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COMPACT OPERATIONS, MULTIPLIERS AND RADON-NIKODYM PROPERTY IN JB^*-TRIPLES

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We study the (weak) compactness of certain algebraic operations on JB^*-triples and we introduce multiplier triples. Applications to structure theory are given and connections with the Radon-Nikodym Property are described.

Introduction. Recently the authors [5] studied the Radon-Nikodym property (RNP) in the dual spaces of some complex Banach spaces known as JB^*-triples. A number of intrinsic characterisations were obtained. One of these was that, if A is a JB^*-triple, then A^* has the RNP if and only if A has a composition series of closed triple ideals (i.e. M-ideals) for which successive quotients can be realised either as spaces of compact operators from one Hilbert space to another or else are reflexive. This hints at a connection between the RNP and compact, and weakly compact, operators on A itself. This paper evolves from an investigation into the form and extent of this connection.

Thus, in a fairly systematic way, we study the (weak) compactness of natural algebraic operations, introduce the notion of a multiplier triple of a JB^*-triple (which may be of independent interest), and explain how the resulting phenomena interweave with the RNP.

JB^*-triples originate in the study of holomorphy in unspecified (possibly infinite) dimension and can be realised as that class of complex Banach spaces whose unit ball is a bounded symmetric domain (in finite dimensions, the classical Cartan domains of complex analysis) [23]. The considerable recent activity and rapid progress in JB^*-triples is due in no small part to fertile applications in, amongst other topics (see [26, 27]), infinite dimensional Lie algebras, mathematical physics and operator spaces. Notably, the image of a contractive projection on a C^*-algebra is, while rarely a C^*-algebra, always a JB^*-triple [15].

1. Preliminaries. Precisely a *JB^*-triple* is a complex Banach space A with a continuous triple product $\{\dots\}: A^3 \to A$ which is linear and symmetric in the outer variables and antilinear in the middle variable, and satisfies

(i) the operator $a \to \{xxa\}$ on A is hermitian with nonnegative spectrum for all x in A;
(ii) $\|\{xxx\}\| = \|x\|^3$;
(iii) the main identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} + \{xy\{abz\}\} - \{x\{bay\}z\}.$$

For $x, y \in A$, the linear operator $a \to \{xya\}$ is denoted by $D(x, y)$; the antilinear operator $a \to \{xay\}$ is denoted by $Q(x, y)$, and by $Q(x)$ if $x = y$.

A JB^*-triple which is a dual Banach space is called a *JBW^*-triple*, in which case the predual is unique and the triple product is separately weak*-continuous. If A is a JB^*-triple then A^{**} is a JBW^*-triple in which, via the canonical embedding, A is a JB^*-subtriple [**2**, **10**, **20**]. An element u of A is called a *tripotent* if $\{uuu\} = u$, with which are associated the *Peirce projections* $P_i(u)\colon A \to A$, $i = 0, 1, 2$, defined by $P_2(u) = Q(u)^2$, $P_1(u) = 2(D(u, u) - Q(u)^2)$, $P_0(u) = I - 2D(u, u) + Q(u)^2$. The tripotent u is said to be *complete* if $P_0(u) = 0$, *minimal* if $\{uAu\} = \mathbb{C}u$, and *unitary* if $\{uAu\} = A$. A subspace I of A is called a *triple ideal* of A if $\{AAI\} + \{AIA\} \subset I$; if merely $\{IAI\} \subset I$ then I is called an *inner ideal* of A. Elements $x, y \in A$ are *orthogonal* if $D(x, y) = 0$. Two triple ideals I and J are *orthogonal* if $D(x, y) = 0$ for all $x \in I$ and $y \in J$; equivalently, if $I \cap J = 0$. A JB^*-triple is *simple* if it has no nontrivial norm closed triple ideals.

A norm closed subspace of a C^*-algebra which is also algebraically closed under the triple product $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$ is a JB^*-triple. The JB^*-triples which can be realised in this way are called *J^*-algebras*. Other examples of JB^*-triples include the *Cartan factors* C_i $(i = 1, \dots, 6)$, where C_4 is a complex spin factor, C_5 consists of 1×2 matrices over the complex Cayley division algebra $\mathbb{O}$ and C_6 the hermitian 3×3 matrices over $\mathbb{O}$. The types C_1, C_2, C_3 are defined as follows for arbitrary Hilbert spaces H and H': $C_1 = B(H, H')$; $C_2 = \{x \in B(H)\colon x = -jx^*j\}$; $C_3 = \{x \in B(H)\colon x = jx^*j\}$, where $j\colon H \to H$ is a conjugation. Correspondingly, we define (as in [**5**]) the elementary JB^*-triples, K_i $(i = 1, \dots, 6)$ as follows: $K_1 = K(H, H')$ (the compact operators); $K_i = C_i \cap K(H)$ for $i = 2, 3$; $K_i = C_i$ for $i = 4, 5, 6$. Each K_i is a simple JB^*-triple and can alternatively be described as the subtriple of C_i generated by the minimal tripotents. Extensive (often tacit) use will be made of the polarisation identities:

$$4\{xya\} = \sum_{k=0}^{3} i^k\{x + i^k y, x + i^k y, a\};$$

$$4\{axa\} = \sum_{k=0}^{3}(-1)^k\{i^k a + x, i^k a + x, i^k a + x\},$$

$$2\{axb\} = \{a + b, x, a + b\} - \{axa\} - \{bxb\}.$$

Further theory of JB^*-triples can be found in [**2**, **8**, **10**, **11**, **14–27**].

