PACIFIC JOURNAL OF MATHEMATICS
Vol. 154, No. 2, 1992

NEW SPECTRAL CHARACTERIZATION THEOREMS FOR S^2

MARTIN ENGMAN

Two theorems are proved: (1) Among surfaces of revolution which are diffeomorphic to S^2, the constant curvature metric is completely characterized by the multiplicities of its eigenvalues and, (2) If the square of an eigenfunction is, again, an eigenfunction then the metric is the standard metric on S^2.

0. Introduction. The Laplacian on a surface of revolution (M, g) "splits" into a sequence of ordinary differential operators $\{L_k\}$, $k \in \mathbf{Z}$, whose sequence of Green's operators is denoted by $\{\Gamma_k\}$. A nice feature of these Green's operators is that we can compute their traces exactly. We prove in §4 that for every surface of revolution diffeomorphic to S^2:

$$\operatorname{trace}(\Gamma_k) = \frac{1}{k}, \quad \text{for } k \neq 0.$$

This formula is used in §5 to prove the following, rather surprising, fact: If the multiplicities of the eigenvalues on (M, g) are the same as those of the standard sphere, then the numerical values of the eigenvalues are the same as those of the standard sphere. One can then use a well-known result of Berger [1] to prove Theorem 5.4. It is paraphrased here as follows.

THEOREM 0.1. *Among surfaces of revolution which are diffeomorphic to S^2, the constant curvature metric is completely characterized by the multiplicities of its eigenvalues.*

Other "spectral characterization" theorems of this sort can be found in the literature (see Berger [1], Brüning and Heintze [4], Cheng [5], Goldberg and Gauchman [10], Obata [14], and Patodi [15]).

The result of Cheng is interesting in that it characterizes the standard sphere with a property of eigenfunctions rather than eigenvalues. He proves: The spheres in 3-dim Euclidean space are characterized by the fact that they have three first eigenfunctions with square sum equal to a constant.

In the setting of surfaces of revolution, we found a slightly simpler eigenfunction characterization theorem, namely:

THEOREM 0.2. *Let (M, g) be a surface of revolution. If h and h^2 are both eigenfunctions of $-\Delta$, then the metric is the standard (i.e. constant curvature) metric on S^2.*

This is proved as Theorem 6.1.

I would like to thank Howard Fegan for his helpful comments during the preparation of this paper.

1. Preliminaries. Let M be a surface of revolution which is diffeomorphic to S^2 and let U be the chart $M\backslash\{np, sp\}$ where np and sp are the poles of the axis of revolution. The usual form of the embedded metric on U is $ds \otimes ds + r^2(s)\, d\theta \otimes d\theta$ where s is the arclength along a generator. In this paper we make a change of variables transforming the metric on the chart U into the form

$$g = \frac{1}{f(x)}\, dx \otimes dx + f(x)\, d\theta \otimes d\theta \tag{1.1}$$

where $(x, \theta) \in (-1, 1) \times [0, 2\pi)$. In order that M be diffeomorphic to S^2, the function $f \in C^\infty(-1, 1) \cap C^0[-1, 1]$ must satisfy the following conditions:

(1) $f(x) > 0$ on $(-1, 1)$,
(2) $f(-1) = 0 = f(1)$,
(3) $f'(-1) = 2 = -f'(1)$.

This metric has 2-volume equal to 4π and its Gauss curvature is given by $\kappa(x) = -\frac{1}{2} f''(x)$. The standard (i.e. constant curvature) sphere is obtained by taking $f(x) = 1 - x^2$ and the metric in this case is denoted g_0.

The Laplace-Beltrami operator on U is given by

$$\Delta = \frac{\partial}{\partial x}\left(f(x)\frac{\partial}{\partial x}\right) + \frac{1}{f(x)}\frac{\partial^2}{\partial \theta^2}, \tag{1.2}$$

and the volume two-form is $\omega = dx \wedge d\theta$, because $\sqrt{\det g} \equiv 1$. As a result, the L^2 inner product for the metric (1.1) is given by:

$$\langle \phi, \psi \rangle = \int_0^{2\pi} \int_{-1}^{1} \phi \overline{\psi}\, dx \wedge d\theta \tag{1.3}$$

for all functions ϕ and ψ.

2. The spectrum of (M, g)**.** The *spectrum of* (M, g) is the set of eigenvalues of $-\Delta$. We denote this set of positive real numbers by the symbol $\operatorname{Spec}(-\Delta)$. We will reduce the problem of finding $\operatorname{Spec}(-\Delta)$ to that of finding the spectrum of certain ordinary differential operators.

