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LOCAL REAL ANALYTIC BOUNDARY REGULARITY OF AN INTEGRAL SOLUTION OPERATOR OF THE $\bar{\partial}$-EQUATION ON CONVEX DOMAINS

ZHENHUA CHEN

In this paper we show that a well known integral solution operator of the $\bar{\partial}$-equation on a convex domain Ω locally preserves the real analyticity of $\bar{\partial}$-closed $(0, 1)$ forms at boundary points near which $\partial\Omega$ is totally convex in the complex tangential directions.

1. Introduction. The real analytic boundary regularity of the canonical solution or Kohn's solution of the $\bar{\partial}$-equation was studied by many researchers [2], [3], [7], [8], [9], [10], [14], [15], [16]. In general, the canonical solution is not explicit. An interesting question is, whether we can find some explicit or computable solution of the $\bar{\partial}$-equation that is real analytically regular up to the boundary of pseudoconvex domains. In this paper we prove the following local real analytic boundary regularity of a well known integral solution operator of the $\bar{\partial}$-equation for totally convex domains (terminology will be defined in §2). The real analytic boundary regularity of the canonical solution of such domains has not been proved yet. The recent work of Boas and Straube [1] gives the global C^∞ boundary regularity of the canonical solution for convex domains.

THEOREM. *Suppose Ω is a bounded convex domain in $\mathbb{C}^n$ with C^2 boundary. If p is a real analytic boundary point of Ω, and $\partial\Omega$ is totally convex at p in the complex tangential directions, then the Henkin operator T locally preserves the real analyticity of $\bar{\partial}$-closed $(0, 1)$ forms up to the boundary point p.*

It had been conjectured for some time that global analytic hypoellipticity of the $\bar{\partial}$-Neumann problem held for weakly pseudoconvex domains, and this remains open. But local results on finite type domains came as a surprise comparing with C^∞ results. A recent counterexample of Christ and Geller [6] shows that local real analytic boundary regularity of the $\bar{\partial}$-equation does not hold for general pseudoconvex domains of finite type. We would like to point out that the domain used by Christ and Geller in their counterexample does not satisfy the total convexity, which is the key assumption in our theorem, at the boundary point $z = 0$.

2. Preliminaries. Let h be a function on a domain $\Omega \subseteq \mathbb{C}^n$ and p be a boundary point of Ω. We say that h is *real analytic* at p if h has a convergent power series expansion in a neighborhood U of p in $\mathbb{C}^n$. We say that a $(0, 1)$ form is a real analytic at p if its coefficients are all real analytic at p. Let E be a subset of $\overline{\Omega}$. We define

$$C^\omega(E) = \text{ all functions that are real analytic at every point of } E\,.$$

Suppose Ω is a bounded convex domain in $\mathbb{C}^n$ with C^2 boundary. Following the standard construction we can obtain an explicit solution of the $\overline{\partial}$-equation $\overline{\partial} u = f$ on Ω as follows. Let

$$\delta(z) = \begin{cases} -\operatorname{dist}(z, \partial\Omega), & z \in \Omega, \\ \operatorname{dist}(z, \partial\Omega), & z \notin \Omega, \end{cases}$$

$$\Phi(\zeta, z) = \sum_{j=1}^{n} \frac{\partial \delta}{\partial \zeta_j}(\zeta)(\zeta_j - z_j),$$

$$B = B(\zeta, z) = \frac{\partial_\zeta(|\zeta - z|^2)}{|\zeta - z|^2}, \qquad W(\zeta, z) = \sum_{j=1}^{n} \frac{(\partial\delta/\partial\zeta_j)(\zeta)}{\Phi(\zeta, z)}\, d\zeta_j,$$

$$\widehat{W} = \widehat{W}(\lambda, \zeta, z) = \lambda W(\zeta, z) + (1 - \lambda) B(\zeta, z),$$

$$\Omega(B(\zeta, z)) = C_n B \wedge (\overline{\partial}_\zeta B)^{n-1}, \qquad \Omega(\widehat{W}) = C_n \widehat{W} \wedge (\overline{\partial}_{\zeta,\lambda} \widehat{W})^{n-1},$$

where

$$\overline{\partial}_{\zeta,\lambda} = \overline{\partial}_\zeta + d_\lambda, \qquad C_n = \frac{1}{(2\pi i)^n}.$$

Let

$$T(h)(z) = \int_{\partial\Omega \times I} h(\zeta) \wedge \Omega(\widehat{W}(\lambda, \zeta, z)) - \int_\Omega h(\zeta) \wedge \Omega(B(\zeta, z)).$$