REMARK 1.1. We will need the following supplementaries:

(a) If $\pi: A \to B$ is a weak* continuous triple homomorphism between JBW^*-triples, then $\pi(A)$ is weak* closed (and hence a JBW^*-subtriple) in B.

(b) If $\pi: A \to B$ is a triple homomorphism, where A is a JB^*-triple and B is a JBW^*-triple, then π has a unique extension to a weak* continuous triple homomorphism $\overline{\pi}: A^{**} \to B$ and $\overline{\pi}(A^{**}) = \overline{\pi(A)}^{\text{weak}^*}$.

It is easily seen that (a) follows from the fact [**20**, Theorem 4.2] that $\ker\pi$ has a complementary weak* closed triple ideal in A together with the Krein-Smulian theorem. To see (b) note that since B is a Banach dual space, there is a (unique) weak* continuous operator $\phi: B^{**} \to B$ whose composition with the natural map $B \to B^{**}$ is the identity on B. By (separate) weak* continuity, ϕ is a triple homomorphism. Then $\overline{\pi} = \phi \circ \pi^{**}$ is seen to fill the requirements.

For later use, we conclude this section with some ideal theory. Given an element x in a JB^*-triple A, we write A_x for the JB^*-subtriple of A generated by x. This is the Banach subspace of A generated by the (triple) monomials in x defined by $x^{(1)} = x$, $x^{(2n+1)} = \{xx^{(2n-1)}x\}$ for $n \geq 1$.

LEMMA 1.2. *Let x be an element of a JB^*-triple A. Then $A_x = A_y$ where $y = x^{(2n+1)}$, for all $n \geq 0$.*

Proof. Let $y = x^{(2n+1)}$. It is enough to show that $x \in A_y$ for $n \geq 1$. A_x can be realised as an abelian C^*-algebra B in which x is nonnegative and generates B as a C^*-algebra (cf. [**22**]). Since in B, $x^{(2n+1)} = x^{2n+1}$, we need only observe [**18**, Lemma 5.7] that there is a sequence (P_k) of polynomials with zero constant term such that, for all $n \geq 1$, $x^{2n-1}P_k(x^2) \to x^{2n-1}$ uniformly in B as $k \to \infty$. So $x^{(2n-1)}$ lies in A_y and, by induction, so does x.

PROPOSITION 1.3. *Let I be a closed subspace of a JB^*-triple A. Then the following conditions are equivalent*:

(i) *I is a triple ideal of A*;
(ii) $\{AAI\} \subset I$;
(iii) $\{AIA\} \subset I$;
(iv) $\{AII\} \subset I$.

Proof. The conditions are progressively weaker and (i) $\Leftrightarrow$ (ii) is proved in [**11**, Proposition 1.4]. So it is enough to show that (iv) $\Rightarrow$ (ii). Suppose then that $\{AII\} \subset I$ and let $x \in I$, $a \in A$. Let $n \geq 0$ and note that I is a JB^*-subtriple of A. Using [**25**, JP1], we have

$$\{xax^{(2n+3)}\} = \{xa\{xx^{(2n+1)}x\}\} = \{x\{axx^{(2n+1)}\}x\} \in I.$$

Therefore $\{xaA_{\{xxx\}}\} \subset I$ and hence $\{xax\} \in I$, by Lemma 1.2. Thus, using [**25**, JP10], we have

$$\{aa\{xxx\}\} = 2\{x\{axx\}a\} - \{ax\{xax\}\} \in I,$$

which, by appropriate use of the polarisation identities, means that $\{AA\{III\}\} \subset I$ and hence that $\{AAI\} \subset I$, as required.

If X is an extremally disconnected compact space and I is a norm closed inner ideal of $C(X)$, then $I = \{f \in C(X) \colon f(Y) = 0\}$ for some closed subspace $Y \subset X$. Given a nonzero element $g \in I$, the sets $U_n = \{x \in X \colon |g(x)| > \frac{1}{n}\}$ and $E_n = \overline{U}_n$ are open in X with $E_n \cap Y = \varnothing$, so the characteristic function χ_{E_n} lies in I. Moreover $\|g - g\chi_{E_n}\| < \frac{1}{n}$ for all $n \geq 1$.

LEMMA 1.4. *Let A be a JBW^*-triple and let I be a norm closed inner ideal of A. For $x \in I$, there is a sequence (u_n) of tripotents in I such that $\{u_n u_n x\} = \{u_n x u_n\} \to x$ uniformly.*

Proof. Let M be the weak* closure of A_x in A. Then M can be represented as an abelian W^*-algebra, W, in such a way that $x \geq 0$ in W. In this way $I \cap M$ corresponds to a norm closed ideal J in W. By the preceding remarks, a sequence of projections in J, and hence tripotents in $I \cap M$, can be chosen in the way required.