From formula (1.2) we see that an eigenfunction $h \in C^\infty(M)$ of $-\Delta$ with eigenvalue $\lambda \in \operatorname{Spec}(-\Delta)$ satisfies the partial differential equation:

$$\frac{\partial}{\partial x}\left(f(x)\frac{\partial h}{\partial x}\right) + \frac{1}{f(x)}\frac{\partial^2 h}{\partial \theta^2} = -\lambda h. \tag{2.1}$$

The S^1 action on M induces irreducible representations of complex dimension 1 on each eigenspace E_λ. These representations act via multiplication by $\exp(ik\theta)$ for $k \in \mathbf{Z}$; hence every eigenfunction $h \in E_\lambda$ has an expansion

$$h(x, \theta) = \sum_{j=1}^{n} a_j u_j(x) \exp(ik_j\theta). \tag{2.2}$$

Each of the real-valued functions u_j above is a solution of the ordinary differential equation

$$\frac{d}{dx}\left(f(x)\frac{du}{dx}\right) - \frac{k^2}{f(x)}u = -\lambda u \tag{2.3}$$

for $k = k_j$. Each term in the expansion (2.2) is itself an eigenfunction and falls into one of two categories as follows: If $k = 0$ then we call the eigenfunction S^1 *invariant* because it is independent of θ; if $k \neq 0$ then it is called k *equivariant*, or simply *of type* k. For a type $k \neq 0$ eigenfunction it is necessary that $u(-1) = u(1) = 0$, for otherwise it is not well defined at the poles $(-1, \theta)$ and $(1, \theta)$. The S^1 invariant eigenfunctions require only bounded conditions at the endpoints. It is easy to prove that the solutions of (2.3), with either bounded ($k = 0$) or zero ($k \neq 0$) boundary conditions, are unique up to multiplication by constants.

We will use the differential equation (2.3) in our investigation of the spectrum of $-\Delta$. For each $k \in \mathbf{Z}$ define the ordinary differential operator L_k by the formula:

$$L_k = -\frac{d}{dx}\left(f(x)\frac{d}{dx}\right) + \frac{k^2}{f(x)}.$$

Then the boundary value problem (2.3), $u(-1) = 0 = u(1)$, can be formulated as:

$$L_k u = \lambda u, \qquad u(-1) = 0 = u(1) \tag{2.4}$$

and the S^1 invariant problem as:

$$L_0 u = \lambda u, \qquad u(-1) + u(1) = 2A, \tag{2.5}$$

where A is a constant depending on λ. We can now define $\operatorname{Spec}(L_k)$ to be the eigenvalues of the problem (2.4) or (2.5) depending on whether $k \neq 0$ or $k = 0$. From formula (2.2) we see that $\lambda \in \operatorname{Spec}(-\Delta)$ implies that for some $k \in \mathbf{Z}$ $\lambda \in \operatorname{Spec}(L_k)$. Conversely, $\lambda \in \operatorname{Spec}(L_k)$ implies there exists a function u satisfying (2.4) or (2.5); hence, the function $u(x)\exp(ik\theta) \in C^\infty(M)$ is an eigenfunction of the Laplacian with eigenvalue λ. We summarize the above discussion in

PROPOSITION 2.1. *As sets of numbers,*

$$\operatorname{Spec}(-\Delta) = \bigcup_{k \in \mathbf{Z}} \{\operatorname{Spec}(L_k)\}. \qquad \square$$

The reader should be aware that the set union in Proposition 2.1 is not, in general, disjoint as it fails to take into account multiplicities of eigenvalues. In fact, the extent to which this union fails to be disjoint occupies much of the upcoming discussion.

Of course, the multiplicity of λ is simply the dimension of E_λ (denoted $\dim E_\lambda$). The next two results produce an estimate for $\dim E_\lambda$.

PROPOSITION 2.2. *Assume $k \neq 0$. A necessary condition for the existence of a non-trivial solution of (2.4) is that $k^2 < \lambda \sup f$.*

Proof. Assume a solution $u \in C^2(-1, 1) \cap C^0[-1, 1]$ exists for the boundary value problem (2.4). We may multiply equation (2.4) by $u(x)$, rearrange the terms, and integrate by parts to obtain:

$$\int_{-1}^{1} f(x)\left(\frac{du(x)}{dx}\right)^2 dx = \int_{-1}^{1} \frac{\lambda f(x) - k^2}{f(x)} u^2(x)\, dx,$$

and because the integrand on the left is non-negative there must exist a point $\xi \in (-1, 1)$ such that $\lambda f(\xi) - k^2 > 0$. Consequently we have $k^2 < \lambda \sup f$. $\square$

PROPOSITION 2.3. *Let λ be an eigenvalue of $-\Delta$, E_λ, its eigenspace, and $\dim E_\lambda$, the dimension of E_λ. Then*

$$\dim E_\lambda \leq 2[\sqrt{\lambda \sup f}] + 1,$$

where [ ] denotes the greatest integer function.

