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COBCAT AND SINGULAR BORDISM

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Dold proved that a homomorphism $\phi: H^n(\mathrm{BO}) \to \mathbb{Z}_2$ corresponds to a manifold M^n if and only if $\phi(\mathrm{Sq}^p u + v_p \cdot u) = 0$, $\forall p \geq 0$ and $\forall u \in H^{n-p}(\mathrm{BO})$, v_p being the Wu class. The object of the present work is to have a singular analogue of this result and to study the bordism classification of singular manifolds in BO.

1. Introduction. Singh [1] has developed the notion of cobcat for a manifold M^n and has classified, upto bordism, all manifolds M^n with $\mathrm{cobcat}(M^n) \leq 3$. $\mathrm{Cobcat}(M^n)$ was defined to be the smallest positive integer k such that the number $\langle W_{i_1} \cdots W_{i_p}, [M^n]\rangle = 0$ for all partitions $i_1 + \cdots + i_p = n$ with $k \leq p \leq n$.

Here we develop the notion of cobcat for a singular manifold (M^n, f) in a space X and discuss the bordism classification of all singular manifolds (M^n, f) in BO with $\mathrm{cobcat}(M^n, f) \leq 3$, $n = 2^r$.

Here all the manifolds are to be unoriented, smooth and closed, and all the homology and cohomology coefficients are to be in $\mathbb{Z}_2$. The space X is such that for each n, $H_n(X)$ and hence $H^n(X)$ is a finite dimensional vector space over $\mathbb{Z}_2$.

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2. Preliminaries. Consider the set $N_n(X)$ of bordism classes of n-dimensional singular manifolds (M^n, f) in X, $f: M^n \to X$ being a continuous map. We know that $N_n(X)$ is an abelian group under the operation "disjoint union"

$$[M_1^n, f_1] + [M_2^n, f_2] = [M_1^n \sqcup M_2^n, f_1 \sqcup f_2],$$

where $f_1 \sqcup f_2: M_1^n \sqcup M_2^n \to X$ is given by

$$f_1 \sqcup f_2(x) = \begin{cases} f_1(x) & \text{if } x \in M_1^n, \\ f_2(x) & \text{if } x \in M_2^n. \end{cases}$$

Further, we have

$$N_*(X) = \bigoplus_{n \geq 0} N_n(X).$$

We know that for a point, $N_*(pt) = N_*$, the unoriented bordism ring, and there is a N_*-module structure in $N_*(X)$ given by

$$[M^n, f] \times [N^m] = [M^n \times N^m, f\pi],$$

where $\pi\colon M^n \times N^m \to M^n$ is the projection.

For a singular manifold (M^n, f) in X let $\tau\colon M^n \to \mathrm{BO}$ be the classifying map of the tangent bundle over M^n. Then there is defined a homomorphism $t\colon H^n(\mathrm{BO} \times X) \to \mathbb{Z}_2$ given by

$$t(w \otimes x) = \langle (\tau, f)^*(w \otimes x), [M^n] \rangle = \langle \tau^*(w) f^*(x), [M^n] \rangle,$$

where $w \otimes x \in H^n(\mathrm{BO} \times X) = \bigoplus_{i=0}^n H^{n-i}(\mathrm{BO}) \otimes H^i(X)$ and $(\tau, f)\colon M^n \to \mathrm{BO} \times X$ is given by $(\tau, f)(z) = (\tau(z), f(z))$.

The number $\langle W_{i_1} \cdots W_{i_r} f^*(x_{n-p}), [M^n] \rangle$ is called the Stiefel-Whitney number of (M^n, f) associated to the cohomology class $x_{n-p} \in H^{n-p}(X)$ corresponding to the partition $i_1 + \cdots + i_r = p$. Moreover, this number is as usual bordism invariant [2].

Analogous to [1], given a singular manifold (M^n, f) in X there is associated a Poincaré algebra P^* given as follows:

Let $J = \{z \in H^*(\mathrm{BO} \times X)\colon$ either $\dim z > n$, or for all $z' \in H^{n-\dim z}(\mathrm{BO} \times X)$, $\langle (\tau, f)^* z (\tau, f)^* z', [M^n] \rangle = 0\}$.

It is easy to see that J is an ideal of the graded algebra $H^*(\mathrm{BO} \times X)$. Set

$$P^* = \frac{H^*(\mathrm{BO} \times X)}{J}, \text{ the quotient algebra.}$$

Let $q\colon H^*(\mathrm{BO} \times X) \to P^*$ be the quotient map. Clearly, $P^* = 0$ if and only if (M^n, f) bounds. Let $z \in H^*(\mathrm{BO} \times X)$; we say, "$z = 0$ in P^*" if $q(z) = 0$.

As in [1], we have the following proposition, whose verification is a routine matter.