We note that the above proves that if I and J are norm closed inner ideals in a JBW^*-triple having the same tripotents, then $I = J$.

2. Multipliers. Given a JB^*-triple B and a closed subtriple $A \subset B$, we define

$$M(A, B) = \{x \in B \colon \{xAA\} \subset A\}$$

and call it the set of *multipliers* of A in B. For the special case $A \subset A^{**}$, we write $M(A) = M(A, A^{**})$.

Note that if A and B above are C^*-algebras and $x \in B$ is such that $\{xAA\} \subset B$, then for each $a \in A_+$, we have

$$xa^2 = \{xaa\} + \{xa^{1/2}a^{1/2}\}a - a\{xa^{1/2}a^{1/2}\} \in A.$$

Therefore $xA \subset A$. Similarly $Ax \subset A$. Thus $M(A, B)$ is the idealiser of A in B and $M(A)$ is the multiplier algebra of A.

THEOREM 2.1. *Let A be a JB^*-subtriple of a JB^*-triple B. Then $M(A, B)$ is a JB^*-subtriple of B. It is the largest JB^*-subtriple of B which contains A as a triple ideal.*

Proof. Since $M(A, B)$ is clearly norm closed, the second statement will follow from the first, by Proposition 1.2. Let $x \in M(A, B)$ and $a \in A$. Then we have $\{axa^{(2n+3)}\} = \{a\{xaa^{(2n+1)}\}a\} \in A$ for all $n \geq 0$. By Lemma 1.2, this means that $\{axa\} \in A$ and hence that $\{AxA\} \subset A$, upon polarising. In turn, this shows that

$$\{xx\{aaa\}\} = 2\{a\{xaa\}\} - \{\{axa\}ax\} \in A$$

where we have used [25, JP10]. Therefore $\{xx\{AAA\}\} \subset A$ and $\{xxA\} \subset A$.

In addition, the main identity and then [25, JP2] gives

$$\begin{aligned}\{x\{aaa\}\} &= 2\{\{aax\}ax\} - \{aa\{xax\}\} \\ &= 2\{\{aax\}ax\} - \{a\{axa\}x\} \in A\end{aligned}$$

which implies that $\{xAx\} \subset A$. Consequently,

$$\{\{xxx\}aa\} = 2\{xx\{aaa\}\} - \{x\{aax\}x\} \in A$$

from which we deduce that $\{\{xxx\}AA\} \subset A$, so that $\{xxx\} \in M(A, B)$. Hence $M(A, B)$ is a subtriple of B.

LEMMA 2.2. *Let A be a weak* dense JB^*-subtriple of a JBW^*-triple B and let I be a nonzero inner ideal (not necessarily norm closed) of $M(A, B)$. Then $A \cap I \neq 0$.*

Proof. Let x be a nonzero element of I. Then $\{xA\} \subset I \cap A$, using Theorem 2.1. But $\{xAx\} \neq 0$ else, by separate weak* continuity of the triple product, we would have $\{xxx\} = 0$ and hence $x = 0$, a contradiction.

LEMMA 2.3. *Let A and B be JB^*-triples, $\pi\colon A \to B$ an isomorphism of A into B and $\pi(A)$ a triple ideal of B such that $\pi(A) \cap I \neq 0$ for all nontrivial closed triple ideals I of B. Then there is an isomorphism $\beta\colon B \to M(A)$ such that $\beta\pi$ is the identity on A.*

Proof. Consider the composition $A \xrightarrow{\pi} B \xrightarrow{p} \pi(A)^{**} \xrightarrow{\alpha} A^{**}$ where α is the natural isomorphism and p is the restriction of the natural projection $B^{**} \to \pi(A)^{**}$. Then $\beta = \alpha p$ is injective because $\ker p \cap \pi(A) = 0$ implies $\ker p = 0$. Since $A = \beta\pi(A)$ is a triple ideal of $\beta(B)$, we have $\beta(B) \subset M(A)$.

THEOREM 2.4. *Let $\pi\colon A \to B$ be a triple isomorphism of a JB^*-triple A onto a weak* dense JB^*-subtriple of a JBW^*-triple B. Then the weak* continuous extension $\bar{\pi}\colon A^{**} \to B$ maps $M(A)$ isometrically onto $M(\pi(A), B)$.*

Proof. $\bar{\pi}$ is isometric on $M(A)$ by Lemma 2.2, and $\bar{\pi}M(A) \subset M(\pi(A), B)$ by Theorem 2.1. It follows from Lemma 2.2 and Lemma 2.3 that there is a triple isomorphism $\beta\colon M(\pi(A), B) \to M(A)$ such that $\bar{\pi}\beta$ is the identity on $\pi(A)$. Thus, given $x \in M(\pi(A), B)$ and $y \in \pi(A)$, we have $\{xyy\} \in \pi(A)$ and $\{(\bar{\pi}\beta(x) - x), y, y\} = \pi\beta\{xyy\} - \{xyy\} = 0$. Since $\pi(A)$ is weak* dense in B, this means that $\{(\bar{\pi}\beta(x) - x)BB\} = 0$ and hence that $x = \bar{\pi}\beta(x) \in \bar{\pi}M(A)$. So $\bar{\pi}M(A) = M(\pi(A), B)$.

