PACIFIC JOURNAL OF MATHEMATICS
Vol. 157, No. 1, 1993

A REMARK ON LERAY'S INEQUALITY

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In this paper we recapitulate briefly the significance of Leray's inequality in his proof of the existence of stationary solutions to the Navier-Stokes equations and show that in some simple cases it is equivalent to the flux condition on the boundary value.

1. Leray's inequality. The problem about whether or not there exist stationary solutions to the Navier-Stokes equations has been an open problem despite of a lot of efforts of many mathematicians. What has been so far obtained for this equation in this respect is an existence theorem due to Leray [2] under the condition which we call "flux condition" to be explained below.

Let D be a bounded domain with C^∞ boundary Γ in $\mathbf{R}^n$ $(n \geq 2)$. The stationary Navier-Stokes equation in D is expressed as

$$(1)\qquad \begin{cases} \Delta X - (X \cdot \nabla)X - \operatorname{grad} p = F & \text{in } D, \\ \operatorname{div} X = 0 & \text{in } D, \\ X = B & \text{on } \Gamma, \end{cases}$$

where $X = (X_1, \dots, X_n)$ is the velocity vector field, p the pressure, F the exterior force and B is the boundary condition. Δ is the Laplacian, $(\Delta X)_i = \Delta X_i$, and

$$((X \cdot \nabla)Y)_i = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j} Y_i .$$

The boundary condition B cannot be given arbitrarily. As a necessary condition of the solenoidalness condition $\operatorname{div} X = 0$ and the Gauss-Stokes formula, B should satisfy the following compatibility condition

$$(2)\qquad \int_\Gamma B \cdot \vec{n}\, dS = 0,$$

where $\vec{n}$ is the unit outer normal to the boundary Γ and dS is the surface element. The problem is whether equation (1) admits a solution (X, p) under the compatibility condition (2).

In his celebrated 1933 thesis, Leray succeeded in giving an affirmative answer to this problem under a condition which is stronger than (2) namely,

$$\int_{\Gamma_i} B \cdot \vec{n}\, dS = 0, \qquad i = 1, \ldots, N, \tag{3}$$

where Γ_i is the connected component of the boundary Γ and $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$. We shall call this condition (3) "flux condition." (Leray says that condition (2) which does not satisfy (3) is unphysical, and he did not go further to investigate what would happen if the flux condition (3) is not satisfied. In this respect, see Takeshita [**3**].)

The crucial point of Leray's arguments is the following inequality which is due essentially to him and we call Leray's inequality.

Leray's inequality. Let B be a C^∞ vector field defined on Γ satisfying flux condition (3). Then for any $\varepsilon > 0$, there exists a solenoidal extension B_ε of class C^∞ into domain D such that

$$|((X \cdot \nabla) B_\varepsilon, X)| \le \varepsilon \|\nabla X\|^2 \tag{4}$$

for any solenoidal C^∞ vector field X with compact support in D.

Here we have used the notations

$$(X, Y) = \sum_{i=1}^{n} \int_D X_i Y_i\, dx, \quad \|\nabla X\|^2 = \sum_{i,j=1}^{n} \int_D \left(\frac{\partial}{\partial x_i} X_j\right)^2 dx.$$

This inequality of Leray enables us to obtain an a priori bound for the possible solutions to the stationary Navier-Stokes equation in question and thereby to apply topological method (which is again due to Leray (and Schauder)) to prove existence theorem of stationary solutions. Thus we find that Leray's inequality is the most basic in his proof of existence of solutions.

2. Condition (L). The next problem to study after Leray is to prove or disprove the existence of solutions only under compatibility condition (2) on the boundary value without assuming flux condition (3).

Since in Leray's arguments what is needed to prove the existence of solutions is Leray's inequality and not the flux condition itself, and it seems that there would be some gap between the flux condition and Leray's inequality, one might quite well hope that one might be able to prove Leray's inequality even in the case in which the flux condition is dropped.

The aim of this short note is to study this problem.

First we make clear our problem. By $C_{0,\sigma}^{\infty}(D)$ we denote the totality of all the solenoidal C^{∞} vector fields with compact supports in D.

Problem. Let B be a C^{∞} vector field defined on Γ. What conditions should B satisfy in order that for any $\varepsilon > 0$, B admits a C^{∞} solenoidal extension B_{ε} into D such that

$$|((X\cdot\nabla)B_{\varepsilon}, X)| \le \varepsilon\|\nabla X\|^2$$

holds for all $X \in C_{0,\sigma}^{\infty}(D)$? When this holds, we shall say that B satisfies condition (L).

As for condition (L), we can prove the following

LEMMA 1. *Let B, B' be C^{∞} vector fields defined on Γ such that*

(i) $\int_{\Gamma} B\cdot\vec{n}\,dS = 0, \quad \int_{\Gamma} B'\cdot\vec{n}\,dS = 0,$

(ii) $\int_{\Gamma_i} B\cdot\vec{n}\,dS = \int_{\Gamma_i} B'\cdot\vec{n}\,dS, \qquad i = 1, \dots, N,$

(ii) *B satisfies the condition* (L).