Proof. By Proposition 2.1 there exists a k such that $\lambda \in \operatorname{Spec}(L_k)$ and, since the solutions to (2.4) or (2.5) are unique, we need only count the number of integers satisfying the conclusion of Proposition 2.2. Let k_0 be the largest integer satisfying the inequality $k^2 < \lambda \sup f$. Clearly, $k_0 \geq 0$ and $k_0 < \sqrt{\lambda \sup f}$ so that $k_0 \leq [\sqrt{\lambda \sup f}]$. Every integer k satisfying $|k| \leq k_0$ also satisfies $k^2 < \lambda \sup f$, and there are $2k_0 + 1$ of them, including $k = 0$. As a result, $\dim E_\lambda \leq 2[\sqrt{\lambda \sup f}] + 1$. □

According to Proposition 2.1, we need only study $\operatorname{Spec}(L_k)$ for all $k \in \mathbf{Z}$. To facilitate this process, fix $k \in \mathbf{Z}$ and let λ_k^j denote the *jth eigenvalue* of L_k. The *eigenfunction* associated with λ_k^j will be denoted by u_k^j. Hence, $\operatorname{Spec}(L_k) = \{\lambda_k^1, \lambda_k^2, \dots, \lambda_k^j, \dots\}$ for all k. (Note that this is a positive, increasing sequence in j.)

3. Green's operators on (M, g). The inhomogeneous differential equation, $L_k v = h$, is a singular Sturm-Liouville equation and it is equivalent to a Fredholm integral equation whose integral operator has eigenvalues $\{1/\lambda_k^1; 1/\lambda_k^2; \cdots; 1/\lambda_k^j, \cdots\}$. We will find it useful to formulate the equivalent integral equation, and this requires a discussion of Green's functions. In this paper, we restrict ourselves to the case $k \neq 0$.

The eigenfunctions for the case $k \neq 0$ vanish at the endpoints, so it is appropriate to restrict ourselves to the space T defined by:

$$T \equiv L_0^2[-1, 1] \cap C^\infty(-1, 1),$$

where $L_0^2[-1, 1] \equiv \{h \in L^2[-1, 1] : h(-1) = 0 = h(1)\}$, and then solve the inhomogeneous problem $L_k v = h$ for $h \in T$. To this end, define

$$h_k(x) \equiv \exp\left(k \int_0^x \frac{1}{f(t)}\, dt\right) \quad \text{and} \quad h_{-k}(x) \equiv \exp\left(-k \int_0^x \frac{1}{f(t)}\, dt\right).$$

We can now define the Green's function by the formula:

$$G_k(x, y) = \frac{1}{2k} \begin{cases} h_k(y) h_{-k}(x) & \text{if } -1 < y \leq x, \\ h_k(x) h_{-k}(y) & \text{if } x < y < 1. \end{cases} \tag{3.1}$$

The Green's operator $\Gamma_k : T \to L_0^2[-1, 1]$ is defined as the integral operator with (3.1) as its kernel. Explicitly,

$$\Gamma_k(h)(x) \equiv \int_{-1}^1 G_k(x, y) h(y)\, dy. \tag{3.2}$$

The operator defined in equation (3.2) inverts the differential operator L_k in the following sense:

PROPOSITION 3.1. *Let* $h \in T$. *A function* $v \in T$ *is a solution of* $L_k v = h$ *if and only if*

$$v(x) = \Gamma_k(h)(x).$$

The proof is, mostly, a routine exercise in the theory of Green's functions. The standard proof can be found in Weinberger **[16]** and the details for this case, which involves improper integrals, can be found in Engman [7].

And now it is easy to see that:

COROLLARY 3.2. $\lambda \in \operatorname{Spec}(L_k)$ *if and only if* $\frac{1}{\lambda} \in \operatorname{Spec}(\Gamma_k)$. *Furthermore,* L_k *and* Γ_k *have the same eigenfunctions.*

Proof. By Proposition 3.1 $L_k u = \lambda u$ if and only if $\Gamma_k(\lambda u) = u$, i.e. $\Gamma_k(u) = \frac{1}{\lambda} u$. □

4. The trace of the Green's operator. In this section we obtain a double eigenfunction expansion for the Green's function of §3. This leads to a formula for the trace of the Green's operators.

PROPOSITION 4.1. *Let* $G_k(x, y)$ *be given by* (3.1). $G_k(x, y)$ *is bounded, continuous, and positive on the open square* $(-1, 1)\times(-1, 1)$. *Furthermore, for each fixed* $x \in (-1, 1)$, $\lim_{y\to -1^+} G_k(x, y) = 0$ *and* $\lim_{y\to 1^-} G_k(x, y) = 0$.