COROLLARY 2.5. *Let A be a JB^*-triple. Then the natural projection $p\colon A^{**} \to (A^{**})_a$ where $(A^{**})_a$ is the atomic part of A^{**}, maps $M(A)$ isometrically onto $M(p(A), (A^{**})_a)$.*

Proof. This is immediate from Theorem 2.4 because p is isometric on A by [**17**, Proposition 1].

COROLLARY 2.6. *Let A be a JB^*-triple. Then $M(A)$ is a JBW^*-triple if and only if A is a norm closed triple ideal in a JBW^*-triple.*

Proof. Let A be a norm closed triple ideal in a JBW^*-triple B. We may suppose that A is weak* dense in B. Then $B = M(A, B) \cong M(A)$ by Theorem 2.4. The converse is immediate from Theorem 2.1.

REMARK 2.7. For any elementary JB^*-triple K_i, we have that K_i is a triple ideal of $K_i^{**} = C_i$ and so $M(K_i) = K_i^{**}$. At the other extreme, we have:

PROPOSITION 2.8. *Let A be a JB^*triple with a complete tripotent. Then $M(A) = A$.*

Proof. Let u be a complete tripotent of A. The Peirce projections $P_i(u): A^{**} \to A^{**}$ $(i = 0, 1, 2)$ are weak* continuous restrict to the corresponding Peirce projections $A \to A$. Since $P_0(u)A = 0$, we have $P_0(u)(A^{**}) = 0$. Since A is a triple ideal of $M(A)$, we clearly have $P_i(u)M(A) \subset A$ for $i = 1, 2$. So $M(A) = (P_2(u) + P_1(u))M(A) \subset A$ by Pierce decomposition.

3. Weakly compact and compact JB^*-triples. Given a JB^*-triple A, we let $K_0(A)$ denote the Banach subspace of A generated by the minimal tripotents of A. If $x \in A$, then $A(x)$ denotes the norm closed triple ideal in A generated by x.

If $T: X \to X$ is antilinear, we define $T^*: X^* \to X^*$ by $T^*(\rho) = \overline{\rho} \circ T$ where $\overline{\rho}$ is the conjugate of ρ in X^*. Note that T^* is also antilinear. We employ the standard corresponding definitions and notation for linear operators. This does not lead to conflict. For example, if $S, T: X \to X$ are antilinear (so that ST is linear), then $(ST)^* = T^*S^*$.

DEFINITION 3.1. A JB^*-triple is defined to be *weakly compact* if the (antilinear) operator $Q(x): A \to A$ is weakly compact for all $x \in A$; and to be *compact* if $Q(x)$ is compact for all x in A.

LEMMA 3.2. *If A is a weakly compact (respectively, compact) JB^*-triple, then so is every JB^*-subtriple and every quotient of A by a closed triple ideal.*

Proof. This is an elementary consequence of the definitions.

LEMMA 3.3. *Let u be a minimal tripotent in a JB^*-triple A. Then*

(i) *$A(u)$ is the closed subspace of A generated by $\{AuA\}$;*
(ii) *$A(u)$ is elementary;*
(iii) *$K_0(A)$ is a triple ideal of A equal to the c_0-sum of all elementary triple ideals of A.*

Proof. (ii) u is a minimal tripotent of A^{**}, so $A(u)^{**}$ is a Cartan factor by [**8**, p. 302]. For (iii), let $\{A_i\}$ be the family of all elementary triple ideals of A. The A_i are mutually orthogonal by simplicity, each A_i is itself the closed linear span of minimal tripotents and by (ii), every minimal tripotent of A is contained in one of them. Hence $K_0(A) = (\sum_i A_i)_{c_0}$.

(i) The elements of the linear space V generated by $\{AuA\}$ are linear combinations of elements of the form $\{aua\}$ with $a \in A$. We have $\{uAu\} = \mathbb{C}u \subset V$. Given $a, b \in A$,

$$D(a, b)u = 2\{\{abu\}uu\} - \{u\{bau\}u\} \in V$$

and so, from [**25**, JP21], we have

$$\begin{aligned} Q(b)Q(a)u = (&4Q(\{bau\}) + 2Q(Q(b)Q(a)u, u) \\ &- Q(u)Q(a)Q(b) - 4D(b, a)Q(u)D(a, b))u \end{aligned}$$

which is in V. Therefore $\{AVA\} \subset V$. Hence the norm closure of V is a triple ideal, by Proposition 1.2 (iii) $\Rightarrow$ (i), which must equal $A(u)$.

THEOREM 3.4. *The following statements are equivalent for a JB^*-triple A.*

(i) *A is weakly compact;*
(ii) *$D(x, x): A \to A$ is weakly compact for all x in A;*
(iii) *A is an inner ideal of A^{**};*
(iv) *$M(A) = A^{**}$;*
(v) *$K_0(A) = K_0(A^{**})$;*
(vi) *$K_0(A) = A$.*

Proof. (i) $\Leftrightarrow$ (iii). Given $x \in A$, $Q(x): A \to A$ is weakly compact if and only if $Q(x)^{**}A^{**} \subset A$ [**13**, p. 482]. By weak* continuity, $Q(x)^{**} = Q(x)$ on A^{**}. Thus A is weakly compact if and only if $\{xA^{**}x\} \subset A$ for all x in A.