It is well known that for any given $\overline{\partial}$-closed $(0, 1)$ form f with coefficients in $C^1(\overline{\Omega})$, $u(z) = T(f)(z)$ is a solution of the $\overline{\partial}$-equation $\overline{\partial} u = f$ in Ω (see, for example, [13]). We call T the *Henkin operator.*

We say that the Henkin operator T *globally preserves the real analyticity* of $\overline{\partial}$-closed $(0, 1)$ forms up to the boundary of Ω if for any $\overline{\partial}$-closed $(0, 1)$ form f with coefficients in $C^\omega(\overline{\Omega})$ it follows that $T(f) \in C^\omega(\overline{\Omega})$. We say that the Henkin operator T *locally preserves the real analyticity* of $\overline{\partial}$ closed $(0, 1)$ forms up to the boundary point

p if for any $\bar\partial$-closed $(0, 1)$ form f with coefficients in $C^1(\overline{\Omega})$ and real analytic at p it follows that $T(f)$ is also real analytic at p. It is clear that if the Henkin operator T locally preserves the real analyticity of $\bar\partial$ closed $(0, 1)$ forms up to the boundary of Ω then T globally preserves the real analyticity of $\bar\partial$-closed $(0, 1)$ forms up to the boundary of Ω.

The following total convexity was used by Range **[12]** for studying the Carathéodory metric and holomorphic mappings. Here we use it as the basic assumption for our local real analytic boundary regularity result.

DEFINITION. Let Ω be a convex domain in $\mathbb{C}^n$ and p a C^1 boundary point of Ω. We say that $\partial\Omega$ is *totally convex at* p *in the complex tangential directions* if there exists a neighborhood U of p in $\mathbb{C}^n$ such that

$$\overline{\Omega} \cap (H_\zeta(\partial\Omega) + \{\zeta\}) = \{\zeta\},$$

for any $\zeta \in \partial\Omega \cap U$, where $H_\zeta(\partial\Omega)$ is the complex tangent space of $\partial\Omega$ at ζ.

3. Real analytic boundary regularity of Cauchy-Fantappie transform. The following "integration by parts" lemma was proved in **[5]**:

LEMMA 1. *Let* Ω *be a bounded convex domain in* $\mathbb{C}^n$ *with* C^2 *boundary, and* p *be a boundary point of* Ω. *If there exists a neighborhood* U *of* p *in* $\mathbb{C}^n$ *such that* $\partial\delta/\partial\zeta_n \neq 0$ *in* U *and*

$$a_n(\zeta, z) \equiv \frac{\partial\Phi}{\partial\zeta_n}(\zeta, z) - \frac{\partial\delta}{\partial\zeta_n}(\zeta)\left(\frac{\partial\delta}{\partial\bar\zeta_n}(\zeta)\right)^{-1}\frac{\partial\Phi}{\partial\bar\zeta_n}(\zeta, z) \neq 0, \quad \text{in } U \times U,$$

then for any positive integer k *and any function* $h \in C^\infty(\overline{\Omega})$ *with support in* $U \cup \overline{\Omega}$ *we have*

$$\begin{aligned}\int_{\partial\Omega} \frac{h(\zeta)}{\Phi^k(\zeta, z)}\, d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge d\bar\zeta_1 \wedge \cdots \wedge d\bar\zeta_{n-1} \\ = \frac{(-1)^{k-1}}{(k-1)!}\int_{\partial\Omega} (L_n)^{k+1}(h)(\zeta)\Phi(\zeta, z)\log\Phi(\zeta, z)\, d\zeta_1 \\ \wedge \cdots \wedge d\zeta_n \wedge d\bar\zeta_1 \wedge \cdots \wedge d\bar\zeta_{n-1} \\ + \frac{(-1)^{n+k-1}}{(k-1)!}\int_{\partial\Omega} (L_n)^k(h)(\zeta)\, d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge d\bar\zeta_1 \wedge \cdots \wedge d\bar\zeta_{n-1},\end{aligned}$$

for any $z \in U \cap \Omega$, *where*

$$L_n = \left(\frac{\partial \delta}{\partial \overline{\zeta}_n}\right)^{-1} \left(\frac{\partial \delta}{\partial \overline{\zeta}_n}\right) \frac{\partial}{\partial \overline{\zeta}_n} - \frac{\partial}{\partial \zeta_n} + \left(\frac{\partial \delta}{\partial \overline{\zeta}_n}\right)^{-1} \left(\frac{\partial \delta}{\partial \overline{\zeta}_n}\right) \frac{\partial (a_n^{-1})}{\partial \overline{\zeta}_n} - \frac{\partial (a_n^{-1})}{\partial \zeta_n}$$

is a first order differential operator.