Proof. Observe that from equation (3.1) and the definitions of h_k and h_{-k} that:

$$G_k(x, y) = \frac{1}{2k}\begin{cases} \exp\left(-k\int_y^x \frac{1}{f(t)}\,dt\right) & \text{if } -1 < y \le x, \\ \exp\left(k\int_y^x \frac{1}{f(t)}\,dt\right) & \text{if } x < y < 1. \end{cases} \tag{4.1}$$

This formula displays the continuity of $G_k(x, y)$ on the open rectangle, Now, because $\frac{1}{f(x)} > 0$ on $(-1, 1)$, we have

$$0 < \exp\left(-k\int_y^x \frac{1}{f(t)}\,dt\right) \le 1 \quad \text{for } y \le x,$$

and

$$0 < \exp\left(k\int_y^x \frac{1}{f(t)}\,dt\right) < 1 \quad \text{for } x < y.$$

Therefore, it has been shown that:

$$0 < G_k(x, y) \le \frac{1}{2k} \quad \text{for all } (x, y) \in (-1, 1) \times (-1, 1).$$

For the second part of the proposition, observe that for a fixed $x \in (-1, 1)$, $\lim_{y \to -1^+} \int_y^x \frac{1}{f(t)}\, dt = \infty$ because the order of the pole is 1. Now:

$$\lim_{y \to -1^+} G_k(x, y) = \frac{1}{2k} \exp\left(-k \lim_{y \to -1^+} \int_y^x \frac{1}{f(t)}\, dt\right) = 0.$$

The argument that $\lim_{y \to 1^-} G_k(x, y) = 0$ is similar. □

COROLLARY 4.2. *For each* $x \in (-1, 1)$, $G_k(x, x) = \frac{1}{2k}$.

Proof. This comes from setting $y = x$ in equation (4.1). □

The goal at this point is to obtain a double eigenfunction expansion for $G_k(x, y)$. This can be accomplished after proving the following lemma. A similar proof can be found in Weinberger [**16**]. In the proof of the lemma we use the inner product on $L^2[-1, 1]$. It is defined by the equation $\langle \phi, \psi \rangle_2 = \int_{-1}^1 \phi \overline{\psi}\, dx$ for ϕ and $\psi \in L^2[-1, 1]$ and the corresponding norm is denoted by $\|\cdot\|_2$.

LEMMA 4.3. *Let* ν *and* ω *be continuous functions in* $L_0^2[-1, 1]$ *which satisfy the following*:

(i) $\sqrt{f}\nu'$ *and* $\sqrt{f}\omega'$ *are in* $L^2[-1, 1]$, *and they are piecewise continuous on the interval* $[-1, 1]$;

(ii) $\nu/\sqrt{f}$ *and* $\omega/\sqrt{f}$ *are in* $L^2[-1, 1]$; *and*

(iii) $\nu \sim \sum_{j=1}^\infty c_j u_k^j(x)$ *and* $\omega \sim \sum_{j=1}^\infty d_j u_k^j(x)$.

Then

$$\int_{-1}^1 \left(f\nu'\omega' + \frac{k^2}{f}\nu\omega\right) dx = \sum_{j=1}^\infty \lambda_k^j c_j d_j.$$

Proof. First of all, conditions (i) and (ii) and the Cauchy-Schwarz inequality insure that $f\nu'\omega'$ and $\nu\omega/f$ are integrable; hence

$$\int_{-1}^1 \left(f\nu'\omega' + \frac{k^2}{f}\nu\omega\right) dx < \infty.$$

Now let ψ be a twice continuously differentiable function. Then if $\psi \sim \sum_{j=1}^\infty c_j u_k^j(x)$ we also have that $L_k\psi \sim \sum_{j=1}^\infty \lambda_k^j c_j u_k^j(x)$. So by

Parseval's equation,

$$\langle L_k \psi , \psi \rangle = \sum_{j=1}^{\infty} \lambda_k^j c_j^2 ,$$

and after one integration by parts,

$$\int_{-1}^{1} \left(f(\psi')^2 + \frac{k^2}{f}\psi^2 \right) dx = \sum_{j=1}^{\infty} \lambda_k^j c_j^2 .$$

By approximating with twice continuously differentiable functions we can show that this formula holds for the functions ν and ω, also. Finally, this equation is applied to the functions $\nu - \omega$ and $\nu + \omega$, and the results are subtracted to obtain:

$$\int_{-1}^{1} \left(f\nu'\omega' + \frac{k^2}{f}\nu\omega \right) dx = \sum_{j=1}^{\infty} \lambda_k^j c_j d_j. \qquad \square$$

We are now prepared to prove a version of Mercer's Theorem.