(ii) $\Leftrightarrow$ (iv). In the same way (ii) holds if and only if $\{xxA^{**}\} \subset A$ for all x in A. By Proposition 1.3, this is equivalent to A being a triple ideal of A^{**} and, by Theorem 2.1, to $M(A) = A^{**}$.

(iv) $\Rightarrow$ (iii). Immediate from Theorem 2.1.

(iii) $\Rightarrow$ (v). Let u be any minimal tripotent of A^{**}. Then $J = A^{**}(u)$ is an elementary triple ideal of A^{**} by Lemma 3.3. Suppose (iii) holds. Then $I = A \cap J$ is a nonzero triple ideal of A since $0 \neq \{AuA\} \subset I$. Now I^{**} is a weak* closed triple ideal of the Cartan factor $\overline{J}^{w^*}$. Hence $I^{**} = \overline{J}^{w^*}$. Therefore I is elementary by [**5**, Lemma 3.2]. In particular, I is a triple ideal of I^{**} and hence of J, and so it is equal to J, by simplicity. So $u \in A$.

(v) $\Rightarrow$ (vi). Suppose (v) holds. Then $K_0(A)^{**}$ is the atomic part of A^{**}. Therefore $(A/K_0(A))^{**}$ which can be identified with

$A^{**}/K_0(A)^{**}$, has no nonzero minimal tripotents and so must be trivial. Hence $A = K_0(A)$.

(vi) $\Rightarrow$ (iv). Suppose that A is the c_0-sum of a family of elementary JB^*-triples A_i. Then A_i is a triple ideal of A_i^{**} for each i. Hence A is a triple ideal of $A^{**} = (\sum_i A_i^{**})_{l_\infty}$ and so $M(A) = A^{**}$ by Theorem 2.1. The proof is complete.

If A is a weakly compact JB^*-triple, we will call the elementary triple ideals of A the *components* of A. In this way A is the c_0-sum of its components.

COROLLARY 3.5. *Let A be a JB^*-triple. Then $K_0(A) = K_0(A^{**}) \cap A$ and is the largest inner ideal of A which is also an inner ideal of A^{**}. It is also the largest weakly compact (closed) inner ideal of A. Further $K_0(J) = K_0(A) \cap J$ for every norm closed inner ideal J of A.*

Proof. By Theorem 3.4 (vi) $\Rightarrow$ (i) and Lemma 3.3 (iii), $K_0(A)$ is a weakly compact triple (hence inner) ideal of A. So by Theorem 2.1 together with Theorem 3.4 (iv) $\Rightarrow$ (i), $K_0(A)$ is a triple ideal of $K_0(A)^{**}$ and hence of A^{**}. The same citations show that if I is a norm closed inner ideal of A, then it is weakly compact if and only if it is an inner ideal of A^{**} in which case, since minimal tripotents of I are also minimal tripotents of A, $I = K_0(I) \subset K_0(A)$. It follows from this, together with Lemma 3.1, that $K_0(A) = K_0(A^{**}) \cap A$. The last claim in the statement is similarly proved.

THEOREM 3.6. *Let A be a JB^*-triple. Then the following are equivalent*:

(i) *A is compact*;
(ii) *A is weakly compact with no infinite dimensional C_4 components*;
(iii) *A is isomorphic to a subtriple of $K(H) \oplus C_0(S, C_6)$ for some complex Hilbert space H and discrete topological space S.*

Proof. (i) $\Rightarrow$ (ii). Let A be compact. Then it is weakly compact. If I is a C_4 component of A, then it contains a unitary tripotent u. But then $Q(u)^2: I \to I$ is both compact and the identity operator. Hence I is finite-dimensional.

(ii) $\Rightarrow$ (iii). The components of A of type K_i $(i = 1, 2, 3)$ can each be realised as a subtriple of compact operators on a Hilbert space, and the same is clearly true for any finite dimensional C_4 component.

Since $C_5 \subset C_6$, we see that (iii) follows from (ii) by Theorem 3.4 (i) $\Rightarrow$ (vi).

(iii) $\Rightarrow$ (i). Assume (iii). For each $x \in K(H)$, the linear operator $a \to xax$ is compact on $K(H)$ (cf. [**3**, p. 174]). So $a \to xa^*x = Q(x)a$ is a compact antilinear operator on $K(H)$. In other words $K(H)$ is a compact JB^*-triple. But so is $C_0(S, C_6)$, as follows easily from the finite dimensionality of C_6 and the discreteness of S. Since the l_∞-sum of two compact JB^*-triples is clearly compact, (i) follows from Lemma 3.1.