PROPOSITION 2.1. *If (M^n, f) is not a boundary then*
(a) *P^* is an n-dimensional graded algebra with Poincaré duality,*
(b) *the Steenrod algebra acts on P^* with the action given by*

$$\mathrm{Sq}^i(q(z)) = q(\mathrm{Sq}^i(z)),$$

(c) *if $z \in H^n(\mathrm{BO} \times X)$ then $q(z) = 0$ if and only if*

$$t(z) = \langle (\tau, f)^* z, [M^n] \rangle = 0. \qquad \square$$

It is easy to see that for all $p \geq 0$, and for all $z \in H^{n-p}(\mathrm{BO} \times X)$,

$$t(\mathrm{Sq}^p(z) + (v_p \otimes 1) z) = 0,$$

where $v_p \in H^p(\mathrm{BO})$ is the Wu class. So, in view of the above proposition, we have $\mathrm{Sq}^p(z) = (v_p \otimes 1) z$ in P^*.

3. Extension of Dold's and Milnor's results to singular case. Dold [3] has proved the following

Result 3.1. For each integer $n \geq 0$, if $\phi: H^n(\mathrm{BO}) \to \mathbb{Z}_2$ is a homomorphism then there is an n-dimensional closed manifold M^n with $\phi(w) = \langle \tau(w), [M^n] \rangle$ for all $w \in H^n(\mathrm{BO})$ if and only if $\phi(\mathrm{Sq}^p(u) + v_p \cdot u) = 0$ for all $u \in H^{n-p}(\mathrm{BO})$ and for all $p \geq 0$, $v_p \in H^p(\mathrm{BO})$ being the Wu class. □

Here we shall extend this result to the singular case as follows:

THEOREM 3.2. *For each $n \geq 0$, if $h: H^n(\mathrm{BO} \times X) \to \mathbb{Z}_2$ is a homomorphism then there is an n-dimensional singular manifold (M^n, f) with $h(w \otimes x) = \langle \tau^*(w) f^*(x), [M^n] \rangle$ for all $w \otimes x \in H^n(\mathrm{BO} \times X)$ if and only if $h(\mathrm{Sq}^p(u \otimes y) + (v_p \cdot u) \otimes y) = 0$ for all $u \otimes y \in H^{n-p}(\mathrm{BO} \times X)$ and for all $p \geq 0$, $v_p \in H^p(\mathrm{BO})$ being the Wu class.*

Proof. It is easy to see that the condition is necessary. We prove that the condition is sufficient also. Let $\{c_{m,i}\}_{i \in I_m}$ denote a basis for the vector space $H_m(X)$ over $\mathbb{Z}_2$, $m \geq 0$. Let $c^{m,i} \in H^m(X)$ be the cohomology class dual to $c_{m,i}$ i.e. $\langle c^{m,i}, c_{m,j} \rangle = \delta_{ij}$. Note that $\{c^{m,i}\}$ forms a basis for $H^m(X)$. Now, for each $c_{m,j}$ we can choose a singular manifold (M_j^m, f_j^m) with $(f_j^m)_*([M_j^m]) = c_{m,j}$, [2]. Thus, we have

$$\langle (f_j^m)^* c^{m,i}, [M_j^m] \rangle = \delta_{ij}.$$

Now,

$$H^n(\mathrm{BO} \times X) = \bigoplus_{l=0}^{n} H^l(\mathrm{BO}) \otimes H^{n-l}(X).$$

Define,

$$h_i^0: H^0(\mathrm{BO}) \to \mathbb{Z}_2$$

by

$$h_i^0(1) = h(1 \otimes c^{n,i}).$$

Clearly, h_i^0 satisfies the condition given in Result (3.1) and so there exists a manifold V_i^0 such that

$$h_i^0(1) = \langle \tau^*(1), [V_i^0] \rangle = \langle 1, [V_i^0] \rangle,$$

for each $i \in I_n$.

Define $h_i^1 \colon H^1(\mathrm{BO}) \to \mathbb{Z}_2$ by

$$\begin{aligned} h_i^1(w) &= h(w \otimes c^{n-1,i}) \\ &\quad + \sum_{j\in I_n} \langle \tau^*(w)(f_j^n\pi)^*(c^{n-1,i}), [M_j^n \times V_j^0]\rangle \\ &= h(w \otimes c^{n-1,i}) \\ &\quad + \sum_j \langle \tau^* w \cdot ((f_j^n)^* c^{n-1,i} \otimes 1), [M_j^n \times V_j^0]\rangle . \end{aligned}$$

