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LUSTERNIK-SCHNIRELMANN INVARIANTS IN PROPER HOMOTOPY THEORY

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We introduce and study proper homotopy invariants of the Lusternik-Schnirelmann type, p-cat (–), p-Cat(–), and $\operatorname{cat}\varepsilon(-)$ in the category of T_2-locally compact spaces and proper maps. As an application, $\mathbb{R}^n$ $(n \neq 3)$ is characterized as (i) the unique open manifold X with p-Cat$(X) = 2$, or (ii) the unique open manifold with one strong end and p-cat$(x) = 2$.

Introduction. The category $\operatorname{cat}(X)$ of a space X in the sense of Lusternik and Schnirelmann (L-S category) is the smallest number k such that there exists an open covering $\{X_1, \dots, X_k\}$ of X for which each inclusion $X_j \subseteq X$ is nullhomotopic in X. This concept was introduced by the quoted authors in their studies on calculus of variations [**16**] and they used it as a lower bound for the number of critical points of a differentiable real function on a manifold. The basic work on the homotopical significance of cat is due to Borsuk (see [**5**]). Borsuk's work was continued by Fox [**10**].

Here we present the definition and the basic properties of a new numerical topological invariant for T_2-locally compact spaces which agrees with the notion of L-S category for T_2-compact spaces. This invariant, denoted p-cat(X), is called the proper L-S category of X and turns out to be a proper homotopy invariant of X. Hence, p-cat(X) is a finer invariant than $\operatorname{cat}(X)$.

In [**10**] several generalizations of L-S category are suggested. More explicitly, a general notion of L-S $\mathscr{A}$-category with respect to a class $\mathscr{A}$ of spaces has been developed by Puppe and Clapp in [**6**]. Our work shares some common points with [**6**] but does not fit into the notion of L-S $\mathscr{A}$-category since we entirely deal with proper maps instead of ordinary continuous maps.

Another generalization of L-S category has been given in [**1**], where L-S category for pro-objects in pro-$\mathscr{T}\!op$ is defined. This idea is related to proper L-S category by the Edwards-Hastings embedding (see [**8**]) which provides a close link between proper homotopy theory and homotopy in pro-$\mathscr{T}\!op$.

We shall work entirely in the category $\mathfrak{P}_\infty$ of non-compact T_2-locally compact σ-compact spaces and proper maps. Notice that any T_2-compact space X can be regarded in $\mathfrak{P}_\infty$ as the wedge $X \vee J$, where J will stand for the half-line $[0, \infty)$.

We recall that a proper map (p-map) is a continuous map $f: X \to Y$ such that $f^{-1}(K)$ is compact for each compact $K \subseteq Y$. Proper homotopy (p-homotopy), proper deformation (p-deformation), etc. can be defined in the natural way. The symbols "$\simeq$", "$\simeq_p$" and "$\cong$" stand for homotopy equivalence, p-homotopy equivalence and homeomorphism respectively.

Finally we also recall the notion of end in proper homotopy. A Freudenthal end of $X \in \mathfrak{P}_\infty$ is an element of the inverse limit $\mathcal{F}(X) = \varprojlim \pi_0(X-K)$ where K ranges over the family of compact subsets of X and π_0 stands for the set of connected components. The topology of X can be enlarged to a topology on $X \cup \mathcal{F}(X)$ in such a way that $\mathcal{F}(X)$ turns out to be homeomorphic to a closed set of the Cantor set (see **[11]** for details). It is easy to check that any p-map $f: X \to Y$ induces a continuous map $f_*: \mathcal{F}(X) \to \mathcal{F}(Y)$ such that f_* is a homeomorphism when f is a p-homotopy equivalence.

1. Basic properties. In **[8]** the following lemma is proven,

1.1. Lemma (**[8**; 6.3.5]). *Any space X in $\mathfrak{P}_\infty$ admits a p-map $r: X \to J$ unique up to p-homotopy.*

1.2. Definition. A closed subset $C \subseteq X$ is said to be properly deformable to J in X if there exists a diagram in $\mathfrak{P}_\infty$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "r"'] & X \\ J \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutative up to p-homotopy. Notice that we may use r as the restriction of a p-map $X \to J$ given by Lemma 1.1.

1.3. Definition. Given a space X in $\mathfrak{P}_\infty$, $A \subseteq X$ is said to be properly categorical (p-categorical) in X if there is a closed neighbourhood of A properly deformable to J in X.

An open covering $\{U_\alpha\}$ of X is said to be p-categorical if each U_α is p-categorical in X (i.e. $\overline{U}_\alpha$ is properly deformable to J in X).

The p-category of X, p-cat(X), is the least number n such that X admits a p-categorical open covering with n elements. If no finite p-categorical covering exists then p-cat$(X) = \infty$.