THEOREM 3.7. *Let A be a JB^*-triple. Then $D(x, x): A \to A$ is compact for all $x \in A$ if and only if A is a c_0-sum of finite-dimensional JB^*-triples.*

Proof. Suppose that $D(x, x): A \to A$ is compact for all x in A. Then A is weakly compact, by Theorem 3.4, and we may suppose it to be a J^*-algebra contained in $B(H)$, say. Given a minimal tripotent u of A, the operator $S: A \to B(H)$ defined by

$$S(x) = uu^*x = 2uu^*D(u, u)x - Q(u)^2(x)$$

is compact. But the subspace of $B(H)$, uu^*A, is norm closed. Indeed, suppose (b_n) is a sequence in A such that $uu^*b_n \to b \in B(H)$. By the compactness of $D(u, u)$, we may suppose that $uu^*b_n + b_nu^*u \to a \in A$. So $uu^*b_n + uu^*b_nu^*u \to uu^*a$. Since $uu^*b_nu^*u = uu^*(uu^*b_nu^*u) \in uu^*A$, it follows that $uu^*b_n \in uu^*A$. Now observe that the identity operator on uu^*A, which is multiplication on the left by uu^*, is compact because S is. Hence uu^*A is finite-dimensional as, similarly, is Au^*u. It follows that the linear span of $Au^*A = Au^*uu^*A$ has finite dimension as therefore does the subspace of A generated by $\{AuA\}$. Therefore $A(u)$ is finite dimensional by Lemma 3.3 (i) and, since all components of A are of this form for some minimal tripotent u, the proof is complete.

4. The RNP and compact elements. Given a JB^*-triple A, a necessary and sufficient condition for A^* to have the RNP is that A^{**} be atomic [**6**, Theorem 2], whereas A has the RNP if, and only if, A is reflexive [**7**, Theorem 6]. So the implications

A has the RNP $\Rightarrow$ A is weakly compact $\Rightarrow$ A^* has the RNP

are clear from the results of §3, for example. We will examine the relationship more closely. In addition, we will exploit the global results of §3 in order to study the effect of the (weak) compactness of the operators $Q(x)$ and $D(x, x)$ for *individual* elements x.

Recall that as well as having a largest weakly compact triple ideal $K_0(A)$, A also has a largest closed triple ideal I with the property that I^* has the RNP [5, Proposition 3.7].

PROPOSITION 4.1. *Let A be a JB^*-triple. The following are equivalent:*

(i) *A is weakly compact;*
(ii) *A^* has the RNP and $M(A)$ is a JBW^*-triple;*
(iii) *$P_2(u)A$ has the RNP for all tripotents u of A and $M(A)$ is a JBW^*-triple.*

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 3.4 and the above remarks. (ii) $\Rightarrow$ (iii). Suppose (ii) holds and let u be a tripotent of A. Then $P_2(u)A = \{uAu\} = \{uM(A)u\}$ because A is a triple ideal of $M(A)$. Therefore by assumption $\{uAu\}$ is a JBW^*-triple. But $\{uAu\}^*$ has the RNP because A^* does. Hence $\{uAu\}$ is reflexive by [7, Theorem 6].

(iii) $\Rightarrow$ (i). Assume (iii). Let u be a tripotent in A. Having the RNP, and hence being reflexive, the closed inner ideal $\{uAu\}$ is weakly compact. So $\{uAu\} \subset K_0(A)$ by Corollary 3.5. But by Lemma 1.4 and the assumption, given $x \in A$, there is a sequence (u_n) of tripotents of A such that $x = \lim_n \{u_n x u_n\} \in K_0(A)$. Condition (i) now follows from Theorem 3.4 (vi) $\Rightarrow$ (i).

COROLLARY 4.2. *If A is a JBW^*-triple, then $K_0(A)$ is the largest closed triple ideal I of A for which I^* has the RNP.*

COROLLARY 4.3. *Let A be a norm separable JB^*-triple. Then A is weakly compact if and only if $M(A)$ is a JBW^*-triple.*

Proof. The necessity being obvious (from Theorem 3.4). Suppose that $M(A)$ is a JBW^*-triple, and let u be any tripotent in A. From the proof of Proposition 4.1, we see that $\{uAu\}$ is norm separable and is the dual of a Banach space, which means that it has the RNP (cf. [9]). So A is weakly compact by Proposition 4.1.

The following should be compared with Corollary 2.5 and Theorem 3.4.

PROPOSITION 4.4. *Let A be a JB^*-triple such that A^* has the RNP and let $p_0: A^{**} \to K_0(A)^{**}$ be the natural projection. Then p_0 is the identity on $K_0(A)$ and maps $M(A)$ isometrically onto $M(p_0(A), K_0(A)^{**})$.*

Proof. Let I be any nonzero norm closed triple ideal of $M(A)$. Then $J = I \cap A \neq 0$ by Lemma 2.2, and, since J^* has the RNP, contains a nonzero minimal tripotent [**5**, Theorem 3.4]. Therefore $I \cap K_0(A) \neq 0$. Thus applying Lemma 2.3 and its proof to the inclusion $K_0(A) \subset M(A)$, we see that $p_0: M(A) \to K_0(A)^{**}$ is isometric and is the identity on $K_0(A)$. Now applying Theorem 2.4 to $p_0: A \to K_0(A)^{**}$, we have $p_0 M(A) = M(p_0(A), K_0(A)^{**})$.