Notice that $H_\zeta(\partial\Omega) + \{\zeta\} = \{z \in \mathbb{C}^n : \Phi(\zeta, z) = 0\}$. By the continuity of the function $\Phi(\zeta, z)$, we have

LEMMA 2. *Let* Ω *be a bounded convex domain in* $\mathbb{C}^n$ *with* C^1 *boundary. If* $\partial\Omega$ *is totally convex at a boundary point* p *of* Ω *in the complex tangential directions, then there exists a neighborhood* V *of* p *in* $\mathbb{C}^n$ *such that for any neighborhood* $U \subset\subset V$ *of* p *in* $\mathbb{C}^n$ *there exists a constant* $m > 0$ *such that*

$$|\Phi(\zeta, z)| \geq m \quad \text{for any } \zeta \in \partial\Omega \cap (\mathbb{C}^n \setminus V) \text{ and any } z \in \overline{\Omega} \cap \overline{U}.$$

Since cut-off functions are not real analytic, when we localize our results we need the following result of Ehrenpreis (for a proof see [**15**]):

LEMMA 3. *Given* $U \subset\subset V \subset\subset \mathbb{C}^n$ *there exist constants* $C, M > 0$ *such that for any integer* k *we can find* $\psi_k \in C^\infty(\mathbb{C}^n)$ *such that* $\psi_k \equiv 1$ *in* U, $\operatorname{Supp} \psi_k \subset V$ *and*

$$\left| \frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \overline{z}^\beta} \psi_k(z) \right| \leq C M^{|\alpha|+|\beta|} k^{|\alpha|+|\beta|},$$

for any z *in* $\mathbb{C}^n$, *and any* $\alpha, \beta : |\alpha| + |\beta| \leq k$.

We now prove a local real analytic boundary regularity result about the Cauchy-Fantappie transform which may be of independent interest. A similar result was obtained by S.-C. Chen [**4**]. In [**4**] S.-C. Chen used an estimate of $\Phi(\zeta, z)$. Such an estimate can be obtained by using a fundamental theorem of Lojasiewicz about analytic varieties [**11**]. In our proof we do not use the fundamental theorem of Lojasiewicz.

PROPOSITION 4. *Suppose* Ω *is a bounded convex domain in* $\mathbb{C}^n$ *with* C^2 *boundary. If* p *is a real analytic boundary point of* Ω, *and* $\partial\Omega$

is totally convex at p *in the complex tangential directions, then the Cauchy-Fantappie transform* C

$$C(h)(z) = \int_{\partial\Omega} h(\zeta)\Omega(W(\zeta, z))$$

locally preserves the real analyticity up to the boundary point p.

Proof. Let h be any function that is in $C^1(\overline{\Omega})$ and real analytic at p. We show that $C(h)$ is real analytic at p. It suffices to show that for some neighborhood U of p in $\mathbb{C}^n$ there exist constants $C, M > 0$ such that

$$\left|\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \overline{z}^\beta} C(h)(z)\right| \leq CM^{|\alpha|+|\beta|}(|\alpha| + |\beta|)!,$$

for any z in $U \cup \Omega$, and any multi-indices α, β. (For our convenience, constants C and M may take different values at different occurrences.)

Since $\Phi(\zeta, z)$ is a holomorphic function of z in Ω. When $\beta > 0$ we have

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \overline{z}^\beta} C(h)(z) = 0,$$

for any $z \in \Omega$. Hence we only need to estimate

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \overline{z}^\beta} C(h)$$

with $\beta = 0$.