THEOREM 4.4. *Let $G_k(x , y)$ be defined by equation* (3.1), *and let $\{u_k^j(x)\}$ and $\{\lambda_k^j\}$ be the eigenfunctions and eigenvalues for L_k. Then*:

$$G_k(x , y) = \sum_{j=1}^{\infty} \frac{1}{\lambda_k^j} u_k^j(x) u_k^j(y)$$

for all $(x , y) \in (-1 , 1) \times (-1 , 1)$. In other words, the series converges pointwise to $G_k(x , y)$ on the open set $(-1 , 1) \times (-1 , 1)$.

Proof. The proof is another routine exercise in the theory of Green's functions and is accomplished by applying Lemma 4.3 to the functions $\nu(\xi) = G_k(x , \xi)$ and $\omega(\xi) = G_k(y , \xi)$ where we define $\nu(\pm 1) = 0 = \omega(\pm 1)$. It suffices to show that $f(\nu')^2$, $f(\omega')^2$, ν^2/f, and ω^2/f have, at worst, integrable singularities at -1 and 1 so that conditions (i) and (ii) of the lemma hold. The proof of this can be found in Engman [7]. The Fourier coefficients c_j for ν, and d_j for ω are given by

$$c_j = \langle \nu , u_k^j \rangle = \Gamma_k(u_k^j)(x) = \frac{1}{\lambda_k^j} u_k^j(x)$$

and,

$$d_j = \langle \omega , u_k^j \rangle = \Gamma_k(u_k^j)(y) = \frac{1}{\lambda_k^j} u_k^j(y) .$$

So that by Lemma 4.3:

$$\int_{-1}^{1}\left(f\nu'\omega' + \frac{k^2}{f}\nu\omega\right) d\xi = \sum_{j=1}^{\infty}\frac{1}{\lambda_k^j}u_k^j(x)u_k^j(y). \tag{4.2}$$

A calculation shows that:

$$\int_{-1}^{1}\left(f\nu'\omega' + \frac{k^2}{f}\nu\omega\right) d\xi = G_k(x, y).$$

And when we put this equation together with equation (4.2) the theorem is proved. □

COROLLARY 4.5. *For all* $x \in (-1, 1)$

$$\frac{1}{2k} = \sum_{j=1}^{\infty}\frac{1}{\lambda_k^j}[u_k^j(x)]^2.$$

Proof. This formula is obtained by putting $y = x$ in the formula of Theorem 4.4 and then using Corollary 4.2. □

The reader should observe that Corollary 4.5 fails to hold at the endpoints because at these points the series sums to zero.

COROLLARY 4.6. *Let* λ_k^j *be the* j*th eigenvalue of the operator* L_k. *Then*

$$\frac{1}{k} = \sum_{j=1}^{\infty}\frac{1}{\lambda_k^j}. \tag{4.3}$$

Proof. From Corollary 4.5, it is easy to see that for each $n \in N$, the finite series $\sum_{j=1}^{n}(1/\lambda_k^j)[u_k^j(x)]^2$ is bounded above by $\frac{1}{2k}$. Furthermore, each of these finite series defines a Lebesgue measurable function, so we use the Lebesgue Dominated Convergence Theorem to obtain:

$$\int_{-1}^{1}\frac{1}{2k}\,dx = \sum_{j=1}^{\infty}\frac{1}{\lambda_k^j}\int_{-1}^{1}[u_k^j(x)]^2\,dx.$$

The eigenfunctions are normalized and the left side of the equation is obviously $\frac{1}{k}$, so that

$$\frac{1}{k} = \sum_{j=1}^{\infty}\frac{1}{\lambda_k^j}.$$ □

It should be observed that formula (4.3) is simply a trace formula for the Green's operator Γ_k since the jth eigenvalue of Γ_k is, of course, $1/\lambda_k^j$, and the trace of such an operator is the sum of its eigenvalues. So we can reformulate (4.3) in the form:

$$\operatorname{trace}(\Gamma_k) = \frac{1}{k}. \tag{4.4}$$

5. Characterizing S^2 with multiplicities. In this section we will use the formula (4.3) and the following theorem of Berger to characterize the standard sphere by the multiplicities of its eigenvalues.

THEOREM 5.1 (*Berger* '63). *Let* $\operatorname{Spec}(\Delta, M_1) = \operatorname{Spec}(\Delta, M_2)$. *If* $\dim M \leq 4$ *and* M_1 *is isometric to the standard sphere then so is* M_2.

We only need this theorem for the case $\dim M = 2$. The proof in the 2-dimensional case is quite simple. It uses the fact that the volume and Euler characteristic are determined by the spectrum via the asymptotic expansion of the heat kernel. Then the Cauchy-Schwarz inequality is employed to show the curvature must be constant. (See Berger et al. [2].)

Let's recall that λ_k^m denotes the mth eigenvalue for the operator L_k, so that λ_k^1 is the minimal, non-zero eigenvalue for L_k. The next result discusses the dependence of λ_k^1 on the index k.