Recall [**22**, **23**] that for each element x of a JB^*-triple A, there is a locally compact subspace S_x of $(0, \infty)$ such that $S_x \cup \{0\}$ is compact and there is a surjective triple isomorphism $\phi: A_x \to C_0(S_x)$ with $\phi(x)$ the identity on S_x. Moreover S_x and ϕ are unique with these properties. Spectral theory provides a sharp comparison of the RNP phenomena with weak compactness.

PROPOSITION 4.5. *Let A be a JB^*-triple.*

(i) *A^* has the RNP $\Leftrightarrow S_x$ is countable for all x in A.*
(ii) *A is weakly compact $\Leftrightarrow S_x$ is discrete for all x in A.*
(iii) *A has the RNP $\Leftrightarrow S_x$ is finite for all x in A.*

Proof. (i) This was proved in [**5**, Theorem 3.4].

(ii) If A is weakly compact then, given $x \in A$, so is $A_x = C_0(S_x)$, by Lemma 3.1. Since each component of the latter can only be a copy of $\mathbb{C}$, S_x must be discrete. Conversely suppose that the spectral condition is satisfied by A. Let u be any tripotent of A. Recall that $\{uAu\}$ can be realised as a JB^*-algebra. By spectral theory, $S_x = \sigma(x)\backslash\{0\}$ for every $x \in \{uAu\}_+$. It follows from [**4**, Theorem 3.3] that $\{uAu\}_{sa}$ is a unital dual JB-algebra, so that u is a finite sum of orthogonal minimal projections of $\{uAu\}_{sa}$. Therefore u is a finite sum of minimal tripotents of A. In particular, $u \in K_0(A)$. But by hypothesis every element x of A can be written as a norm convergent sum $x = \sum_{n=1}^{\infty} \lambda_n u_n$ where $\lambda_n \geq 0$ and u_n is a tripotent of $A_x = C_0(S_x)$. Hence $x \in K_0(A)$ and A is weakly compact by Theorem 3.4.

(iii) If A has the RNP, and so is reflexive [7], then $A_x = C_0(S_x)$ is reflexive and hence S_x is finite, for all x in A. If on the other hand S_x is finite for all x in A, then A is weakly compact by (ii). Suppose that (u_n) is an infinite sequence of orthogonal tripotents in A. Then $x = \sum_{n=1}^{\infty} u_n/n^2 \in A$ and the monomials $x^{(2k+1)}$, $k \geq 0$, are clearly linearly independent, implying that A_x is infinite dimensional, a contradiction. Thus A must be a finite sum of elementary triples drawn from the following types: $K(H, H')$ with $\dim H' < \infty$; finite dimensional K_2 and K_3; arbitrary K_4, K_5 and K_6. Hence A is reflexive.

REMARK 4.6. We note from the above proof that if A is weakly compact, then each nonzero element x of A can be written as a norm convergent (possibly finite) sum $x = \sum \lambda_n u_n$, where u_n are mutually orthogonal minimal tripotents of A and $\{\lambda_n\} = S_x \subset (0, \infty)$ (cf. **[18**, Theorem 3.3]).

We say that an element x of a JB^*-triple A is a *weakly compact* (respectively, *compact*) *element* of A if $Q(x)\colon A \to A$ is weakly compact (respectively, compact).

PROPOSITION 4.7. *The set of all weakly compact elements of a JB^*-triple A is the triple ideal $K_0(A)$.*

Proof. Let $x \in A$ be a weakly compact element and consider the norm closed inner ideal $I = \overline{Q(x)A}$, which contains x by Lemma 1.2, for instance. Given $y \in I$, we have $\|y - y_n\| \to 0$ for some $y_n = Q(x)a_n$ where $a_n \in A$. Since $\|Q(y) - Q(y_n)\| \to 0$ and $Q(y_n) = Q(x)Q(a_n)Q(x)\colon A \to A$ is weakly compact, $Q(y)\colon I \to I$ must be weakly compact. Hence $x \in I \subset K_0(A)$ by Corollary 3.5. On the other hand, let $x \in K_0(A)$. By Corollary 3.5 and Remark 4.6, we have $x = \sum \lambda_n u_n$ where u_n are mutually orthogonal minimal tripotents. With $x_n = \lambda_1 u_1 + \cdots + \lambda_n u_n$ and $v_n = u_1 + \cdots + u_n$, we have $Q(x_n) = Q(v_n)Q(x_n)Q(v_n)$ and $\|Q(x_n) - Q(x)\| \to 0$. But v_n is a weakly compact element of $K_0(A)$, by Theorem 3.4, and hence of A since $Q(v_n)^3 = Q(v_n)$ and $K_0(A)$ is a triple ideal. Therefore x is a weakly compact element of A.

COROLLARY 4.8. *An element x of a JB^*-triple A is weakly compact if and only if $D(x, x)\colon A \to A$ is weakly compact.*

Proof. Let x be a weakly compact element of A. Using Proposition 4.7 and viewing $A_x = C_0(S_x) \subset K_0(A)$, we see that $x = y^{(3)}$ for some

$y \in K_0(A)$. Now $D(y, y): K_0(A) \to K_0(A)$ is weakly compact by Theorem 3.4. So, using [**25**, JP13],

$$D(x, x) + Q(y)Q(y, x) = 2D(y, y)D(y, x): A \to A$$

is weakly compact since $D(y, x)A \subset K_0(A)$. But $Q(y): A \to A$ is weakly compact by Proposition 4.7 and hence so is $D(x, x)$.