Now,

$$\begin{aligned} &h_i^1(\mathrm{Sq}^1(1) + v_1 \cdot 1) \\ &\quad = h(v_1 \otimes c^{n-1,i}) + \sum_j \langle \tau^* v_1 \cdot ((f_j^n)^* c^{n-1,i} \otimes 1), [M_j^n \times V_j^0]\rangle \\ &\quad = h(\mathrm{Sq}^1(1 \otimes c^{n-1,i})) + \sum_j \langle \mathrm{Sq}^1((f_j^n)^* c^{n-1,i} \otimes 1), [M_j^n \times V_j^0]\rangle \\ &\quad = h(1 \otimes \mathrm{Sq}^1 c^{n-1,i}) + \sum_j \langle (f_j^n)^* \mathrm{Sq}^1 c^{n-1,i}, [M_j^n]\rangle \langle 1, [V_j^0]\rangle . \end{aligned}$$

Since, $\mathrm{Sq}^1 c^{n-1,i} \in H^n(X)$, there is a subset $K_n \subset I_n$ such that

$$\mathrm{Sq}^1 c^{n-1,i} = \sum_{k\in K_n} c^{n,k} .$$

Therefore,

$$\begin{aligned} &h_i^1(\mathrm{Sq}^1(1) + v_1 \cdot 1) \\ &\quad = \sum_{k\in K_n} h(1 \otimes c^{n,k}) + \sum_j \sum_{k\in K_n} \langle (f_j^n)^* c^{n,k}, [M_j^n]\rangle h_j^0(1) \\ &\quad = \sum_k h_k^0(1) + \sum_k h_k^0(1), \quad \text{since } \langle (f_j^n)^* c^{n,k}, [M_j^n]\rangle = \delta_{kj} \\ &\quad = 0. \end{aligned}$$

So, by Result (3.1), there exists a manifold V_i^1 such that

$$h_i^1(w) = \langle \tau^* w, [V_i^1]\rangle ,$$

for each $i \in I_{n-1}$.

Now, using induction, we define $h_i^l \colon H^l(\mathrm{BO}) \to \mathbb{Z}_2$ by

$$\begin{aligned} h_i^l(w) &= h(w \otimes c^{n-l,i}) \\ &\quad + \sum_{\substack{j\in I_{n-m} \\ 0\le m<1}} \langle \tau^* w \cdot ((f_j^{n-m})^* c^{n-l,i} \otimes 1), [M_j^{n-m} \times V_j^m]\rangle \end{aligned}$$

where $1 \le l \le n$, and each V_j^m is given by h_j^m $(m < l)$.

Now, it is enough to show that h_i^l satisfies the condition given in (3.1). For if it is so, then there exists a family $\{V_i^l\}_{i\in I_{n-l}}$ of manifolds such that for each $i \in I_{n-l}$,

$$h_i^l(w) = \langle \tau^* w, [V_i^l]\rangle \qquad (1 \le l \le n).$$

Also, we already have a family $\{V_i^0\}_{i\in I_n}$ of manifolds such that

$$h_i^0(1) = \langle 1, [V_i^0]\rangle,$$

for each $i \in I_n$. It is then easy to see that the given homomorphism $h: H^n(\mathrm{BO}\times X) \to \mathbb{Z}_2$ corresponds to the singular manifold (M^n, f) given by

$$M^n = \bigsqcup M_i^{n-l} \times V_i^l \quad \text{and} \quad f = \bigsqcup (f_i^{n-l}\pi),$$

where the disjoint union $\bigsqcup$ is taken over all $i \in I_{n-l}$ and all $0 \le l \le n$, and $\pi: M_i^{n-l} \times V_i^l \to M_i^{n-l}$ denotes the projection map.

Note that, for each $p \ge 0$ and for each $u \in H^{l-p}(\mathrm{BO})$,

$$h_i^l(\mathrm{Sq}^p u + v_p \cdot u) = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\begin{aligned}
\mathrm{I} &= h((\mathrm{Sq}^p u + v_p \cdot u) \otimes c^{n-l,i}) \\
&= h(\mathrm{Sq}^p u \otimes c^{n-l,i} + (v_p \cdot u) \otimes c^{n-l,i}) \\
&= h\left(\sum_{\substack{r+s=p\\ r\ne p}} \mathrm{Sq}^r u \otimes \mathrm{Sq}^s c^{n-l,i}\right), \quad \text{by hypothesis on } h, \\
&= \sum_{\substack{r+s=p\\ r\ne p}} h(\mathrm{Sq}^r u \otimes \mathrm{Sq}^s c^{n-l,i}), \\
\mathrm{II} &= \sum_{\substack{j\in I_{n-m}\\ 0\le m<l}} \langle \tau^*(\mathrm{Sq}^p u)((f_j^{n-m})^* c^{n-l,i} \otimes 1), [M_j^{n-m} \times V_j^m]\rangle \\
&= \sum_{j,m} \langle \mathrm{Sq}^p(\tau^* u)((f_j^{n-m})^* c^{n-l,i} \otimes 1), [M_j^{n-m} \times V_j^m]\rangle, \quad \text{and} \\
\mathrm{III} &= \sum_{j,m} \langle \tau^*(v_p \cdot u)((f_j^{n-m})^* c^{n-l,i} \otimes 1), [M_j^{n-m} \times V_j^m]\rangle \\
&= \sum_{j,m} \langle \mathrm{Sq}^p(\tau^* u((f_j^{n-m})^* c^{n-l,i} \otimes 1)), [M_j^{n-m} \times V_j^m]\rangle.
\end{aligned}$$