Then B' also satisfies condition (L).

Proof. Let $\varepsilon > 0$ be given arbitrarily. Since B satisfies the condition (L), B admits a solenoidal C^{∞} extension B_{ε} such that

$$|((X\cdot\nabla)B_{\varepsilon}, X)| \le \frac{\varepsilon}{2}\|\nabla X\|^2 \tag{4}$$

for any $X \in C_{0,\sigma}^{\infty}(D)$. On the other hand, by (ii) we have

$$\int_{\Gamma_i} (B - B')\cdot\vec{n}\,dS = 0, \qquad i = 1, \dots, N.$$

From this we infer that there exists a $C^{\infty}(n-2)$-form φ on D such that $*d\varphi$ is a C^{∞} solenoidal extension of $B - B'$ into $\overline{D}$. Here $*$ is the Hodge star operation and d denotes the exterior derivation. Choosing an appropriate C^{∞} function ρ on D which is identically 1 near Γ and applying Leray's arguments, we can prove that

$$|((X\cdot\nabla)(*d(\rho\varphi)), X)| \le \frac{\varepsilon}{2}\|\nabla X\|^2 \tag{5}$$

holds for any $X \in C_{0,\sigma}^{\infty}(D)$. From (4) and (5) we immediately see that $B'_{\varepsilon} = B_{\varepsilon} - *d(\rho\varphi)$ is a desired extension of B'.

From this lemma we see that condition (L) does not depend on the boundary value B itself but only on the domain D and $\mu_i \equiv \int_{\Gamma_i} B\cdot\vec{n}\,dS$, $i = 1, \dots, N$. This observation leads us to the following

Definition 1. Let $\{\mu_1, \dots, \mu_N\}$ be a set of real numbers such that $\sum_{i=1}^N \mu_i = 0$. We say that a pair $\{D; \mu_1, \dots, \mu_N\}$ satisfies condition (L) if for any $\varepsilon > 0$ and for any C^∞ vector field B defined on Γ with $\int_{\Gamma_i} B \cdot \vec{n}\, dS = \mu_i$, $i = 1, \dots, N$, there exists a solenoidal C^∞ extension B_ε of B such that

$$|((X \cdot \nabla)B_\varepsilon, X)| \le \varepsilon \|\nabla X\|^2$$

for any $X \in C_{0,\sigma}^\infty(D)$.

Definition 2. Let $\{\mu_1, \dots, \mu_N\}$ be as above. We say that a pair $\{D; \mu_1, \dots, \mu_N\}$ satisfies condition (L) if for any $\varepsilon > 0$ there exists a C^∞ solenoidal vector field B_ε on $\overline{D}$ such that

$$\text{(i)} \quad \int_{\Gamma_i} B_\varepsilon \cdot \vec{n}\, dS = \mu_i, \qquad i = 1, \dots, N,$$

$$\text{(ii)} \quad |((X \cdot \nabla)B_\varepsilon, X)| \le \varepsilon \|\nabla X\|^2 \quad \text{for any } X \in C_{0,\sigma}^\infty(D).$$

For these two conditions the following is proved by Leray's arguments.

Lemma 2. *Definitions* 1 *and* 2 *are equivalent.*

Concerning condition (L), it would be of much interest to study the following

Conjecture. The necessary and sufficient condition for a pair $\{D; \mu_1, \dots, \mu_N\}$ to satisfy condition (L) is $\mu_i = 0$, $i = 1, \dots, N$.

3. Some examples. So far the author has not been able to give a complete answer to our conjecture. As a partial answer to this problem we give some affirmative simple examples.

Theorem 1. *Let* $D = \{x \in \mathbf{R}^n; R_1 < |x| < R_2\}$, $0 < R_1 < R_2$ *and* $\Gamma_i = \{x \in \mathbf{R}^n; |x| = R_i\}$, $i = 1, 2$. *In this case the necessary and sufficient condition for a pair* $\{D; -a, a\}$ *to satisfy condition* (L) *is* $a = 0$.

Proof. Sufficiency is trivial by Lemma 2. We give a proof for necessity. Let $\varepsilon > 0$ be given arbitrarily. By the assumption there exists a C^∞ solenoidal vector field B_ε on $\overline{D}$ such that

$$\int_{\Gamma_1} B_\varepsilon \cdot \vec{n}\, dS = -a, \quad \int_{\Gamma_2} B_\varepsilon \cdot \vec{n}\, dS = a, \tag{6}$$

$$|((X \cdot \nabla)B_\varepsilon, X)| \le \varepsilon \|\nabla X\|^2 \quad \text{for any } X \in C_{0,\sigma}^\infty(D). \tag{7}$$

