PACIFIC JOURNAL OF MATHEMATICS
Vol. 156, No. 2, 1992

SURFACES IN THE 3-DIMENSIONAL LORENTZ-MINKOWSKI SPACE SATISFYING $\Delta x = Ax + B$

LUIS J. ALÍAS, ANGEL FERRÁNDEZ AND PASCUAL LUCAS

In this paper we locally classify the surfaces M_s^2 in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ verifying the equation $\Delta x = Ax + B$, where A is an endomorphism of $\mathbb{L}^3$ and B is a constant vector.

We obtain that classification by proving that M_s^2 has constant mean curvature and in a second step we deduce M_s^2 is isoparametric.

0. Introduction. In [**FL90**] the last two authors obtain a classification of surfaces M_s^2 in the 3-dimensional Lorentz-Minkowski space satisfying the condition $\Delta H = \lambda H$, for a real constant λ, where H is the mean curvature vector field. That equation is nothing but a system of partial differential equations, so that the problems quoted in [**FL90**] can be framed in a more general situation: classify semi-Riemannian submanifolds by means of some characteristic differential equations. In this line, the technique of finite type submanifolds, created and developed by B. Y. Chen, has been shown as a fruitful tool to inquire into not only the intrinsic configuration of the submanifold, but also the extrinsic one, because the Laplacian of the isometric immersion is essentially the mean curvature vector field of the submanifold.

Following Chen's idea, Garay [**Gar88**] has obtained a characterization of connected, complete surfaces of revolution in $\mathbb{E}^3$ whose component functions in $\mathbb{E}^3$ are eigenfunctions of its Laplacian with possibly distinct eigenvalues. In a second step, in [**Gar90**], Garay found that the only Euclidean hypersurfaces whose coordinate functions are eigenfunctions for its Laplacian are open pieces of a minimal hypersurface, a hypersphere or a generalized circular cylinder.

More recently, in [**DPV90**], Dillen-Pas-Verstraelen pointed out that Garay's condition is not coordinate invariant as a circular cylinder in $\mathbb{E}^3$ shows. Then they study and classify the surfaces in $\mathbb{E}^3$ which satisfy $\Delta x = Ax + B$, where Δ is the Laplacian on the surface, x represents the isometric immersion in $\mathbb{E}^3$, $A \in \mathbb{E}^{3\times 3}$ and $B \in \mathbb{R}^3$.

It is well known that when the ambient space is the 3-dimensional

Lorentz-Minkowski space $\mathbb{L}^3$, then the surface M_s^2 can be endowed with a Riemannian metric (spacelike surface) or a Lorentzian metric (Lorentzian surface) and therefore, as we pointed out in **[FL90]**, a richer classification is hoped. So, the following geometric question seems to be coming up in a natural way:

> "*Which are the surfaces in* $\mathbb{L}^3$ *satisfying the condition* $\Delta x = Ax + B$, *where* A *is an endomorphism of* $\mathbb{L}^3$ *and* B *is a constant vector?*"

To solve this question we follow the same way of reasoning as in **[FL90]**, which is quite different than that used by Dillen-Pas-Verstraelen in **[DPV90]**. We would like to remark that our proof also works in the Riemannian case, so that the Theorem in **[DPV90]** can be obtained as a consequence of our main result.

1. Some examples. Let $f\colon \mathbb{L}^3 \to \mathbb{R}$ be a real function defined by

$$f(x, y, z) = -\delta_1 x^2 + y^2 + \delta_2 z^2,$$

where δ_1 and δ_2 belong to the set $\{0, 1\}$ and they do not vanish simultaneously. Taking $r > 0$ and $\varepsilon = \pm 1$, the set $f^{-1}(\varepsilon r^2)$ is a surface in $\mathbb{L}^3$ provided that $(\delta_1, \delta_2, \varepsilon) \neq (0, 1, -1)$.