LEMMA 5.2 (*Monotonicity Theorem*). *Let* $0 < k < j$, *and let* λ_k^1 *and* λ_j^1 *be the first eigenvalues of the operators* L_k *and* L_j *respectively. Then* $\lambda_k^1 < \lambda_j^1$.

Proof. By Rayleigh's Theorem we have that $\lambda_k^1 = \inf_{\|\phi\|=1}\langle L_k\phi, \phi\rangle$, but

$$\langle L_k\phi, \phi\rangle = \int_{-1}^{1} \left(f(\phi')^2 + \frac{k^2}{f}\phi^2\right) dx$$

after an integration by parts. So we can now see that $\langle L_k\phi, \phi\rangle < \langle L_j\phi, \phi\rangle$ for all ϕ because $k < j$, and, therefore, $\lambda_k^1 \leq \lambda_j^1$. Now suppose $\lambda_k^1 = \lambda_j^1 \equiv \lambda$ with eigenfunctions ϕ and ψ respectively; then $L_k\phi = \lambda\phi$ and $L_j\psi = \lambda\psi$, so that $\langle L_j\psi, \phi\rangle = \langle L_k\phi, \psi\rangle$. As a result,

$$\left\langle \frac{d}{dx}\left(f\frac{d\psi}{dx}\right), \phi\right\rangle - j^2\left\langle \frac{\psi}{f}, \phi\right\rangle = \left\langle \frac{d}{dx}\left(f\frac{d\phi}{dx}\right), \psi\right\rangle - k^2\left\langle \frac{\psi}{f}, \phi\right\rangle;$$

but $\frac{d}{dx}(f\frac{d}{dx})$ is self-adjoint, so $(k^2 - j^2)\langle \frac{\psi}{f}, \phi\rangle = 0$. But first eigenfunctions are non-negative; hence the expression $\langle \frac{\psi}{f}, \phi\rangle$ is the integral of a non-negative, non-trivial function, so $\langle \frac{\psi}{f}, \phi \rangle > 0$; this means $k = j$. This is a contradiction, so we must have $\lambda_k^1 < \lambda_j^1$. □

Let $D\operatorname{Spec}(-\Delta) = \{\lambda_1, \lambda_2, \dots, \lambda_m, \dots\}$ be the list of distinct non-zero eigenvalues of $-\Delta$, in which $\lambda_i < \lambda_j$ whenever $i < j$. Also, recall that E_λ denotes the eigenspace associated with the eigenvalue λ.

LEMMA 5.3. *Let $\lambda_m \in D\operatorname{Spec}(-\Delta)$ and assume $\dim E_{\lambda_m} = 2m+1$ for all $m \in \mathbf{N}$. Then for each $m \in \mathbf{N}$,*

$$\lambda_m = \lambda_{m-j}^{j+1}$$

for all j such that $0 \le j \le m-1$.

Proof. The proof is by induction on m. For the case $m = 1$ we have $j = 0$ and $\dim E_{\lambda_1} = 3$. As a result, it must be the case that $\lambda_1 = \lambda_r^1$ for some r. Now, suppose $r \neq 1$. Then by the minimality of λ_1 we must have $\lambda_1 < \lambda_1^1$ so that $\lambda_r^1 < \lambda_1^1$ for $r > 1$, but this contradicts Lemma 5.2. So $\lambda_1 = \lambda_1^1$ and the $m = 1$ case is finished.

Now assume for all $m \in \{1, \dots, n\}$ that $\lambda_m = \lambda_{m-j}^{j+1}$ for all j such that $0 \le j \le m-1$. It must be proved that the conclusion holds for the case $m = n+1$. Now, since $\dim E_{\lambda_{n+1}} = 2(n+1)+1$, there exist $n+1$ pairs $(k_i, j_{k_i}) \in \mathbf{N} \times \mathbf{N}$ such that $k_1 < k_2 < \cdots < k_{n+1}$, and $\lambda_{n+1} = \lambda_{k_i}^{j_{k_i}}$ for all $i \in \{1, \dots, n+1\}$. There are $n+1$ of the distinct integers k_i so $k_{n+1} > n$, and by the induction hypothesis there is no λ_m for $m \le n$, which is a type k_{n+1} eigenvalue. So it must be the case that $j_{k_{n+1}} = 1$, and, therefore, $\lambda_{n+1} = \lambda_{k_{n+1}}^1$.