So,

$$\begin{aligned}\mathrm{II}+\mathrm{III} &= \sum_{j\,,m}\sum_{\substack{r+s=p\\ r\neq p}} \langle \mathrm{Sq}^r(\tau^* u)(\mathrm{Sq}^s(f_j^{n-m})^* c^{n-l\,,i}\otimes 1)\,,\,[M_j^{n-m}\times V_j^m]\rangle \\ &= \sum_{j\,,m}\sum_{\substack{r+s=p\\ r\neq p}} \langle \mathrm{Sq}^r(\tau^* u)((f_j^{n-m})^*\,\mathrm{Sq}^s\,c^{n-l\,,i}\otimes 1)\,,\,[M_j^{n-m}\times V_j^m]\rangle\,.\end{aligned}$$

Now, since $\mathrm{Sq}^s\,c^{n-l\,,i}\in H^{n-l+s}(X)$, there is a subset K_{n-l+s} of I_{n-l+s} such that

$$\mathrm{Sq}^s\,c^{n-l\,,i} = \sum_{k\in K_{n-l+s}} c^{n-l+s\,,k}\,,\qquad (1\le s\le p).$$

So,

$$\begin{aligned}(1)\quad & h(\mathrm{Sq}^r\,u\otimes \mathrm{Sq}^s\,c^{n-l\,,i}) \\ &= \sum_{k\in K_{n-l+s}} h(\mathrm{Sq}^r\,u\otimes c^{n-l+s\,,k}) \\ &= \sum_k \left\{ \sum_{\substack{j\,,m\\ (m<l-s)}} \langle \tau^*(\mathrm{Sq}^r\,u)((f_j^{n-m})^* c^{n-l+s\,,k}\otimes 1)\,,\,[M_j^{n-m}\times V_j^m]\rangle \right. \\ &\qquad\qquad \left. +\ \langle \tau^*(\mathrm{Sq}^r\,u)\,,\,[V_k^{l-s}]\rangle \right\}\,,\end{aligned}$$

noting that $l-s<l$

$$\begin{aligned} &= \sum_{\substack{j\,,m\\ (m<l-s)}} \langle \mathrm{Sq}^r(\tau^* u)((f_j^{n-m})^*\,\mathrm{Sq}^s\,c^{n-l\,,i}\otimes 1)\,,\,[M_j^{n-m}\times V_j^m]\rangle \\ &\quad + \sum_k \langle \mathrm{Sq}^r(\tau^* u)\,,\,[V_k^{l-s}]\rangle\,.\end{aligned}$$

Also,

$$\begin{aligned}(*)\quad & \sum_{\substack{j\,,m\\ (l-s\le m<l)}} \langle \mathrm{Sq}^r(\tau^* u)((f_j^{n-m})^*\,\mathrm{Sq}^s\,c^{n-l\,,i}\otimes 1)\,,\,[M_j^{n-m}\times V_j^m]\rangle \\ &= \sum_j \langle \mathrm{Sq}^r(\tau^* u)((f_j^{n-l+s})^*\,\mathrm{Sq}^s\,c^{n-l\,,i}\otimes 1)\,,\,[M_j^{n-l+s}\times V_j^{l-s}]\rangle\,,\end{aligned}$$

by dimensional consideration, since $m\ge l-s$. Further, we note that u is a polynomial in Stiefel-Whitney classes of BO, so that $\mathrm{Sq}^r(\tau^* u)$ is a polynomial in Stiefel-Whitney classes of $M_j^{n-l+s} \times V_j^{l-s}$. Therefore the above expression $(*)$ becomes equal to

$$(2)\quad \sum_j \langle (f_j^{n-l+s})^* \operatorname{Sq}^s c^{n-l,i}, [M_j^{n-l+s}]\rangle \langle \operatorname{Sq}^r(\tau^* u), [V_j^{l-s}]\rangle$$
$$= \sum_j \sum_{k \in K_{n-l+s}} \langle (f_j^{n-l+s})^* c^{n-l+s,k}, [M_j^{n-l+s}]\rangle \langle \operatorname{Sq}^r(\tau^* u), [V_j^{l-s}]\rangle$$
$$= \sum_k \langle \operatorname{Sq}^r(\tau^* u), [V_k^{l-s}]\rangle .$$