A straightforward computation shows that the unit normal vector field is written as $N = (1/r)(\delta_1 x, y, \delta_2 z)$ and the principal curvatures are

$$\mu_1 = -\delta_1/r \quad \text{and} \quad \mu_2 = -\delta_2/r.$$

Then the mean curvature is given by

$$\alpha = (\varepsilon/2)(\mu_1 + \mu_2) = (-\varepsilon/2r)(\delta_1 + \delta_2)$$

and by using the well-known formula $\Delta x = -2H = -2\alpha N$ we obtain $\Delta x = Ax$, where

$$A = \frac{\varepsilon(\delta_1 + \delta_2)}{r^2}\begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta_2 \end{pmatrix}.$$

The adjoint table collects all the above possibilities.

TABLE 1

Equation	Surface	A
$y^2+z^2=r^2$	$\mathbb{L}\times S^1(r)$	$\begin{pmatrix}0&0&0\\0&1/r^2&0\\0&0&1/r^2\end{pmatrix}$
$-x^2+y^2=-r^2$	$H^1(r)\times\mathbb{R}$	$\begin{pmatrix}-1/r^2&0&0\\0&-1/r^2&0\\0&0&0\end{pmatrix}$
$-x^2+y^2=r^2$	$S_1^1(r)\times\mathbb{R}$	$\begin{pmatrix}1/r^2&0&0\\0&1/r^2&0\\0&0&0\end{pmatrix}$
$-x^2+y^2+z^2=-r^2$	$H^2(r)$	$\begin{pmatrix}-2/r^2&0&0\\0&-2/r^2&0\\0&0&-2/r^2\end{pmatrix}$
$-x^2+y^2+z^2=r^2$	$S_1^2(r)$	$\begin{pmatrix}2/r^2&0&0\\0&2/r^2&0\\0&0&2/r^2\end{pmatrix}$

2. Setup. Let M_s^2 be a surface in $\mathbb{L}^3$ with index $s=0,1$. Throughout this paper we will denote by σ, S, H, ∇ and $\overline{\nabla}$ the second fundamental form, the shape operator, the mean curvature vector field, the Levi-Civita connection on M_s^2 and the usual flat connection on $\mathbb{L}^3$, respectively. Let N be a unit vector field normal to M_s^2 and let α be the mean curvature with respect to N, i.e., $H=\alpha N$.

Let $x\colon M_s^2\to\mathbb{L}^3$ be an isometric immersion satisfying the equation

$$\Delta x = Ax + B, \tag{2.1}$$

where A is an endomorphism of $\mathbb{L}^3$ and B is a constant vector in $\mathbb{L}^3$. If we take a covariant derivative in (2.1) and use the well-known equation $\Delta x=-2H$, by applying the Weingarten formula we have

$$AX = 2\alpha SX - 2X(\alpha)N, \tag{2.2}$$

for any vector field X tangent to M_s^2. From here and the self-adjointness of S one easily gets

$$\langle AX, Y\rangle = \langle X, AY\rangle, \tag{2.3}$$

for any tangent vector fields X and Y.

The covariant derivative in (2.3) yields

$$\begin{aligned}\langle A\sigma(X,Z), Y\rangle - \langle A\sigma(Y,Z), X\rangle \\ = \langle\sigma(X,Z), AY\rangle - \langle\sigma(Y,Z), AX\rangle.\end{aligned} \tag{2.4}$$

Now, by applying the Laplacian on both sides of (2.1) and taking into account the formula for ΔH obtained in **[FL90]**, we have

$$AH = 2S(\nabla\alpha) + 2\varepsilon\alpha\nabla\alpha + \{\Delta\alpha + \varepsilon\alpha|S|^2\}N, \tag{2.5}$$

where $\nabla\alpha$ stands for the gradient of α and $\varepsilon = \langle N, N\rangle$.

