $[-1, 1]$ with respect to $d\rho(x)$ and $(1-x^2)\,d\rho(x)$, respectively. Then, on writing $x = (z + z^{-1})/2$, we get, see Szegö [**13**, §11.5],

$$P_n(x) = \frac{2^{-n}}{1 + \Phi_{2n}(0)}\{z^{-n}\Phi_{2n}(z) + z^n\Phi_{2n}(z^{-1})\}, \qquad n \geq 0, \tag{3.2}$$

and

$$Q_{n-1}(x) = \frac{2^{1-n}}{1 - \Phi_{2n}(0)}\{z^{-n}\Phi_{2n}(z) - z^n\Phi_{2n}(z^{-1})\}/(z - z^{-1}), \qquad n > 0. \tag{3.3}$$

Now replace z by z^k and use (3.3) to obtain after some simplification

$$Q_{n-1}(T_k(x)) = \frac{2^{1-n}}{1 - S_{2nk}(0)} \cdot \frac{z^{-nk}S_{2nk}(z) - z^{nk}S_{2nk}(z^{-1})}{z - z^{-1}}\{1/U_{k-1}(x)\},$$

where $\{T_n(x)\}$ and $\{U_n(x)\}$ are Chebyshev polynomials of the first and second kind, respectively. On the other hand starting with symmetric monic polynomial sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$ orthogonal on $[-1, 1]$ with respect to $d\rho(x)$ and $(1 - x^2)\,d\rho(x)$, respectively we can apply (11.5.1) and (11.5.2) in Szegö [**13**] and find a polynomial set $\{\Phi_n(z)\}$ orthogonal on the unit circle with respect to $d\mu(\theta)$ of (3.1). Once the sieved polynomials on the circle $\{S_n(x)\}$ are defined, we can again apply (11.5.1) and (11.5.2) in [**13**] and construct two families $\{p_n(x)\}$ and $\{q_n(x)\}$ of sieved polynomials orthogonal on $[-1, 1]$ with respect to $d\alpha(x)$ and $d\beta(x) = (1 - x^2)\,d\alpha(x)$. Thus we proved the following theorem.

THEOREM 3.4. *Let $\{P_n(x)\}$, $\{Q_n(x)\}$ and $\{\Phi_n(z)\}$ be as above and assume that $\{S_n(x)\}$, $\{p_n(x)\}$ and $\{q_n(x)\}$ are constructed from them following the above procedure. Then we have*

$$p_n(x) = \frac{2^{-n}}{1 + S_{2n}(0)}\{z^{-n}S_{2n}(z) + z^nS_{2n}(z^{-1})\}, \tag{3.5}$$

and

$$q_n(x) = \frac{2^{1-n}}{1 - S_{2n}(0)}\{z^{-n}S_{2n}(z) - z^n S_{2n}(z^{-1})\}/(z - z^{-1}). \tag{3.6}$$

In particular

$$\begin{aligned} 2^{n(k-1)}q_{nk-1}(x) &= U_{k-1}(x)Q_{n-1}(T_k(x)), \\ 2^{n(k-1)}p_{nk}(x) &= P_n(T_k(x)). \end{aligned} \tag{3.7}$$

It can be easily verified that

$$d\alpha(x) = k^{-1}|U_{k-1}(x)|\, d\rho(T_k(x)), \quad d\beta(x) = (1 - x^2)\, d\alpha(x). \tag{3.8}$$

Formula (3.7) *is in* **[3]** *when the* $P_n(x)$ *'s are monic random walk polynomials.*

4. Iterated mappings. In this section we give a generalization of the construction in §2.

We start with a polynomial set $\{\Phi_n(z)\colon n \geq 0\}$ orthogonal on the unit circle whose reflection coefficients are $\{a_n\colon n \geq 0\}$; hence $|a_n| < 1$, $n \geq 0$ and (1.1) holds. Assume further that we are given $k-1$ numbers $b_0, b_1, \dots, b_{k-2}$ such that $|b_j| < 1$, $0 \leq j \leq k-2$. Define a sequence $\{\alpha_n\colon n \geq 0\}$ of reflection coefficients as

$$\alpha_{mk-1} = a_{m-1}, \quad \alpha_{mk+j} = b_j, \qquad j = 0, 1, \dots, k-2. \tag{4.1}$$

The generalized sieved polynomials $\{S_n(z)\}$ are generated by the reflection coefficients $\{\alpha_n\}$, i.e. (1.2) holds. The special pattern of the α_n's enables us to find a representation of $S_n(z)$. This is achieved as follows.

For $j = 1, 2, \dots, k-1$ and for any polynomial $v(z)$ define

$$\begin{aligned} f_j(v) := (z, -\overline{b_{j-1}}) &\begin{bmatrix} z & -\overline{b_{j-2}} \\ -b_{j-2}z & 1 \end{bmatrix} \\ &\cdots \begin{bmatrix} z & -\overline{b_0} \\ -b_0 z & 1 \end{bmatrix} \begin{bmatrix} v(z) \\ v^*(z) \end{bmatrix}. \end{aligned} \tag{4.2}$$

Next define a sequence $\{\zeta_n(z)\colon n \geq 0\}$ as

$$\zeta_0(z) = 1, \quad \zeta_{n+1}(z) = z f_{k-1}(\zeta_n(z)) - \overline{a_n}\, (f_{k-1}(\zeta_n(z)))^*. \tag{4.3}$$

It is straightforward to prove:

PROPOSITION 4.4. *Let* $\{\alpha_n\}$ *and* $\{S_n(z)\}$ *be as in* (4.1) *and* (1.2), *respectively*; *then*

$$S_{nk}(z) = \zeta_n(z) \quad \textit{and} \quad S_{nk+j}(z) = f_j(\zeta_n(z)), \qquad j = 1, 2, \dots, k-1;\ n = 0, 1, \dots. \tag{4.5}$$

In the special case when $b_0 = b_1 = \cdots = b_{k-2} = 0$ *and* $\{\Phi_n(z)\}$ *are the original polynomials then*

$$f_j(\zeta_0(z)) = z^j, \quad f_j(\zeta_1(z)) = z^j\zeta_1(z),$$
$$j = 0, 1, \ldots, k-1, \quad \text{with } \zeta_1(z) = \Phi_1(z^k),$$

and (1.5) *follows.*

One would like to relate the orthogonality measure of the Φ_n 's and the S_n 's in the above construction. Unfortunately our attempts to find such a relationship have not been successful.

Added in proof. After the final version of this paper was sent for publication a paper of Sansigre and Marcellan appeared where they considered the algebraic properties of the case $k = 2$ of our present paper. The reference is G. Sansigre and F. Marcellan, *Orthogonal polynomials on the unit circle; symmetrization and quadratic decomposition*, J. Approximation Theory, **65** (1991), 109–119.

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