Hence, combining I with (1) and II + III with (2), it follows that

$$h_i^l(\operatorname{Sq}^p u + v_p \cdot u) = \mathrm{I} + \mathrm{II} + \mathrm{III} = 0 .$$

That is, h_i^l satisfies the condition given in (3.1). □

Now, consider the universal bundle $\gamma\colon \mathrm{EO} \to \mathrm{BO}$ and the cartesian product $\gamma \times \gamma$ over $\mathrm{BO} \times \mathrm{BO}$. Let $\mu\colon \mathrm{BO} \times \mathrm{BO} \to \mathrm{BO}$ be the classifying map of $\gamma \times \gamma$. μ has the property that

$$\mu^*(W_i) = \sum_{k=0}^{i} W_k \otimes W_{i-k} .$$

The product of two singular manifolds (M^m, f) and (M^n, g) in BO is given by $(M^m \times M^n, \mu \circ (f \times g))$, and this product induces a multiplication in $N_*(\mathrm{BO})$ given by

$$[M^m, f] \times [M^n, g] = [M^m \times M^n, \mu \circ (f \times g)],$$

which makes $N_*(\mathrm{BO})$ an algebra over $\mathbb{Z}_2$.

Analogous to **[4]**, we have

LEMMA 3.3. *The Stiefel-Whitney numbers*

$$\langle W_{2i_1} \cdots W_{2i_r} (\mu \circ (g \times g))^* (W_{2i_{r+1}} \cdots W_{2i_{r+s}}), [N \times N]\rangle$$

of the product $(N, g) \times (N, g)$ *in* BO *are equal to*

$$\langle W_{i_1} \cdots W_{i_r} g^* (W_{i_{r+1}} \cdots W_{i_{r+s}}), [N]\rangle ,$$

while the numbers

$$\langle W_{j_1} \cdots W_{j_p} (\mu \circ (g \times g))^* (W_{j_{p+1}} \cdots W_{j_{p+q}}), [N \times N]\rangle$$

are zero if some j_h *is odd.*

Proof. Routine verification. □

THEOREM 3.4. *Let (M^{2n}, f) be a singular manifold in* BO, *such that*

$$\langle W_{j_1} \cdots W_{j_p} f^*(W_{j_{p+1}} \cdots W_{j_{p+q}}), [M^{2n}] \rangle = 0,$$

whenever some j_h is odd. Then

$$[M^{2n}, f] = [(N^n, g) \times (N^n, g)] \quad \text{in } N_{2n}(\mathrm{BO}).$$

Proof. We shall construct a singular manifold (N^n, g) in BO whose Stiefel-Whitney numbers

$$\langle W_{i_1} \cdots W_{i_r} g^*(W_{i_{r+1}} \cdots W_{i_{r+s}}), [N^n] \rangle$$

are equal to

$$\langle W_{2i_1} \cdots W_{2i_r} f^*(W_{2i_{r+1}} \cdots W_{2i_{r+s}}), [M^{2n}] \rangle .$$

This will imply that (M^{2n}, f) is cobordant to $(N^n, g) \times (N^n, g)$, by (3.3).

Let $R^n \subset H^n(\mathrm{BO} \times \mathrm{BO})$ be the vector space generated by all elements of the form $\mathrm{Sq}^p(x \otimes y) + (v_p \cdot x) \otimes y$. The Stiefel-Whitney numbers of each manifold (N^n, g) determine a homomorphism

$$h_N \colon H^n(\mathrm{BO} \times \mathrm{BO}) \to \mathbb{Z}_2$$

given by $h_N(x \otimes y) = \langle (\tau, g)^*(x \otimes y), [N] \rangle$, and by Theorem (3.2) we know that a given homomorphism $H^n(\mathrm{BO} \times \mathrm{BO}) \to \mathbb{Z}_2$ corresponds to a singular manifold in BO if and only if it annihilates R^n.

Define the "doubling homomorphism"

$$d \colon H^*(\mathrm{BO} \times \mathrm{BO}) \to H^*(\mathrm{BO} \times \mathrm{BO})$$

by

$$d(W_i \otimes W_j) = W_{2i} \otimes W_{2j} .$$

Let (M^{2n}, f) satisfy the hypothesis of Theorem (3.4). Then we shall show that $h_M \circ d \colon H^n(\mathrm{BO} \times \mathrm{BO}) \to \mathbb{Z}_2$ annihilates R^n. This will prove the existence of the required manifold (N^n, g).