As for the structure equations we would like to set the notation that will be used later on. Let $\{E_1, E_2, E_3\}$ be a local orthonormal frame and let $\{\omega^1, \omega^2, \omega^3\}$ and $\{\omega_i^j\}_{i,j}$ be the dual frame and the connection forms, respectively, given by

$$\omega^i(X) = \langle X, E_i\rangle, \quad \omega_i^j(X) = \langle\overline{\nabla}_X E_i, E_j\rangle.$$

Then we have

$$d\omega^i = -\sum_{j=1}^{3}\varepsilon_j\omega_j^i \wedge \omega^j, \quad d\omega_i^j = -\sum_{k=1}^{3}\varepsilon_k\omega_k^j \wedge \omega_i^k.$$

3. The characterization theorem. All exhibited examples in §1 have constant mean curvature. It seems reasonable to ask for surfaces in $\mathbb{L}^3$ satisfying (2.1) having non constant mean curvature. The answer is negative as the following proposition shows.

PROPOSITION 3.1. *Let $x: M_s^2 \to \mathbb{L}^3$ be an isometric immersion satisfying $\Delta x = Ax + B$. Then M_s^2 has constant mean curvature.*

Proof. Let us start with the open set $\mathcal{U} = \{p \in M_s^2 : \nabla\alpha^2(p) \neq 0\}$. We are going to show that $\mathcal{U}$ is empty. Otherwise, we have

$$\sigma(X, Y) = \varepsilon\frac{\langle SX, Y\rangle}{\alpha}H,$$

for any tangent vector fields on $\mathcal{U}$. Then from (2.5) we obtain

$$\langle A\sigma(X, Y), Z\rangle = 2\frac{\langle SX, Y\rangle}{\alpha}(\varepsilon SZ(\alpha) + \alpha Z(\alpha)). \tag{3.6}$$

Now, by applying (2.2), (2.4) and (3.6) we get

$$TX(\alpha)SY = TY(\alpha)SX, \tag{3.7}$$

where T is the self-adjoint operator given by $TX = 2\alpha X + \varepsilon SX$.

Case 1. $T(\nabla\alpha) \neq 0$ on $\mathcal{U}$. Then there exists a tangent vector field X such that $TX(\alpha) \neq 0$, which implies by using (3.7) that S has rank one on $\mathcal{U}$. Thus we can choose a local orthonormal frame $\{E_1, E_2, E_3\}$ with $SE_1 = 2\varepsilon\alpha E_1$, $SE_2 = 0$ and $E_3 = N$. From here and again from (3.7) we have $E_2(\alpha) = 0$. Let $\{\omega^1, \omega^2, \omega^3\}$ and $\{\omega_i^j\}_{i,j}$ be the dual frame and the connection forms, respectively. It is easy to see that

$$\omega_3^1 = -2\varepsilon\alpha\omega^1, \tag{3.8}$$

$$\omega_3^2 = 0, \tag{3.9}$$

$$d\alpha = \varepsilon_1 E_1(\alpha)\omega^1. \tag{3.10}$$

Taking exterior differentiation in (3.8) and using (3.10) and the structure equations we obtain $d\omega^1 = 0$ and therefore we locally have $\omega^1 = du$, for a certain function u. Now, from (3.10) we get $d\alpha \wedge du = 0$ and then α depends on u, $\alpha = \alpha(u)$, and therefore $E_1(\alpha) = \varepsilon_1\alpha'(u)$.

Taking into account (3.9) and $d\omega^1 = 0$ we deduce $\omega_2^1 = 0$. Then we have

$$\Delta\alpha = -\sum_i \varepsilon_i\{E_iE_i(\alpha) - \nabla_{E_i}E_i(\alpha)\} = -\varepsilon_1 E_1E_1(\alpha) = -\varepsilon_1\alpha''. \tag{3.11}$$

On the other hand, from (2.2), (2.5) and (3.11) the associated matrix to the endomorphism A with respect to $\{E_1, E_2, N\}$ is given by

$$\begin{pmatrix} 4\varepsilon\alpha^2 & 0 & 6\varepsilon\alpha' \\ 0 & 0 & 0 \\ -2\varepsilon_1\alpha' & 0 & -\varepsilon_1\frac{\alpha''}{\alpha} + 4\varepsilon\alpha^2 \end{pmatrix}.$$

By considering the invariant elements of A, we obtain the following differential equations:

$$\varepsilon_1\alpha'' = 8\varepsilon\alpha^3 - \lambda_1\alpha, \tag{3.12}$$

$$-4\varepsilon\varepsilon_1\alpha\alpha'' + 16\alpha^4 + 12\varepsilon\varepsilon_1(\alpha')^2 = \lambda_2, \tag{3.13}$$

where λ_1 and λ_2 are two real constants.