LEMMA 1.4 [4; 3.4]. *Let P be a locally compact metrizable space and Q a locally compact ANR. Suppose X is a closed subset of P and $f, g: X \to Q$ are p-homotopic maps. If $\overline{f}$, $\overline{g}$ are extensions of f and g respectively, there exists a closed neighbourhood U of X such that $\overline{f}|U$ and $\overline{g}|U$ are p-homotopic.*

Hence, if we deal with locally finite polyhedra we have

1.5. PROPOSITION. *Given a locally finite polyhedron P,* p-cat(P) *is the smallest number n such that P can be covered by n p-categorical subpolyhedra.*

Proof. Given a p-categorical open covering $\{W_i\}$ let $H^i: \overline{W}_i \times I \to P$ be a p-deformation of $\overline{W}_i$. Since P is ANR by Lemma 1.4 there is a closed neighbourhood Ω_i of $\overline{W}_i$ and a p-extension $\overline{H}^i: \Omega_i \times I \to P$. Now by [23; 3.5] we may take a closed subpolyhedron P_i such that $\overline{W}_i \subseteq P_i \subseteq \operatorname{int}\Omega_i$ and thus $\{P_i\}$ is a covering of P with P_i p-categorical in P. Conversely, if P is covered by m p-categorical subpolyhedra we may use regular neighbourhoods to obtain a p-categorical open covering with m elements.

The following properties of p-cat(–) are straightforwardly checked:

1.6. PROPOSITION. (i) $\text{cat}(X) \leq \text{p-cat}(X)$. *If X is compact the equality holds.*

(ii) $\text{p-cat}(X) = 1$ *if and only if* $X \simeq_{\text{p}} J$.

(iii) *Let $f: X \to Y$ and $g: Y \to X$ be p-maps such that* $fg \simeq_{\text{p}} \text{id}_Y$. *Then* $\text{p-cat}(Y) \leq \text{p-cat}(X)$. *In particular,* p-cat(–) *is a p-homotopy invariant.*

(iv) *If A is a p-retract of X then* $\text{p-cat}(A) \leq \text{p-cat}(X)$.

(v) *If A and B are closed and* $X = \operatorname{int} A \cup \operatorname{int} B$, *then* $\text{p-cat}(X) \leq \text{p-cat}(A) + \text{p-cat}(B)$.

The next proposition gives us an elementary relation between the set of Freudenthal ends of X and p-cat(X).

1.7. PROPOSITION. *Given a space X in $\mathfrak{P}_\infty$ we have* $\text{card}(\mathscr{F}(X)) \leq \text{p-cat}(X)$.

Proof. If $\text{p-cat}(X) = \infty$ there is nothing to prove. Otherwise, if $C_1, C_2, \dots, C_m$ is a p-categorical closed covering of X, the natural inclusions $K_j: C_j \to X$ induce continuous maps $k_j: \mathscr{F}(C_j) \to$

$\mathscr{F}(X)$. Moreover, it is easy to check that $\mathscr{F}(X) = \bigcup \operatorname{Im} k_{j*}$ and since C_j is p-deformable to J we get that each $\operatorname{Im} k_{j*}$ is an one-point set. Hence $\operatorname{card}(\mathscr{F}(X)) \leq \text{p-cat}(X)$.

1.8. EXAMPLES. (a) $\text{p-cat}(\mathbb{R}^n) = 2 > 1 = \operatorname{cat}(\mathbb{R}^n)$.

(b) For any T_2-compact space X,

$$\operatorname{cat}(X) = \text{p-cat}(X \times J) = \text{p-cat}(X \vee J).$$

(c) If $\widetilde{S}^n$ is the space obtained from J by attaching one copy of S^n at each natural number $n \in J$ we have $\text{p-cat}(\widetilde{S}^n) = 2$.

(d) If X is a space in $\mathfrak{P}_\infty$ and $r\colon X \to J$ is a p-map we can define (up to p-homotopy) the proper cone C_pX of X as the push-out of the diagram $X \times I \xleftarrow{i_0} X \xrightarrow{r} J$ and the proper mapping cone C_pf of a p-map f is defined in the natural way (see **[2]** for details). If $f = r$, C_pr turns to be the proper suspension $\sum_p X$ of X. As in the ordinary case it follows from 1.6(iii) and (iv)

$$\text{p-cat}(C_pf) \leq \text{p-cat}(Y) + 1.$$

In particular $\text{p-cat}(\sum_p X) \leq 2$.

(e) As a consequence of (d), $\text{p-cat}(X) \leq \dim X + 1$, if X is a locally finite CW-complex with only one Freudenthal end.

(f) The notion of categorical sequence due to Fox (see **[10]**) can be translated into proper terms. Namely, given a space X in $\mathfrak{P}_\infty$ a sequence of open sets $V_1 \subseteq \cdots \subseteq V_n = X$ for which each difference $V_i - V_{i-1}$ is p-categorical ($V_0 = \varnothing$), is called a p-categorical sequence. It is easy to check that $\text{p-cat}\, X \leq n$ if and only if X admits a p-categorical sequence of length n. By using this result, one shows the inequality $\text{p-cat}(X \times Y) \leq \text{p-cat}(X) + \text{p-cat}(Y) - 1$.

(g) The inequality $\max\{\operatorname{cat} X, \operatorname{cat} Y\} \leq \operatorname{cat}(X \times Y)$ holds in ordinary L-S category but not in proper L-S category as the following example shows. Let $X = \mathbb{R}^2$ and $Y = J$; then $\text{p-cat}(X \times Y) = 1 < \text{p-cat}(\mathbb{R}^2)$.