Let V_1 be a neighborhood of p in $\mathbb{C}^n$ such that h is real analytic on V_1. Since δ is a defining function of Ω, there exist a neighborhood $V_2 \subseteq V_1$ of p in $\mathbb{C}^n$ and some j such that $\partial\delta/\partial\zeta_j \neq 0$ in V_2 and

$$a_j(\zeta, z) = \frac{\partial\Phi}{\partial\zeta_j}(\zeta, z) - \frac{\partial\delta}{\partial\zeta_j}(\zeta)\left(\frac{\partial\delta}{\partial\overline{\zeta}_j}(\zeta)\right)^{-1}\frac{\partial\Phi}{\partial\overline{\zeta}_j}(\zeta, z) \neq 0,$$

for any ζ, $z \in V_2$. By Lemma 2, there exists a neighborhood $V \subset\subset V_2$ of p in $\mathbb{C}^n$ such that for any neighborhood $U \subset\subset V$ of p in $\mathbb{C}^n$ there exists a positive constant m such that

$$|\Phi(\zeta, z)| \geq m, \qquad \text{for any } \zeta \in \partial\Omega \cap (\mathbb{C}^n \setminus V) \text{ and any } z \in U \cap \Omega.$$

By Lemma 3, there exist constants C, $M > 0$ such that for any integer k we can find $\psi_k \in C^\infty(\mathbb{C}^n)$ such that $\psi_k \equiv 1$ in V, Supp $\psi_k \subset V_2$ and

$$\left|\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \overline{z}^\beta} \psi_k(z)\right| \leq CM^{|\alpha|+|\beta|}k^{|\alpha|+|\beta|},$$

for any z in $\mathbb{C}^n$, and any α, β: $|\alpha| + |\beta| \leq k$.

Let U be a neighborhood of p in $\mathbb{C}^n$ such that $U \subset\subset V$. We estimate

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} C(h)$$

on $U \cap \Omega$.

For any multi-index α, we have

$$C(h) = C(\psi_{|\alpha|+n+1} h) + C((1 - \psi_{|\alpha|+n+1})h).$$

We first estimate $C((1 - \psi_{|\alpha|+n+1})h)$:

$$\begin{aligned}
&\left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} C((1 - \psi_{|\alpha|+n+1})h)(z) \right| \\
&= \left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} \int_{\partial\Omega} (1 - \psi_{|\alpha|+n+1}(\zeta)) h(\zeta) \Omega(W(\zeta, z)) \right| \\
&= \left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} \int_{\partial\Omega} \frac{(1 - \psi_{|\alpha|+n+1}(\zeta)) h(\zeta)}{\Phi^n(\zeta, z)} \Omega\left(\sum_{j=1}^n \frac{\partial\delta}{\partial\zeta_j} d\zeta_j \right) \right| \\
&= \left| \int_{\partial\Omega} \frac{\partial^{|\alpha|}}{\partial z^\alpha} \left(\frac{(1 - \psi_{|\alpha|+n+1}(\zeta)) h(\zeta)}{\Phi^n(\zeta, z)} \right) \Omega\left(\sum_{j=1}^n \frac{\partial\delta}{\partial\zeta_j} d\zeta_j \right) \right| \\
&= \left| \int_{\partial\Omega} \frac{n(n+1)\cdots(n+|\alpha|-1) \left(\frac{\partial\delta}{\partial\zeta}\right)^\alpha (1 - \psi_{|\alpha|+n+1}(\zeta)) h(\zeta)}{\Phi^{n+|\alpha|}(\zeta, z)} \right. \\
&\qquad\qquad \left. \times \Omega\left(\sum_{j=1}^n \frac{\partial\delta}{\partial\zeta_j} d\zeta_j \right) \right|.
\end{aligned}$$

Since $1 - \psi_{|\alpha|+n+1} \equiv 0$ in V

$$\begin{aligned}
&= \left| \int_{\partial\Omega \cap (\mathbb{C}^n \setminus (V)} \frac{n(n+1)\cdots(n+|\alpha|-1) \left(\frac{\partial\delta}{\partial\zeta}\right)^\alpha (1 - \psi_{|\alpha|+n+1}(\zeta)) h(\zeta)}{\Phi^{n+|\alpha|}(\zeta, z)} \right. \\
&\qquad\qquad \left. \times \Omega\left(\sum_{j=1}^n \frac{\partial\delta}{\partial\zeta_j} d\zeta_j \right) \right| \\
&\le C M^{|\alpha|} |\alpha|!.
\end{aligned}$$

for any z in $U \cap \Omega$.