Now we want to show that $k_{n+1} = n+1$. Suppose not; then $k_{n+1} > n+1$, and, as a result, one of two situations occurs. Either for some $i < n+1$, $k_i = n+1$ and then $j_{k_i} = 1$, so that

$$\lambda_{n+1} = \lambda_{k_i}^1 = \lambda_{k_{n+1}}^1;$$

or for some $r > 1$, $\lambda_{n+1}^1 = \lambda_{n+r} > \lambda_{n+1}$ so that

$$\lambda_{n+1}^1 > \lambda_{k_{n+1}}^1.$$

In either situation the monotonicity theorem is contradicted, so we must have $k_{n+1} = n+1$. And now, for $i < n+1$ there is no other choice but to have $k_i = i$. But we can re-index so that $k_i = n+1-j$. With the use of the induction hypothesis it is easy to see that $j_{k_i} = j+1$; in other words:

$$\lambda_{n+1} = \lambda_{n+1-j}^{j+1} \quad \text{for all } j \text{ such that } 0 \le j \le n.$$

And the proof by induction is finished. □

THEOREM 5.4. *Let* (M, g) *be a surface of revolution and let* λ_m *be its* m*th distinct eigenvalue. If* $\dim E_{\lambda_m} = 2m+1$ *for all* m, *then the metric* g *is the standard* (*i.e. constant curvature*) *metric on* S^2.

Proof. By Lemma 5.3, it is easy to see for all k and $j \geq 1$ that $\lambda_{k+j-1} = \lambda_k^j$. Substituting this equation into the trace formulae (4.3) for both k and $k+1$ yields

$$\frac{1}{k} = \sum_{j=1}^{\infty} \frac{1}{\lambda_{k+j-1}} \quad \text{and} \quad \frac{1}{k+1} = \sum_{j=1}^{\infty} \frac{1}{\lambda_{k+j}}.$$

The two series can be subtracted term by term, because of the absolute convergence, to obtain:

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{\lambda_k}.$$

As a result, $\lambda_k = k(k+1)$ for all $k \in \mathbf{N}$. But these are the eigenvalues for the standard sphere and, since the multiplicities are those of the sphere, the Berger theorem, i.e. Theorem 5.1, finishes the proof. □

6. A property of eigenfunctions which characterizes S^2. The theorem of this section was discovered during a search for isospectral potentials for the Laplacian on surfaces of revolution.

The problem of finding the spectrum for $-\Delta$ plus a potential has been addressed by many authors including Fegan [**8**], Guillemin and Uribe [**11**], and Weinstein [**17**]. In the first two papers above, isospectral potentials for homogeneous spaces are found. In an example, the second paper addresses the problem on the standard S^2. In this case, it is clear that their technique depends on the fact that products of highest weight eigenfunctions are also eigenfunctions. (In our notation these are the eigenfunctions u_k^1.) So in an attempt to generalize these techniques to other surfaces of revolution one might ask, first of all, is the square of an eigenfunction also an eigenfunction? We found the answer to be *no* in every case except the standard sphere, and this leads to another characterization theorem. (Compare with Cheng [**5**].)

The standard metric on S^2 has the following property: If $u_k^1(x)$ is the eigenfunction for λ_k^1, then $[u_k^1(x)]^2$ is the eigenfunction for λ_{2k}^1. It will now be shown that the standard metric is the only one with this property.

THEOREM 6.1. *Let (M, g) be a surface of revolution with metric (1.1). If h and h^2 are both eigenfunctions of $-\Delta$, then the metric is the standard (i.e. constant curvature) metric on S^2.*

Proof. The expansion (2.2) shows that if $h(x, \theta)$ and $h^2(x, \theta)$ are both eigenfunctions then there exists a k type eigenfunction $u(x)$ such that $u^2(x)$ is a $2k$ type eigenfunction. So the problem reduces to a problem regarding the functions u and u^2. Let λ be the eigenvalue for u, η be the eigenvalue for u^2, and let $A \equiv \sup f$. So by hypothesis we have a system of equations:

$$\frac{d}{dx}\left(f(x)\frac{du}{dx}\right) - \frac{k^2}{f(x)}u = -\lambda u, \tag{6.1}$$

$$\frac{d}{dx}\left(f(x)\frac{du^2}{dx}\right) - \frac{4k^2}{f(x)}u^2 = -\eta u^2. \tag{6.2}$$

We expand (6.2) to obtain

$$2u\frac{d}{dx}\left(f(x)\frac{du}{dx}\right) + 2f\left(\frac{du}{dx}\right)^2 - \frac{4k^2}{f(x)}u^2 = -\eta u^2. \tag{6.3}$$

Multiplying (6.1) by $-2u$ and adding this to (6.3) gives us

$$2f\left(\frac{du}{dx}\right)^2 = \frac{(2\lambda - \eta)f + 2k^2}{f(x)}u^2$$

or equivalently,

$$\left(\frac{du}{dx}\right)^2 = \frac{(2\lambda - \eta)f + 2k^2}{2f^2}u^2. \tag{6.4}$$