The following definition provides a new proper L-S invariant which is the translation of Ganea's strong L-S category into proper homotopy (see **[15]**).

DEFINITION 1.9. given a space X in $\mathfrak{P}_\infty$, the strong L-S category of X is the smallest integer $\text{p-Cat}(X)$ such that there exists a space Y in $\mathfrak{P}_\infty$ p-homotopically equivalent to X which can be covered by p-Cat(X) closed sets each with the same p-homotopy type as J. Obviously p-cat$(X) \leq$ p-Cat(X).

The relation between p-Cat(X) and the 1-LC at ∞ condition is given in the following theorem. We firstly recall that a space X in $\mathfrak{P}_\infty$ is 1-LC at ∞ if given a sequence of compact subsets $\{L_j\}$ with $L_j \subseteq \operatorname{int} L_{j+1}$ and $X = \bigcup L_j$, the inverse sequence

$$(1.9.1) \qquad \pi_1(X - L_1, \alpha(t_1)) \xleftarrow{\theta_1} \pi_1(X - L_2, \alpha(t_2)) \xleftarrow{\theta_2} \cdots$$

is trivial, where $\alpha\colon J \to X$ is a proper map with $\alpha([t_i, \infty)) \subseteq X - L_i$ $(i \geq 1)$ and θ_i is the inclusion induced homomorphism followed by the change of basepoint isomorphism given by $\alpha|[t_i, t_{i+1}]$.

THEOREM 1.10. *Let X be a space in $\mathfrak{P}_\infty$ with one Freudenthal end. If* p-Cat$(X) = 2$, *X is 1-LC at ∞ if and only if the inverse sequence*

$$(1.10.1) \quad H_1(X - L_1) \leftarrow H_1(X - L_2) \leftarrow \cdots \leftarrow H_1(X - L_n) \leftarrow \cdots$$

is trivial.

This theorem is an immediate corollary of

PROPOSITION 1.11. *Let X be as in Theorem* 1.10. *Assume that X can be covered by two one-ended p-categorical closed subsets U and V. Then X is 1-LC at ∞ if and only if the inverse sequence* (1.10.1) *is trivial.*

Proof. Let $\{U, V\}$ be a p-categorical open covering with U and V one-ended. We claim that $U \cap V$ also is one-ended or equivalently that the inverse sequence

$$(1.11.1) \qquad \widetilde{H}_0(U \cap V - L_1) \leftarrow \cdots \leftarrow \widetilde{H}_0(U \cap V - L_n) \leftarrow \cdots$$

is trivial. In the commutative diagram

$$\begin{tikzcd}
H_1(X - L_n) \arrow[d] & H_1(X - L_{n_1}) \arrow[l] \arrow[d] & \\
\widetilde{H}_0(U \cap V - L_n) & \widetilde{H}_0(U \cap V - L_{n_1}) \arrow[l, "(1)"'] \arrow[d] & \widetilde{H}_0(U \cap V - L_{n_2}) \arrow[l, "(2)"'] \arrow[d] \\
 & \widetilde{H}_0(U - L_{n_1}) \oplus \widetilde{H}_0(V - L_{n_1}) & \widetilde{H}_0(U - L_{n_2}) \oplus \widetilde{H}_0(V - L_{n_2}) \arrow[l]
\end{tikzcd}$$

where the vertical arrows are provided by the respective Mayer-Vietoris sequences. We may choose $n_2 > n_1 > n$ such that the upper and lower horizontal arrows are trivial. This implies that the composition of the morphisms (1) and (2) is trivial and so the inverse sequence (1.11.1) is trivial.

We have the following facts:

(i) Since U and V are p-categorical, we may choose $n_3 > n$ such that any loop either in $U - L_{n_3}$ or in $V - L_{n_3}$ is null-homotopic in $X - L_n$.

(ii) On the other hand, since $U \cap V$ is one-ended there is $n_4 > n_3$ such that all the connected components of $U \cap V - L_{n_4}$ are included in the same connected component of $U \cap V - L_{n_3}$.

Given any loop $f\colon (I, \{0, 1\}) \to (X - L_{n_4}, *)$, by Lebesgue's Lemma we may find a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $f|[t_i, t_{i-1}]$ is included either in $U - L_{n_4}$ or in $V - L_{n_4}$. Therefore, by (ii), f is a loop in $X - L_{n_3}$ that can be expressed as a product of loops either in $U - L_{n_3}$ or in $V - L_{n_3}$ and each factor is null-homotopic in $X - L_n$ by (i). Thus the morphism

$$\pi_1(X - L_{n_4}) \to \pi_1(X - L_n)$$

is trivial and X is 1-LC at ∞.

Corollary 1.12. *Let X be a homologically trivial open n-manifold $(n \geq 3)$ with* p-Cat$(X) = 2$. *Then X is 1-LC at ∞.*

Proof. As $n \geq 3$, Poincaré-Lefschetz Duality arguments (see [**7**; 3.2]) show that the inverse sequence (1.10.1) is semistable (i.e. satisfies the Mittag-Leffler condition) and its inverse limit is trivial. Then by [**18**; II.6.2.2] the inverse sequence (1.10.1) is trivial. Now the result follows from Theorem 1.11.