Let us take $\beta = (\alpha')^2$. Then $d\beta/d\alpha = 2\alpha''$ and from (3.12) we have

$$\beta = 4\varepsilon\varepsilon_1\alpha^4 - \lambda_1\varepsilon_1\alpha^2 + C, \tag{3.14}$$

where C is a constant.

Now, from (3.12) and (3.13) we get

$$12\beta = \lambda_2\varepsilon\varepsilon_1 + 16\varepsilon\varepsilon_1\alpha^4 - 4\lambda_1\varepsilon_1\alpha^2. \tag{3.15}$$

Finally, we deduce from (3.14) and (3.15) that α is locally constant on $\mathscr{U}$, which is a contradiction.

Case 2. There exists a point p in $\mathscr{U}$ such that $T(\nabla\alpha)(p) = 0$. Thus from (2.2) and (2.5) we have

$$\langle AH, X\rangle(p) = -2\varepsilon\alpha(p)X(\alpha)(p) = \langle H, AX\rangle(p),$$

which implies, jointly with (2.3), that A is a self-adjoint endomorphism in $\mathbb{L}^3$. Then the above equation remains valid everywhere on

$\mathscr{U}$ and therefore we get

$$S(\nabla\alpha) = -2\varepsilon\alpha\nabla\alpha. \tag{3.16}$$

Since $-2\varepsilon\alpha$ is an eigenvalue of S and $\operatorname{tr} S = 2\varepsilon\alpha$ then S is diagonalizable and we can choose a local orthonormal frame $\{E_1, E_2, E_3\}$ such that $E_3 = N$, $SE_1 = -2\varepsilon\alpha E_1$ with E_1 parallel to $\nabla\alpha$ and $SE_2 = 4\varepsilon\alpha E_2$. Let $\{\omega^1, \omega^2, \omega^3\}$ and $\{\omega_i^j\}_{i,j}$ be the dual frame and the connection forms, respectively. Then

$$\omega_3^1 = 2\varepsilon\alpha\omega^1, \tag{3.17}$$

$$\omega_3^2 = -4\varepsilon\alpha\omega^2, \tag{3.18}$$

$$d\alpha = \varepsilon_1 E_1(\alpha)\omega^1. \tag{3.19}$$

Taking again exterior differentiation in (3.17) and using the structure equations we have $d\omega^1 = 0$. Therefore one locally has $\omega^1 = du$, for some function u, and thus α depends on u, $\alpha = \alpha(u)$ and $E_1(\alpha) = \varepsilon_1\alpha'$.

By exterior differentiation in (3.18) and using again the structure equations we obtain

$$3\varepsilon_1\alpha\omega_2^1 = 2\alpha'\omega^2. \tag{3.20}$$

A straightforward computation from (3.20) leads to

$$3\alpha\alpha'' = 5(\alpha')^2 - 36\varepsilon\varepsilon_1\alpha^4. \tag{3.21}$$

If we put $\beta = (\alpha')^2$ then the last equation can be rewritten as

$$\frac{3}{2}\alpha\frac{d\beta}{d\alpha} = 5\beta - 36\varepsilon\varepsilon_1\alpha^4, \tag{3.22}$$

whose solution is given by

$$\beta = C\alpha^{10/3} - 36\varepsilon\varepsilon_1\alpha^4, \tag{3.23}$$

where C is a constant.