From equation (6.4) it can be shown that the eigenvalue η depends on λ as follows (see Engman [7] for a proof):

$$\eta = 2\left(\lambda + \frac{k^2}{A}\right).$$

So that equation (6.4) simplifies to:

$$\left(f\frac{du}{dx}\right)^2 = k^2\left(1 - \frac{f(x)}{A}\right)u^2, \tag{6.5}$$

and this equation is differentiated to obtain:

$$2f\frac{du}{dx}\frac{d}{dx}\left(f(x)\frac{du}{dx}\right) = -\frac{k^2}{A}\frac{df}{dx}u^2 + 2k^2\left(1 - \frac{f(x)}{A}\right)u\frac{du}{dx}.$$

Now substitute equation (6.1) into the left-hand side of this equation and we get:

$$\frac{k^2}{A}\frac{df}{dx}u^2 = \left(2\lambda - \frac{2k^2}{A}\right) f u \frac{du}{dx}.$$

Next, we square both sides and substitute (6.5) into the right-hand side of the result to obtain:

$$\frac{k^4}{A^2}\left(\frac{df}{dx}\right)^2 u^4 = k^2\left(2\lambda - \frac{2k^2}{A}\right)^2 \left(1 - \frac{f(x)}{A}\right) u^4,$$

and because $u^4(x) > 0$ a.e. on $(-1, 1)$ we can divide by u^4 a.e. on this interval and simplify to get the following equation for smooth functions f:

$$\left(\frac{df}{dx}\right)^2 + \frac{A}{k^2}\left(2\lambda - \frac{2k^2}{A}\right)^2 f = \frac{A^2}{k^2}\left(2\lambda - \frac{2k^2}{A}\right)^2. \tag{6.6}$$

By putting $c = (A/k^2)(2\lambda - 2k^2/A)^2$ we see that (6.6) simplifies to:

$$\left(\frac{df}{dx}\right)^2 = c(A - f). \tag{6.7}$$

Differentiating one more time produces:

$$2\frac{df}{dx}\frac{d^2f}{dx^2} = -c\frac{df}{dx};$$

so the smooth solutions that we seek are those satisfying the boundary value problem:

$$\frac{d^2f}{dx^2} = -\frac{c}{2},$$

$$f(-1) = 0 = f(1) \quad \text{and} \quad f'(-1) = 2 = -f'(1).$$

Clearly, the differential equation has a quadratic solution and the boundary conditions show that the solution is:

$$f(x) = (1 - x^2).$$

As we saw in §1 this function gives us the standard metric, i.e. the one with Gauss curvature, $\kappa(x) \equiv 1$.

This ends the proof. □

References

[1] M. Berger, *Sur le Spectre d'une Variété Riemannienne*, C. R. Acad. Sci. Paris, **263** (1963), 13–16.

[2] M. Berger, P. Gauduchon, and E. Mazet, *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Math., vol. 194, Springer-Verlag, Berlin, 1971.

[3] A. Besse, *Manifolds All of Whose Geodesics are Closed*, Springer-Verlag, Berlin, 1978.

[4] J. Brüning and E. Heintze, *Spektrale Starrheit gewisser Drehflächen*, Math. Ann., **269** (1984), 95–101.

[5] S. Cheng, *A characterization of the 2-sphere by eigenfunctions*, Proc. Amer. Math. Soc., **55** (1976), 379–381.

[6] J. Dieudonné, *Foundations of Modern Analysis*, Treatise on Analysis, vol. 10-I, Academic Press, New York, 1969.

[7] M. Engman, *The Spectrum of a Surface of Revolution*, Ph.D. Dissertation, University of New Mexico, 1990.

[8] H. Fegan, *Special function potentials for the Laplacian*, Canad. J. Math., **34** (1982), 1183–1194.

[9] ——, *Introduction to Compact Lie Groups*, World Scientific Publishing, to appear.

[10] S. Goldberg and H. Gauchman, *Characterizing S^m by the spectrum of the Laplacian on 2-forms*, Proc. Amer. Math. Soc., **99** no. 4 (1987), 750–756.

[11] V. Guillemin and A. Uribe, *Spectral properties of a certain class of complex potentials*, Trans. Amer. Math. Soc., **279** (1983), 759–771.

[12] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly, **73**, Part II (1966), 1–23.

[13] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience Tracts in Pure and Applied Math., **15** (1963).

[14] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan, **14** (1962), 333–340.

[15] V. Patodi, *Curvature and the fundamental solution of the heat operator*, J. Indian Math. Soc., **34** (1970), 269–285.

[16] H. Weinberger, *A First Course in Partial Differential Equations*, Wiley, 1965.

[17] A. Weinstein, *Asymptotics of eigenvalue clusters for the Laplacian plus a potential*, Duke Math. J., **44** (1977), 883–892.

Received March 14, 1991.

UNIVERSITY OF NEW MEXICO
ALBUQUERQUE, NM 87131