2. Proper L-S category and L-S category in pro-Top. We recall that $\mathrm{cat}(f) \leq n$ for a continuous map f from X to Y, if there is an open covering $\{V_1, \dots, V_n\}$ of X such that $f|V_j$ is homotopically trivial for $1 \leq j \leq n$.

In [**1**] a notion of L-S category for inverse systems is defined. Namely, given an inverse systems $\chi = \{X_\alpha; p_{\alpha\beta}\}$, the L-S category of χ, $\mathrm{cat}(\chi)$, is $\leq n$ if for each α there exists $\beta > \alpha$ such that $\mathrm{cat}(p_{\alpha\beta}) \leq n$.

If pro-$\mathcal{T}op$ is the category of topological inverse systems and pro-morphisms, Edwards and Hastings (see [**8**]) have proven that there exists a natural functor $\varepsilon\colon \mathfrak{P}_\infty \to$ pro-$\mathcal{T}op$, $\varepsilon(X)$ being the inverse system $\{U_1 \leftarrow U_2 \leftarrow \cdots\}$ where $U_k = X - C_k$ and C_k is an increasing sequence of compacts with $C_k \subseteq \operatorname{int} C_{k+1}$ and $X = \bigcup C_k$. A relation between p-cat(X) and cat$(\varepsilon(X))$ is given by

2.1. THEOREM. *If X is a space in $\mathfrak{P}_\infty$,* p-cat$(X) \geq$ cat$(\varepsilon(X))$.

Proof. If $\{W_1, \dots, W_n\}$ is a p-categorical open covering of X, we consider the covering $\{W_j \cap U_k\}$ of U_k. Let $r_j \colon \overline{W}_j \to J$ and $\alpha_j \colon J \to X$ be p-maps such that there are p-homotopies $H^j \colon \overline{W}_j \times I \to X$ between $\alpha_j \circ r_j$ and the inclusion $\overline{W}_j \subseteq X$. For each k there exists $s(k) \geq k$ such that $H^j(W_j \cap U_{s(k)} \times I) \subseteq U_k$. But since J is contractible it follows that $U_{s(k)}$ is null-homotopic in each $W_j \cap U_{s(k)}$. So cat$(\varepsilon(X)) \leq$ p-cat(X).

2.2. REMARKS. (a) We now consider the category (pro-$\mathcal{T}op$, $\mathcal{T}op$) whose objects are arrows $f \colon \chi \to A$ where χ is an object in pro-$\mathcal{T}op$ and A is a space regarded as the constant inverse system. Morphisms are pairs (ϕ, h): where $\phi \colon \chi \to \xi$ is a pro-morphism and $h \colon A \to B$ is a continuous map compatible with ϕ via the bonding maps. Edwards and Hastings proved that $\varepsilon \colon \mathfrak{P}_\infty \to$ (pro-$\mathcal{T}op$, $\mathcal{T}op$) given by $\varepsilon(X) = X \leftarrow U_1 \leftarrow U_2 \cdots$ is a full embedding (see **[8]**). We can also prove that cat$(\varepsilon(X)) \leq$ p-cat(X) in this case.

(b) There exist spaces with cat$(\varepsilon(X)) <$ p-cat(X). Indeed, let X be the punctured torus. Then p-cat$(X) = 3$ according to Corollary 3.3 but obviously cat$(\varepsilon(X)) = $ cat$(S^1) = 2$.

2.3. PROPOSITION. *If M is a compact connected triangulable manifold with boundary and $W = M - \partial M$ then* cat$(\partial M) \leq$ p-cat$(W) \leq$ cat$(\partial M)+$cat(M). *Furthermore, if M is contractible then* p-cat$(W) =$ cat(∂M).

Proof. The inverse system $\varepsilon(W)$ in (pro-$\mathcal{T}op$, $\mathcal{T}op$) is

$$W \leftarrow \partial M \times [0, \infty) \leftarrow \partial M \times [1, \infty) \cdots .$$

Now we have cat$[i \colon \partial M \times [0, \infty) \to W] \leq$ cat$(\partial M \times [0, \infty)) =$ cat(∂M) and cat$(\varepsilon(X)) = \max\{cat(i)$, cat$(\partial M)\} =$ cat(∂M). So, by Remark 2.2(a) cat$(\partial M) \leq$ p-cat(W).

On the other hand if we take a copy of M, $M' \subseteq W$ then p-cat(W) $\leq$ p-cat$(M' \vee J)$ + p-cat$(\partial M' \times J)$ = cat(M) + cat(∂M) by 1.6(viii) and 1.8(b).