On the other hand, from the definition of $\Delta\alpha$, the fact that E_1 is parallel to $\nabla\alpha$ and (3.20) we obtain

$$\alpha\Delta\alpha = -\varepsilon_1\alpha\alpha'' + \frac{2\varepsilon_1}{3}(\alpha')^2. \tag{3.24}$$

Now, from (2.2) and (2.5) it is easy to get

$$\alpha\Delta\alpha = \lambda\alpha^2 - 24\varepsilon\alpha^4, \qquad \lambda = \operatorname{tr}(A), w$$

that jointly with (3.24) yields

$$3\alpha\alpha'' = 72\varepsilon\varepsilon_1\alpha^4 - 3\lambda\varepsilon_1\alpha^2 + 2(\alpha')^2. \tag{3.25}$$

Finally, a similar reasoning as in Case 1 by using now (3.21), (3.23) and (3.25) leads to α is locally constant on $\mathscr{U}$, which is again a contradiction with the definition of $\mathscr{U}$.

Anyway, we deduce $\mathscr{U}$ is empty and then M_s^2 has constant mean curvature. □

Now, we are ready to show the main theorem of this paper.

THEOREM 3.2. *Let $x\colon M_s^2 \to \mathbb{L}^3$ be an isometric immersion. Then $\Delta x = Ax + B$ if and only if one of the following statements holds true:*

(1) *M_s^2 has zero mean curvature everywhere.*

(2) *M_s^2 is an open piece of one of the following surfaces: $\mathbb{L} \times S^1(r)$, $H^1(r) \times \mathbb{R}$, $S_1^1(r) \times \mathbb{R}$, $H^2(r)$, $S_1^2(r)$.*

Proof. Let M_s^2 be a surface in $\mathbb{L}^3$ such that $\Delta x = Ax + B$. From Proposition 3.1 we know M_s^2 has constant mean curvature α. If $\alpha = 0$ there is nothing to prove. So, suppose $\alpha \neq 0$. Then from (2.2) and (2.5) we get

$$\begin{cases} AX = 2\alpha SX, \\ AN = \varepsilon|S|^2 N \end{cases} \tag{3.26}$$

and therefore

$$\operatorname{tr}(A) = 2\alpha \operatorname{tr}(S) + \varepsilon|S|^2 = 4\varepsilon\alpha^2 + \varepsilon|S|^2,$$

from which we deduce $|S|^2$ is constant and then M_s^2 is an isoparametric surface. If $s = 0$, M is an open piece of $H^2(r)$ or $H^1(r) \times \mathbb{R}$. When $s = 1$, it follows from **[Mag85]** that M is an open piece of one of the following surfaces: $S_1^2(r)$, $S_1^1(r) \times \mathbb{R}$, $\mathbb{L} \times S^1(r)$ and a B-scroll. However a straightforward calculation shows that the B-scroll does not satisfy the condition $\Delta x = Ax + B$. □

As we have pointed out in the Introduction, our proof also works when the ambient space is $\mathbb{E}^3$. Then the Theorem of Dillen-Pas-Verstraelen in **[DPV90]** can be viewed as a consequence of our Theorem:

COROLLARY 3.3. *Let $x\colon M^2 \to \mathbb{E}^3$ be an isometric immersion. Then M satisfies $\Delta x = Ax + B$ if and only if it is an open piece of a minimal surface, a sphere or a circular cylinder.*

References

[DPV90] F. Dillen, J. Pas and L. Verstraelen, *On surfaces of finite type in Euclidean 3-space*, Kodai Math. J., **13** (1990), 10-21.

[FL90] A. Ferrández and P. Lucas, *On surfaces in the 3- dimensional Lorentz-Minkowski space*, 1990. To appear in Pacific J. Math.

[Gar88] O. J. Garay, *On a certain class of finite type surfaces of revolution*, Kodai Math. J., **11** (1988), 25–31.

[Gar90] ——, *An extension of Takahashi's theorem*, Geometriae Dedicata, **34** (1990), 105–112.

[Mag85] M. A. Magid, *Lorentzian isoparametric hypersurfaces*, Pacific J. Math., **118** (1985), 165–197.

Received April 30, 1991. The first author was supported by an FPI Grant, DGICYT, 1990. The second and third authors were partially supported by a DGICYT Grant No. PS 87-0115-C03-03.

Departamento de Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Espinardo, Murcia, Spain