Let $I \subset H^*(\mathrm{BO} \times \mathrm{BO})$ denote the ideal generated by the family $\{W_i \otimes 1, 1 \otimes W_i\}_{i \text{ odd}}$. Note that

$$\begin{aligned} \mathrm{Sq}^{2i} d(W_j \otimes 1) &= (\mathrm{Sq}^{2i} W_{2j}) \otimes 1 \\ &= \left(\sum_{k=0}^{2i} \binom{2j - 2i + k - 1}{k} W_{2i-k} W_{2j+k} \right) \otimes 1 , \end{aligned}$$

where $\binom{p}{q}$ denotes the binomial coefficients reduced modulo 2. Therefore using the fact that

$$\binom{2j-2i+2l-1}{2l} = \binom{j-i+l-1}{l}$$

we get

$$\begin{aligned} \mathrm{Sq}^{2i} d(W_j \otimes 1) &\equiv \left(\sum_{l=0}^{i} \binom{j-i+l-1}{l} W_{2i-2l} W_{2j+2l} \right) \otimes 1 \mod I \\ &\equiv d(\mathrm{Sq}^i(W_j \otimes 1)) \mod I. \end{aligned}$$

Similarly, $\mathrm{Sq}^{2i} d(1 \otimes W_j) \equiv d(\mathrm{Sq}^i(1 \otimes W_j)) \mod I$. Further, if

$$\begin{aligned} &\mathrm{Sq}^{2i} d(x \otimes y) \equiv d(\mathrm{Sq}^i(x \otimes y)) \mod I \quad \text{and} \\ &\mathrm{Sq}^{2i} d(x \otimes y) \equiv d(\mathrm{Sq}^i(x' \otimes y')) \mod I, \end{aligned}$$

then

$$\begin{aligned} &\mathrm{Sq}^{2i} d((x \otimes y)(x' \otimes y')) \\ &\quad \equiv \sum_{p+q=i} (\mathrm{Sq}^{2p} d(x \otimes y))(\mathrm{Sq}^{2q} d(x' \otimes y')) \mod I \\ &\quad \equiv \sum_{p+q=i} (d(\mathrm{Sq}^p(x \otimes y)))(d(\mathrm{Sq}^q(x' \otimes y'))) \mod I \\ &\quad \equiv d(\mathrm{Sq}^i((x \otimes y)(x' \otimes y'))) \mod I. \end{aligned}$$

Hence, by induction, it follows that

$$\mathrm{Sq}^{2i} d(x \otimes y) \equiv d(\mathrm{Sq}^i(x \otimes y)) \mod I,$$

for each $x \otimes y \in H^*(\mathrm{BO} \times \mathrm{BO})$.

It is simple to verify that I is closed under Steenrod squaring operation. Applying induction on p, one gets

$$d(v_p \otimes 1) \equiv (v_{2p} \otimes 1) \mod I.$$

Now, consider the manifold (M^{2n}, f). By the hypothesis on (M^{2n}, f) we have

$$h_M(I^{2n}) = 0, \quad \text{where } I^{2n} = I \cap H^{2n}(\mathrm{BO} \times \mathrm{BO}).$$

Therefore, for any generator $\mathrm{Sq}^p(x \otimes y) + (v_p \otimes 1)(x \otimes y)$ of R^n we have, using the congruences established above,

$$\begin{aligned} &(h_M \circ d)(\mathrm{Sq}^p(x \otimes y) + (v_p \otimes 1)(x \otimes y)) \\ &\quad = h_M(\mathrm{Sq}^{2p}(d(x \otimes y)) + (v_{2p} \otimes 1)\, d(x \otimes y) + (\text{terms in } I^{2n})) \\ &\quad = 0. \end{aligned}$$

That is, $h_M \circ d$ annihilates R^n and so by Theorem (3.2) there exists a singular manifold (N^n, g) in BO such that $h_M \circ d = h_N$. Hence, the theorem follows. □

4. Cobcat and singular bordism in BO. Analogous to **[1]**, we define the cobcat for a singular manifold (M^n, f) in X as follows

Definition. Cobcat(M^n, f) is the smallest positive integer k such that for each m, $0 \le m \le n$, the number

$$\langle W_{i_1} \cdots W_{i_p} f^*(x_{j_1} \cdots x_{j_q}), [M^n]\rangle = 0$$

for all partitions $i_1 + \cdots + i_p$ of m and for all partitions $j_1 + \cdots + j_q$ of $n - m$, with $k \le p + q \le n$ ($x_{j_h} \in H^{j_h}(X)$ for all j_h). If no such k exists define cobcat$(M^n, f) = n + 1$.

Remark 4.1. (a) Cobcat$(M^n) \le$ cobcat(M^n, f),
(b) cobcat$(M^n, f) = 1$ if and only if (M^n, f) bounds,
(c) cobcat$(M^n, f) \le$ nil(Im$(\tau, f)^*) \le$ cat$(\tau, f) \le$ cat(M^n),
(d)

$$\text{cobcat}(M_1^n \sqcup M_2^n, f_1 \sqcup f_2) \le \max\{\text{cobcat}(M_1^n, f_1),\ \text{cobcat}(M_2^n, f_2)\}.$$

Now we shall discuss the singular version of some results proved in **[1]**. Let P^* be the Poincaré algebra associated to the singular manifold (M^n, f) in X. As in **[1]**, an element z of any graded algebra A^* will be called k-decomposable if it is zero or is the sum of the products $z_1 \cdot z_2 \cdot \cdots \cdot z_p$ where $z_i \in A^*$ with $\dim z_i > 0$ for each i, and $p \ge k$.