If we assume that M is contractible, we get $\operatorname{cat}(\partial M) \leq \text{p-cat}(W) \leq \operatorname{cat}(\partial M) + 1$. But actually $\text{p-cat}(W) \leq \operatorname{cat}(\partial M)$. Indeed, if $\{W_j\}_{j \leq n}$ is a categorical covering of ∂M, we may assume that $\overline{W}_1$ is a subpolyhedron (see Proposition 1.5). By [22; 6.30] we extend the proper deformation of $\overline{W}_1 \times J$ into $\{*\} \times J$ to a proper deformation of $\overline{W}_1 \times J \cup M$. Thus $\{\overline{W} \times J \cup M, W_2 \times J, \dots, W_n \times J\}$ is a p-categorical covering of W.

The invariant $\operatorname{cat}\varepsilon(X)$ can be used to give some results on the behaviour of the inverse sequence

$$\pi_1(X - K_1) \leftarrow \pi_1(X - K_2) \leftarrow \cdots \leftarrow \pi_1(X - K_n) \leftarrow \cdots \tag{2.3.1}$$

where $\{K_i\}$ is a sequence of compact subsets with $K_i \subseteq \operatorname{int} K_{i+1}$, and $X = \bigcup K_i$.

2.4. THEOREM. *Let X be a locally finite polyhedron with one Freudenthal end and* $\operatorname{cat}\varepsilon(X) = 2$. *Assume that the inverse sequence of abelian groups*

$$H_1(X - K_1) \leftarrow H_1(X - K_2) \leftarrow \cdots \leftarrow H_1(X - K_n) \leftarrow \cdots$$

is trivial. Then the inverse sequence (2.3.1) *is pro-isomorphic in* pro-$\mathcal{G}r$ *to an inverse sequence of finitely generated groups.*

THEOREM 2.5. *Under the hypotheses of Theorem* 2.4, *X is* 1-*LC at* ∞ *if and only if the inverse sequence* (2.3.1) *is semistable, i.e.* $\lim^1 \pi_1(X - K_j) = *$.

Proof of Theorem 2.4. We may choose the sequence $\{K_j\}$ with $U_j = \overline{X - K_j}$ subpolyhedron of X. Up to pro-isomorphism we may replace $\pi_1(X - K_j)$ by $G_j = \pi_1(U_j)$ in (2.3.1). By using the 1-skeleton of X, it is easy to check that there is a commutative diagram

$$\begin{tikzcd}
F(L_1) \arrow[d] & F(L_2) \arrow[l] \arrow[d] & F(L_3) \arrow[l] \arrow[d] & \cdots \arrow[l] & F(L_n) \arrow[l] \arrow[d] & \arrow[l] \\
G_1 & G_2 \arrow[l] & G_3 \arrow[l] & \cdots \arrow[l] & G_n \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{2.4.1}$$

where $F(L_n)$ denotes the corresponding free group of basis L_n, the differences $L_n - L_{n+1}$ are finite, and the bonding morphisms are the natural inclusions.

Since the inverse sequence

$$G_1^{\mathrm{ab}} \leftarrow G_2^{\mathrm{ab}} \leftarrow G_3^{\mathrm{ab}} \leftarrow \cdots \leftarrow G_n^{\mathrm{ab}} \leftarrow \cdots$$

is trivial, by abelianizing (2.4.1) we readily check that G_n^{ab} is finitely generated for $n \geq 1$.

Now, since

$$\operatorname{cat} \varepsilon(X) = \sup\{\operatorname{cat}[i_n : U_n \to U_{n-1}]\} = 2$$

we get from [**15**; 7.3] that for any $n \geq 2$, there is μ_n such that

$$\begin{array}{ccc} U_n & \xrightarrow[\mu_n]{} & U_{n-1} \vee U_{n-1} \\ \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle l_{n-1}} \\ U_{n-1} & \xrightarrow{\Delta} & U_{n-1} \times U_{n-1} \end{array}$$

is commutative up to homotopy. This diagram induces a commutative diagram

$$(2.4.2) \qquad \begin{array}{ccc} G_n & \xrightarrow[\mu_{n*}]{} & G_{n-1} * G_{n-1} \\ \downarrow{\scriptstyle i_{n*}} & & \downarrow{\scriptstyle l_{n-1*}} \\ G_{n-1} & \xrightarrow{\Delta} & G_{n-1} \times G_{n-1} \end{array}$$

where l_{n*} induces the natural epimorphism $a * b \to (a, b)$.

From the Kurosh Subgroup Theorem (see [**17**]) the group $l_{n-1*}^{-1}\Delta i_{n*}(G_n)$ is a free group. Therefore, $\mu_{n*} : G_n \to \operatorname{Im} \mu_{n*}$ is an epimorphism onto a free group and so $(\operatorname{Im} \mu_{n*})^{\mathrm{ab}}$ is a finitely generated free abelian group. Hence, $\operatorname{Im} \mu_{n*}$ is a finitely generated free group.

On the other hand, Δ is injective and so the commutativity of (2.4.2) yields a natural epimorphism $\operatorname{Im} \mu_{n*} \twoheadrightarrow \operatorname{Im} i_{n*}$. Thus, $\operatorname{Im} i_{n*}$ is a finitely generated group.