We now estimate $C(\psi_{|\alpha|+n+1}h)$: Without loss of generality, we assume $j = n$. Set

$$\Omega\left(\sum_{j=1}^{n}\frac{\partial\delta}{\partial\zeta_j}\,d\zeta_j\right) = R(\zeta)\,d\zeta_1\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge d\overline{\zeta}_{n-1}.$$

Then

$$\begin{aligned}&\frac{\partial^{|\alpha|}}{\partial z^\alpha}C(\psi_{|\alpha|+n+1}h)(z)\\ &= \int_{\partial\Omega}\frac{n(n+1)\cdots(n+|\alpha|-1)\left(\frac{\partial\delta}{\partial\zeta}\right)^\alpha\psi_{|\alpha|+n+1}hR}{\Phi^{n+|\alpha|}(\zeta,z)}\,d\zeta_1\\ &\qquad\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge d\overline{\zeta}_{n-1}.\end{aligned}$$

Notice that $\psi_{|\alpha|+n+1}$ is supported in $V\cap\overline{\Omega}$. By Lemma 1:

$$\begin{aligned}&= \frac{(-1)^{|\alpha|+n}}{(|\alpha|+n)!}\int_{\partial\Omega}(L_n)^{n+|\alpha|+1}\Bigg(n(n+1)\cdots(n+|\alpha|-1)\\ &\qquad\times\left(\frac{\partial\delta}{\partial\zeta}\right)^\alpha\psi_{|\alpha|+n+1}hR\Bigg)\Phi\log\Phi\\ &\qquad\times d\zeta_1\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge d\overline{\zeta}_{n-1}\\ &\quad+\frac{(-1)^{|\alpha|}}{(|\alpha|+n)!}\int_{\partial\Omega}(L_n)^{n+|\alpha|}\Bigg(n(n+1)\cdots(n+|\alpha|-1)\\ &\qquad\times\left(\frac{\partial\delta}{\partial\zeta}\right)^\alpha\psi_{|\alpha|+n+1}hR\Bigg)\\ &\qquad\times d\zeta_1\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge d\overline{\zeta}_{n-1},\end{aligned}$$

for any $z\in V\cap\Omega$.

Since h, δ and R are real analytic, and

$$|\Phi(\zeta,z)\log\Phi(\zeta,z)|\le M,$$

for any $\zeta\in V\cap\partial\Omega$ and any $z\in V\cap\Omega$, we have

$$\left|\frac{\partial^{|\alpha|}}{\partial z^\alpha}C(\psi_{|\alpha|+n+1}h)(z)\right|\le CM^{|\alpha|}|\alpha|!,$$

for any z in $V\cap\Omega$.

Hence

$$\left|\frac{\partial^{|\alpha|}}{\partial z^{\alpha}} C(h)(z)\right| \leq C M^{|\alpha|}|\alpha|,$$

for any z in $U \cap \Omega$.

This completes the proof.

4. Proof of the theorem. Let f be a $\bar{\partial}$-closed $(0, 1)$ form with coefficients in $C^1(\overline{\Omega})$ and real analytic at the boundary point p. We show that $T(f)$ is also real analytic at p.

Let U be a small ball with center p such that f is real analytic in U. By the Cauchy-Kovalevsky theorem, there exists a solution u_p in $C^{\omega}(U)$ of the equation $\bar{\partial} u_p = f$ in U.

Let $V \subset\subset U$ be a neighborhood of p in $\mathbb{C}^n$, and $\psi \in C^{\infty}(\mathbf{C}^n)$ such that $\psi \equiv 1$ in V and $\psi \equiv 0$ in $\mathbb{C}^n \setminus U$. Then

$$T(f) = T(\psi f) + T((1 - \psi) f).$$

Since the kernel of T is real analytic in the z variable when the variable ζ is restricted to the support of $(1 - \psi) f$, by Lemma 2, $T((1 - \psi) f)$ is real analytic at p.

Notice that

$$T(\psi f) = T(\psi \bar{\partial} u_p) = T(\bar{\partial}(\psi T u_p)) - T((\bar{\partial} \psi) u_p).$$

By a standard homotopy argument, we have

$$T(\bar{\partial}(\psi u_p)) = \psi u_p - C(\psi u_p).$$

Hence

$$T(\psi f) = \psi u_p - C(\psi u_p) - T((\bar{\partial} \psi) u_p).$$

Clearly, ψu_p is real analytic at p. By Proposition 4, $C(\psi u_p)$ is real analytic at p. Since the kernel of T is real analytic in the z variable when the variable ζ is restricted to the support of $(\bar{\partial} \psi) u_p$, by Lemma 2, $T((\bar{\partial} \psi) u_p)$ is real analytic at p. Hence $T(\psi f)$ is real analytic at p. Since both $T(\psi f)$ and $T((1 - \psi) f)$ are real analytic at p it follows that $T(f)$ is real analytic at p.

This completes the proof.

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University of Michigan
Ann Arbor, MI 48109