Proposition 4.2. *Let* cobcat$(M^n, f) \le k$.
(a) *If $z \in P^*$ is k-decomposable, then z is zero.*
(b) *If $z \in P^*$ is $(k-1)$-decomposable and* $\dim z < n$ *then z is zero.*

Proof. Note that any k-decomposable element z of $H^*(\text{BO} \times X) \cong H^*(\text{BO}) \otimes H^*(X)$ can be written as a sum of the products $z_1 \cdot z_2 \cdot \cdots \cdot z_p$, where each z_i is of the type $W_j \otimes 1$ or $1 \otimes x_j$, and $p \ge k$. Also for any $z' \in H^{n-\dim z}(\text{BO} \times X)$ we have

$$\langle (\tau, f)^* z_1 (\tau, f)^* z_2 \cdots (\tau, f)^* z_p (\tau, f)^* z', [M^n]\rangle = 0,$$

since cobcat$(M^n, f) \le k$ and $p \ge k$. Hence (a) follows, using the fact that any k-decomposable element of P^* can be obtained from a k-decomposable element of $H^*(\text{BO} \times X)$.

As in ([**1**], 1.2), (b) follows from (a). □

From now on, the ambient space X will be taken to be the universal base space BO and (M^n, f) will denote a singular manifold in BO with $\operatorname{cobcat}(M^n, f) \leq 3$.

LEMMA 4.3. (a) *If* $z(W_i \otimes 1) = 0$ *and* $z(1 \otimes W_i) = 0$ *in* P^*, *where* $z \in H^{n-i}(\mathrm{BO} \times \mathrm{BO})$ *and* $0 < i < n$, *then* $z = 0$ *in* P^*.

(b) *For* $j > 0$,

$$W_{2j+1} \otimes 1 = \begin{cases} \mathrm{Sq}^1(W_{2j} \otimes 1) & \text{if } 2j+1 < n, \\ 0 & \text{if } 2j+1 = n, \end{cases}$$

and

$$1 \otimes W_{2j+1} = \mathrm{Sq}^1(1 \otimes W_{2j}) \quad \text{if } 2j+1 < n$$

in P^*.

Proof. (a) By the hypothesis, the last proposition and the fact that $\operatorname{cobcat}(M^n, f) \leq 3$, we have

$$\langle (\tau, f)^* z (\tau, f)^* z', [M^n] \rangle = \langle (\tau, f)^* (z \cdot z'), [M^n] \rangle = 0$$

for all $z' \in H^i(\mathrm{BO} \times \mathrm{BO})$. So $z = 0$ in P^*.

(b) Note that

$$\mathrm{Sq}^1(W_{2j} \otimes 1) = (W_1 \otimes 1)(W_{2j} \otimes 1) + W_{2j+1} \otimes 1.$$

If $2j+1 < n$ then $(W_1 \otimes 1)(W_{2j} \otimes 1)$, being decomposable, is zero in P^*. If $2j+1 = n$, then $(W_1 \otimes 1)(W_{2j} \otimes 1) = \mathrm{Sq}^1(W_{2j} \otimes 1)$ in P^*. For the last part of (b) one has

$$\mathrm{Sq}^1(1 \otimes W_{2j}) = (1 \otimes W_1)(1 \otimes W_{2j}) + 1 \otimes W_{2j+1}. \qquad \square$$

LEMMA 4.4. *If* n *is even and* $n > 2$, *then* $W_i \otimes 1 = 0$ *and* $1 \otimes W_i = 0$ *in* P^* *for all odd* i.

Proof. By Lemma (4.3), it is enough to show that

(a) $(W_i \otimes 1)(W_{n-i} \otimes 1) = 0$,

(b) $(W_i \otimes 1)(1 \otimes W_{n-i}) = 0$,

(c) $(1 \otimes W_i)(1 \otimes W_{n-i}) = 0$ in P^* for all odd i.