It is a well-known fact that (2.3.1) is pro-isomorphic to the inverse sequence

$$(2.4.3) \qquad \operatorname{Im} i_{1*} \leftarrow \operatorname{Im} i_{2*} \leftarrow \cdots \leftarrow \operatorname{Im} i_{n*} \leftarrow \cdots.$$

Proof of Theorem 2.5. If in addition $\lim^1 G_j = *$, we know that the inverse sequence (2.3.1) satisfies the Mittag-Leffler condition (see [**18**; p. 174]) and we may assume that the bonding morphisms in (2.4.3) are onto. We shall denote $\operatorname{Im} i_{n*}$ by H_n.

Since the sequences (2.3.1) and (2.4.3) are pro-isomorphic, the abelianization of (2.4.3) is trivial. Furthermore, as the bonding morphisms are onto we get that H_n is its own commutator subgroup. Thus, $H_n^{\mathrm{ab}} = 0$.

On the other hand, diagram (2.4.2) induces a commutative diagram

$$\begin{array}{ccc} H_n & \xrightarrow[\mu']{} & H_{n-2} * H_{n-2} \\ \Big\downarrow{\scriptstyle (1)} & & \Big\downarrow{\scriptstyle i_{n-2*}} \\ H_{n-2} & \xrightarrow{\Delta} & H_{n-2} \times H_{n-2} \end{array} \tag{2.4.4}$$

where (1) is the natural bonding morphism and $\mu' = (i_{n-1} * i_{n-1}) \circ \mu : H_n \to H_{n-2} * H_{n-2}$.

As in the proof of Theorem 2.4, $\operatorname{Im} \mu'$ is a free group and the epimorphism $H_n^{\mathrm{ab}} = 0 \twoheadrightarrow (\operatorname{Im} \mu')^{\mathrm{ab}}$ implies that $\operatorname{Im} \mu'$ is trivial. Since Δ is injective, the morphism on the left side of (2.4.4) is trivial. This completes the proof.

COROLLARY 2.6. *Let X be a homologically trivial open n-manifold $(n \geq 3)$ with only one strong end and* $\operatorname{cat}(\varepsilon(X)) = 2$. *Then X is 1-LC at ∞.*

We recall that the strong ends of X are the elements in the set of p-homotopy classes $[J\,;X]_p$. It is a well-known fact that $[J\,;X]_p \cong \lim^1 \pi_1(X - K_j)$. Now the proof of Corollary 2.6 is similar to the proof of Corollary 1.12.

REMARK 2.7. Notice that Theorem 2.1 and Theorem 2.4 imply that the contractible open 3-manifolds described in [**20**; Theorem 1] have p-cat ≥ 3. These manifolds have infinitely many strong ends.

Indeed, the inverse sequence of π_1's of those manifolds are of the type

$$\mathcal{S} = \left\{ \mathop{\bigstar}_{0}^{\infty} G_i \leftarrow \mathop{\bigstar}_{1}^{\infty} G_i \leftarrow \cdots \leftarrow \mathop{\bigstar}_{n}^{\infty} G_i \cdots \right\}$$

where "$\bigstar$" stands for the free product of groups and the bonding morphisms are the natural inclusions (see [**24**] and [**20**]). The sequence $\mathcal{S}$ is not pro-isomorphic to a sequence of finitely generated groups since the existence of such an isomorphism would yield an inclusion $\bigstar_n^\infty G_i \subseteq F \subseteq \bigstar_0^\infty G_i$ for some finitely generated group F, if $n \geq n_0$ for some n_0.

REMARK 2.8. The following simple example shows that Theorem 2.4 does not imply the condition 1-LC at ∞.

Let $F(x, y)$ be the free group with generators x and y, and let $f: S^1 \vee S^1 \to S^1 \vee S^1$ be a continuous realization of the morphism $\phi: F(x, y) \to F(x, y)$ given by $\phi(x) = [x, y]$ and $\phi(y) = [x^{-1}, y]$, with $[x, y] = xyx^{-1}y^{-1}$.

We now define T as the telescope

$$T = X \times [0, 1] \cup_f X \times [1, 2] \cup_f \cdots \cup_f X \times [n, n+1] \cdots$$

with $X = S^1 \vee S^1$. It is clear that T is under the hypotheses of Theorem 2.4 but the inverse sequence (2.3.1) agrees with

$$F(x, y) \xleftarrow{\phi} F(x, y) \xleftarrow{\phi} F(x, y) \xleftarrow{\phi} \cdots$$

whose $\lim^1$ is not trivial.

3. Cohomology and p-category. Several p-homotopy invariant cohomologies have appeared in the literature. We next give some examples.

3.1. EXAMPLES. (a) Various cohomologies with compact supports (singular, Čech type, Alexander-Spanier, etc.). See **[19]** for details.

(b) Cohomology of the end of X, $H_e^*(X)$. Namely, $H_e^*(X)$ is defined as the cohomology of the cochain complex $C^*(X)/C_c^*(X)$ where $C_c^*(X)$ stands for the subgroup of singular cochains with compact supports. Alternatively, $H_e^*(X) = \varinjlim \{H^*(U_j)\}$ where $\{U_j\}$ is a system of ∞-neighbourhoods of X. See **[13]** for details.