Conversely, suppose that $D(x, x)$ is weakly compact on A for some nonzero element x in A. Identifying $A_x = C_0(S_x)$, we have $D(x, x): C_0(S_x) \to C_0(S_x)$ $(y \to x^2 y)$ is weakly compact. It follows that all left multiplications on $C_0(S_x)$ are weakly compact ($x \geq 0$ and it generates $C_0(S_x)$). Hence S_x is discrete by [**12**, 4.7.20]. Hence we can write, as a norm-convergent sum, $x = \sum \lambda_n u_n$ where $\lambda_n > 0$ and u_n are mutually orthogonal tripotents of $C_0(S_x)$. Using [**25**, JP4], $Q(x\{xxx\}) = D(x, x)Q(x): A \to A$ is weakly compact. By the rule that $Q(u)Q(u, v)Q(u) = 0$, whenever u and v are orthogonal tripotents, we see that, for each n,

$$\lambda_n^4 Q(u_n) = Q(u_n)Q(x, \{xxx\})Q(u_n): A \to A$$

is weakly compact. Hence x is a weakly compact element of A by Proposition 4.7.

COROLLARY 4.9. *Let V be the set of compact elements of a JB^*-triple A. The following conditions are equivalent*:

(i) *V is a linear subspace of A*;
(ii) *$V = K_0(A)$*;
(iii) *$K_0(A)$ is a compact JB^*-triple.*

Proof. (iii) $\Rightarrow$ (ii) is proved as in the second half of Proposition 4.4, and (ii) $\Rightarrow$ (i) is trivial. Note that $V \subset K_0(A)$, by Proposition 4.7. If $K_0(A)$ is not compact, then by Theorem 3.6, it has an infinite dimensional C_4 component B, say. We have $u = u_1 + u_2$ where u is the unitary element of B and u_1, u_2 are minimal tripotents. Obviously u_1 and u_2 are compact elements of A, but u is not, else B is finite dimensional. Hence V is not a linear space. This proves (i) $\Rightarrow$ (iii).

COROLLARY 4.10. *Let A be a JB^*-triple. The set $\{x \in A: D(x, x)$ is compact$\}$ is equal to the norm closed triple ideal I of A generated by the class of all finite dimensional triple ideals of A.*

Proof. We note that I is the c_0-sum of all finite dimensional components of $K_0(A)$. Thus given $x \in I$, $Q(x)$ and $D(x, x): I \to$

I are compact by Theorem 3.6 and Theorem 3.7. The argument in the first half of Corollary 4.8, transparently adapted, proves that $D(x, x): A \to A$ is compact.

Conversely, suppose that $D(x, x)$ is compact on A where $x \in A\backslash\{0\}$. Then by Proposition 4.7, the closed triple ideal $A(x)$ is contained in $K_0(A)$ and it is weakly compact by Lemma 3.1. Thus $A(x) = (\sum A_i)_{c_0}$ where each A_i is an elementary triple ideal of A. Writing $x = \sum x_i$ with $x_i \in A_i$, the map $D(x_i, x_i): A_i \to A_i$ is compact for each i. It is enough to show that each A_i is finite dimensional. We may suppose therefore that $A(x)$ is an elementary J^*-algebra in $B(H)$, say. Observe that the argument used in the second half of Corollary 4.8 shows $Q(x): A(x) \to A(x)$ to be compact. Consider the spectral decomposition $x = \sum \lambda_n u_n$ in $A(x)$ where the $\lambda_n > 0$ and the u_n are mutually orthogonal minimal tripotents. Now with $y = \lambda_1^{-1}x$, we see that the map $S: A(x) \to B(H)$ defined by

$$S(a) = u_1 u_1^* a = u_1 u_1^*(yy^*a + ay^*y) - u_1 y_1^*(yy^*ay^*y)$$

is compact. Hence $A(x) = A(u_i)$ is finite dimensional as in the proof of Theorem 3.7, and the proof is complete.

We conclude with the following relationship between the RNP and the (weak) compact operations.

THEOREM 4.11. *Let A be a JB^*-triple. The following are equivalent*:

(i) *A^* has the RNP*;
(ii) *A has a composition series $\{I_\rho\}_{0\le\rho\le\beta}$ such that $I_{\rho+1}/I_\rho$ is weakly compact for each $\rho < \beta$*;
(iii) *Every JB^*-triple quotient of A contains a nonzero weakly compact element*;
(iv) *Every JB^*-triple quotient of A contains a nonzero compact element.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow from [**5**, Theorem 3.4] and Theorem 3.4, whereas (iii) $\Rightarrow$ (iv) is trivial. Assume that (iv) holds. Let I be the largest closed triple ideal of A for which I^* has the RNP. If A^* does not have the RNP, then $I \neq A$ and it follows [**5**, Corollary 3.7] that $K_0(A/I) = 0$. By Proposition 4.7 and Corollary 4.8, this means that A/I cannot contain a nonzero compact element, a contradiction. Hence A^* has the RNP.

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