For (a), let $i = 1$; then

$$(W_1 \otimes 1)(W_{n-1} \otimes 1) = \mathrm{Sq}^1(W_{n-1} \otimes 1) = \mathrm{Sq}^1 \mathrm{Sq}^1(W_{n-2} \otimes 1) = 0$$

in P^*, using (4.3) and the fact that $\mathrm{Sq}^1 \mathrm{Sq}^1 = 0$. Now let $i = 2j+1$,

$j > 0$ and $n - i = 2k + 1$, $k > 0$; then by (4.3)

$$\begin{aligned}(W_i \otimes 1)(W_{n-i} \otimes 1) &= \mathrm{Sq}^1(W_{2j} \otimes 1)\,\mathrm{Sq}^1(W_{2k} \otimes 1)\\ &= \mathrm{Sq}^1((W_{2j} \otimes 1)\,\mathrm{Sq}^1(W_{2k} \otimes 1))\\ &= (W_1 \otimes 1)(W_{2j} \otimes 1)\,\mathrm{Sq}^1(W_{2k} \otimes 1) = 0\end{aligned}$$

in P^*, as it is 3-decomposable. Hence (a) follows. For (b), using the same technique as in (a), we have $(W_i \otimes 1)(1 \otimes W_{n-i}) = 0$ in P^* for all odd $i < n-1$. However, for $i = n-1$, we have by (4.3)

$$\begin{aligned}(W_{n-1} \otimes 1)(1 \otimes W_1) &= \mathrm{Sq}^1(W_{n-2} \otimes 1)(1 \otimes W_1)\\ &= \mathrm{Sq}^1((W_{n-2} \otimes 1)(1 \otimes W_1)) + (W_{n-2} \otimes 1)\,\mathrm{Sq}^1(1 \otimes W_1)\\ &= (W_1 \otimes 1)(W_{n-2} \otimes 1)(1 \otimes W_1)\\ &\quad + (W_{n-2} \otimes 1)(1 \otimes W_1)(1 \otimes W_1) = 0\end{aligned}$$

in P^*, as it is 3-decomposable. Thus (b) follows. Now, (c) can be proved by the same technique used in (a) and (b) above. □

PROPOSITION 4.5. *If (M^n, f) is a non-bounding n-dimensional singular manifold in* BO *with* cobcat$(M^n, f) \leq 3$, *where n is even and $n > 2$, then (M^n, f) is cobordant to a product $(N, g) \times (N, g)$ in* BO, *where (N, g) is also non-bounding and* cobcat$(N, g) \leq 3$.

Proof. By Theorem (3.4) and Lemma (4.4), there exists a singular manifold (N, g) is BO of dimension $n/2$ such that (M^n, f) is cobordant to the product $(N, g) \times (N, g)$. Also,

$$\begin{aligned}&\langle W_{i_1} \cdots W_{i_p} g^*(W_{i_{p+1}} \cdots W_{i_{p+q}}), [N]\rangle\\ &\quad = \langle W_{2i_1} \cdots W_{2i_p} f^*(W_{2i_{p+1}} \cdots W_{2i_{p+q}}), [M]\rangle,\end{aligned}$$

where $i_1 + \cdots + i_{p+q} = n/2$ is a partition of $n/2$. Hence the proposition follows. □

In the above proposition one can observe that if the underlying manifold M^n in (M^n, f) were a boundary then N in (N, g) would also be a boundary. Further, using induction, one can easily get the following

COROLLARY 4.6. *Let (M^n, f) be an n-dimensional singular manifold in* BO *with* cobcat$(M^n, f) \leq 3$. *Let $n = 2^r \cdot m$ where either m is odd and $m \geq 3$, or $m = 2$. Then either (M^n, f) is a boundary or else (M^n, f) is cobordant to $(N, g)^{2^r}$, where (N, g) is a non-bounding m-dimensional singular manifold in* BO *with* cobcat$(N, g) \leq 3$. □

Finally, in view of the above results and the fact that $N_2(\mathrm{BO})$ is generated by the bordism classes $[(RP^1, \tau_1)^2]$, $[RP^2, \tau_1]$ and $[RP^2, c]$, where $\tau_1: RP^i \to \mathrm{BO}$ is the classifying map of the canonical line bundle over RP^i $(i = 1, 2)$ and $c: RP^2 \to \mathrm{BO}$ is the constant map, we can make the following remarks.

REMARK 4.7. Let (M^n, f) be as in (4.6) and $n = 2^r$, $r \geq 1$. Then either (M^n, f) is a boundary or else (M^n, f) is cobordant to $(N, g)^{2^{r-1}}$, where (N, g) is a 2-dimensional singular manifold generated by $(RP^1, \tau_l)^2$, (RP^2, τ_l) and/or (RP^2, c). □

REMARK 4.8. In Remark (4.7) if, in addition, the underlying manifold M^n in (M^n, f) were a boundary then (N, g) would be equal to exactly one of the following

(i) $(RP^1, \tau_l)^2$,
(ii) $(RP^2, c) \sqcup (RP^2, \tau_l)$, or
(iii) $(RP^1, \tau_l)^2 \sqcup (RP^2, c) \sqcup (RP^2, \tau_l)$,

where $(M_1, f_1) \sqcup (M_2, f_2) = (M_1 \sqcup M_2, f_1 \sqcup f_2)$. □

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