(c) Pro-cohomology of X with coefficients in the inverse system $\mathscr{S}$. Given $\{U_j\}$ as above, $\mathscr{H}^*(X; \mathscr{S})$ is the cohomology of the cochain complex $C^*(X; \mathscr{S}) = (\text{pro-}\mathscr{Ab}, \mathscr{Ab})(\mathscr{C}_*(X); \mathscr{S})$, where $\mathscr{C}_*(X)$ denotes the inverse system $\{C_*(X) \leftarrow C_*(U_1) \leftarrow \cdots\}$. Here $\mathscr{Ab}$ stands for the category of abelian groups and $(\text{pro-}\mathscr{Ab}, \mathscr{Ab})$ is constructed from $\mathscr{Ab}$ as $(\text{pro-}\mathscr{Top}, \mathscr{Top})$ is constructed from $\mathscr{Top}$ in §3. See **[14]** for details.

If $\mathfrak{H}^*$ is any of the above cohomologies, relative groups $\mathfrak{H}^*(X, A)$ are easily defined and restricted morphisms $\rho: \mathfrak{H}^*(X, A) \to \mathfrak{H}^*(X, B)$ are natural when $B \subseteq A$. If A is closed there is an exact sequence

$$\cdots \to \mathfrak{H}^n(X, A) \xrightarrow{\rho} \mathfrak{H}^n(X) \xrightarrow{i^*} \mathfrak{H}^n(A) \xrightarrow{\delta} \mathfrak{H}^{n+1}(X, A) \to \cdots$$

where $i: A \subseteq X$ is the inclusion. Also cup-products can be defined for $\mathfrak{H}^*$.

As in the ordinary case, p-invariant cohomologies provide a lower bound for proper L-S category. Namely, 3.2. Proposition. *Given X in $\mathfrak{P}_\infty$, let $l(\mathfrak{H}^*(X))$ be* $\sup\{n \in \mathbb{N};\ \exists\alpha_1, \alpha_2, \dots, \alpha_n \in \mathfrak{H}^*(X)$ *with* $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n \neq 0\}$. *Then* $\text{p-cat}(X) \geq l(\mathfrak{H}^*(X)) + 1$.

Proof. Let $A_1, A_2, \dots, A_n$ be a p-categorical open covering of X. Since $i^*\colon \mathfrak{H}^*(X) \to \mathfrak{H}^*(\overline{A}_k)$ is trivial for each $k \leq n$, given $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathfrak{H}^*(X)$, there are elements $\alpha'_k \in \mathfrak{H}^*(X, A_k)$ with $\rho(\alpha'_k) = \alpha_k$ $(k \leq n)$ in the diagram

$$\begin{tikzcd}
& \mathfrak{H}^*(X, A_k) \arrow[d, "\rho"] \arrow[dl, "\rho'"'] & \\
\mathfrak{H}^*(X, \overline{A}_k) \arrow[r, "\rho''"] \arrow[ur, bend left, "\rho'"] & \mathfrak{H}^*(X) \arrow[r, "i^*=0"] & \mathfrak{H}^*(\overline{A}_k)
\end{tikzcd}$$

Now the commutative diagram

$$\begin{tikzcd}
\mathfrak{H}^*(X, A_1) \otimes \mathfrak{H}^*(X, A_2) \otimes \cdots \otimes \mathfrak{H}^*(X, A_n) \arrow[r, "\cup"] \arrow[d, "\rho"] & \mathfrak{H}^*(X, A_1 \cup A_2 \cup \cdots \cup A_n) = 0 \arrow[d, "\rho"] \\
\mathfrak{H}^*(X) \otimes \mathfrak{H}^*(X) \otimes \cdots \otimes \mathfrak{H}^*(X) \arrow[r, "\cup"] & \mathfrak{H}^*(X)
\end{tikzcd}$$

yields $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n = 0$.

3.3. Corollary. *If W is a non-compact surface with only one Freudenthal end then*

$$\text{p-cat}(W) = \begin{cases} 2 & \text{if } W \cong \mathbb{R}^2, \\ 3 & \text{otherwise}. \end{cases}$$

Proof. If $W \ncong \mathbb{R}^2$ we can find elements $\alpha, \beta \in H^1_c(W; \mathbb{Z}_2)$ with $\alpha \cup \beta \neq 0$ and then $\text{p-cat}(W) \geq 3$. By 1.8(e) we conclude $\text{p-cat}(W) = 3$.

Also we may use the cohomology with compact supports in the following

3.4. Example. There are contractible spaces with only one Freudenthal end and infinite proper L-S category. Let T^n be the n-dimensional torus ($n \geq 0$, $T^0 = *$) and $\pi_n\colon T^n \to T^{n-1}$ $(n \geq 1)$ the natural projection. Then $X = \bigcup X_n$, where X_n is the mapping cylinder of π_n, is a contractible space and $H^1_c(X; \mathbb{Z})$ is the exterior algebra $\Lambda(x_1, \dots, x_n, \dots)$ with infinitely many generators. Then $\text{p-cat}(X) = \infty$.

Now we characterize $\mathbb{R}^n$ among all the manifolds by using proper L-S category. Namely, THEOREM 3.5. *Let* X *be a connected open manifold* $(n \geq 1)$. *Then* $X \simeq_{\mathrm{p}} \mathbb{R}^n$ *if and only if one of the following statements holds*:

(i) $\text{p-Cat}(X) = 2$

(ii) $\text{p-cat}(X) = 2$ *and* X *has only one strong end.*

Proof. Since $\operatorname{cat}(X) \leq \text{p-cat}(X) = 2$, it follows that $\pi_1(X)$ is a free group (see [**15**, §4]) and so X is orientable.

For any $\mathbb{Z}_p$ with p prime we have the isomorphisms

$$H_c^q(X; \mathbb{Z}_p) \overset{D}{\cong} H_{n-q}(X; \mathbb{Z}_p) \overset{\alpha}{\cong} H^{n-q}(X; \mathbb{Z}_p)$$

given by the Poincaré Duality (see [**19**; 11.2]) and the ordinary algebraic duality respectively. Then, if $\varepsilon \neq 0 \in H_c^q(X; \mathbb{Z}_p)$ and $\mu_X \in H_n^{\mathrm{II}}(X; \mathbb{Z}_p)$ is the fundamental class of X in the homology of "infinite chains", we get

$$0 = \mu_X \cap (\varepsilon \cup \alpha(D(\varepsilon))) = D(\varepsilon) \cap \alpha(D(\varepsilon)) = 1$$

where "$\cap$" and "$\cup$" are the natural cap and cup products $H_p^{\mathrm{II}} \otimes H_c^r \to H_{p-r}$ and $H_c^q \otimes H^r \to H_c^{q+r}$ respectively (see [**19**; 10.4] for details).

Therefore, $H_{n-q}(X; \mathbb{Z}_p) = H_c^q(X; \mathbb{Z}_p) = 0$ for all q and the Universal Coefficient Theorem proves that X is homologically trivial. In particular $\pi_1(X) = 1$ and X is contractible by the Whitehead Theorem. Then $X = \mathbb{R}^n$ if $n = 1, 2$.

On the other hand, if $n \geq 3$ we have the Poincaré Duality $H_q^{\mathrm{II}}(X; \mathbb{Z}) \cong H^{n-q}(X; \mathbb{Z}) = 0$ (see [**19**; 11.4]). Since X is properly 1-connected at infinity by Corollary 1.12 or Corollary 2.6, a convenient proper Hurewicz Theorem (see [**3**; 3.4]) yields that $H_n^{\mathrm{II}}(X; \mathbb{Z}) \cong [\mathbb{R}^n; X]_p$ and the fundamental class μ_X can be represented by a p-map $f \colon \mathbb{R}^n \to X$. It is easily checked that $f_* \colon H_*^{\mathrm{II}}(\mathbb{R}^n) \to H_*^{\mathrm{II}}(X)$ is an isomorphism and again by [**19**; 11.4], $f^* \colon H_c^*(X) \to H_c^*(\mathbb{R}^n)$ is an isomorphism. Now, f is a p-homotopy equivalence by [**9**; 4.9].

By using the results of Siebenmann [**21**] and Freedman [**12**] we get

THEOREM 3.6. *If* X *is a connected open n-manifold* $(n \neq 3)$, *then* X *is homeomorphic to* $\mathbb{R}^n$ *if and only if* 3.5(i) *or* 3.5(ii) *holds.*

REMARK 3.7. We do not know whether Theorem 3.5(ii) is true without the condition on strong ends.

Given an embedding $i \colon J \to X$, we can define a new L-S type proper invariant $\text{hp-cat}(X, i)$ as follows (cf. [**15**; §4 and §5]).

We define the nth canonical proper fat wedge of (X, i) as the union

$$T_p^n(X, i) = \bigcup\{A_k, i \leq k \leq n\}$$

where A_k is the product of $(n-1)$ copies of X with J, where J is placed in the kth factor. Then hp-cat$(X, i) \leq n$ if there is a diagram in $\mathfrak{P}_\infty$

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\eta"] & \Pi^n X \\ T_p^n(X, i) \arrow[ur, "\varphi(i)"'] & \end{tikzcd}$$

commutative up to the p-homotopy. Here Δ is the diagonal map and $\varphi(i)$ is the natural embedding induced by i.

In contrast with Theorem 3.6, the following result holds.

3.7. Proposition. *For any contractible open manifold W, and any embedding $i: J \to W$,* hp-cat$(W, i) = 2$.

Proof. It is clear that hp-cat$(W) \geq 2$ and hp-cat$(\mathbb{R}) = 2$. For $\dim(W) = n \geq 2$, it is known that $W \times W$ is 1-connected at infinity (see [**13**; 1.8]). And by [**12**] and [**21**] $W \times W \overset{h}{\cong} \mathbb{R}^{2n}$. Finally we apply the Proper Cellular Approximation Theorem (see [**9**]) to deform $h \circ \Delta: W \to \mathbb{R}^{2n}$ onto the n-skeleton of $\mathbb{R}^{2n}$ which may be regarded as J since $\mathbb{R}^{2n}$ is properly $(2n-2)$-connected. So, hp-cat$(W) \leq 2$.

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