We shall make use of the averaging method with respect to $G = \mathrm{SO}(n)$, the n-dimensional rotation group. For this purpose we define an action of G on B_ε and its mean $M(B_\varepsilon)$ by

$$(T_g B_\varepsilon)(x) = g B_\varepsilon(g^{-1}x), \quad x \in D, \quad g \in G, \quad M(B_\varepsilon) = \int_G T_g B_\varepsilon \, dg$$

where dg is the normalized Haar measure on G. Then by the isometry of the action of G and (7), $M(B_\varepsilon)$ is solenoidal and we get

$$\int_{\Gamma_1} M(B_\varepsilon) \cdot \vec{n} \, dS = -a, \quad \int_{\Gamma_2} M(B_\varepsilon) \cdot \vec{n} \, dS = a, \tag{8}$$

$$|((X \cdot \nabla) M(B_\varepsilon), X)| \le \varepsilon \|\nabla X\|^2 \quad \text{for any } X \in C^\infty_{0,\sigma}(D). \tag{9}$$

By virtue of averaging with respect to G, $M(B_\varepsilon)$ is G-invariant and consequently has a very simple form.

Before determining the form of $M(B_\varepsilon)$, we give a proof of inequality (9) since it appears rather nontrivial.

For vector fields X, Y on D we set at $x \in D$, $\langle X, Y\rangle(x) = \sum_{i=1}^n X_i(x) Y_i(x)$,

$$\langle \nabla X, \nabla Y\rangle(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} X_j(x) \frac{\partial}{\partial x_i} Y_j(x).$$

For vector fields X, Y, B and any $g \in G$ we have at any $x \in D$

$$\langle \nabla T_g X, \nabla T_g Y\rangle(x) = \langle \nabla X, \nabla Y\rangle(g^{-1}x). \tag{10}$$

$$\langle ((T_g X) \cdot \nabla) T_g B, T_g Y\rangle(x) = \langle (X \cdot \nabla) B, Y\rangle(g^{-1}x). \tag{11}$$

Now we prove inequality (9). For short we denote by $\int$ the integration $\int_G dg$. Then

(12)

$$((X \cdot \nabla) M(B_\varepsilon), X) = \left((X \cdot \nabla) \int T_g B_\varepsilon, X\right) = \int ((X \cdot \nabla) T_g B_\varepsilon, X).$$

Here we have used the Fubini theorem on the interchange of the order of integrations. On the other hand we have, for any fixed $g \in G$, by (11)

$$\begin{aligned} ((X \cdot \nabla) T_g B_\varepsilon, X) &= \int_D dx \langle (X \cdot \nabla) T_g B_\varepsilon, X\rangle(x) \\ &= \int_D dx \langle ((T_g^{-1} X) \cdot \nabla) B_\varepsilon, T_g^{-1} X\rangle(g^{-1}x) \\ &= \int_D dy \langle ((T_g^{-1} X) \cdot \nabla) B_\varepsilon, T_g^{-1} X\rangle(y) \\ &= (((T_g^{-1} X) \cdot \nabla) B_\varepsilon, T_g^{-1} X). \end{aligned} \tag{13}$$

Combining (12) and (13) we get

$$\begin{aligned} |((X \cdot \nabla)M(B_\varepsilon), X)| &= \left| \int (((T_g^{-1}X) \cdot \nabla)B_\varepsilon, T_g^{-1}X) \right| \\ &\leq \int |(((T_g^{-1}X) \cdot \nabla)B_\varepsilon, T_g^{-1}X)| \\ &\leq \int \varepsilon \|\nabla(T_g^{-1}X)\|^2 = \int \varepsilon \|\nabla X\|^2 = \varepsilon \|\nabla X\|^2 . \end{aligned}$$

Here we have used $\|\nabla T_g^{-1} Y\| = \|\nabla Y\|$ which is a direct consequence of (10). Thus we have proved inequality (9).

In what follows we discuss the case $n = 2$ and the case $n \geq 3$ separately.

First we discuss the case $n = 2$. In $\mathbf{R}^2$ we use the polar coordinate system (r, θ) and define vector fields e_r and e_θ to be ones with directions along r, θ respectively and with length 1. Then the SO(2)-invariant vector field $M(B_\varepsilon)$ is found to be of the form

$$M(B_\varepsilon) = b_r(r)e_r + b_\theta(r)e_\theta$$

with real-valued functions $b_r(r), b_\theta(r)$ depending only on r. The solenoidalness of $M(B_\varepsilon)$ is equivalent to

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(rb_r) + \frac{1}{r}\frac{\partial}{\partial \theta}b_\theta \leftrightarrow \frac{\partial}{\partial r}(rb_r) = 0 \leftrightarrow b_r = \frac{c}{r} .$$

Constant c is determined by (8) to be $c = a/2\pi$. Therefore, we see that $M(B_\varepsilon)$ has the form

$$M(B_\varepsilon) = \frac{a}{2\pi}\frac{1}{r}e_r + b_\theta(r)e_\theta .$$

Next we calculate the left-hand side of (9). First we calculate the deformation matrix $\mathscr{D}$ associated with $M(B_\varepsilon)$. It is given by

$$\mathscr{D} = \begin{bmatrix} -\frac{a}{\pi}\frac{1}{r^2} & b_\theta'(r) - \frac{1}{r}b_\theta(r) \\ b_\theta'(r) - \frac{1}{r}b_\theta(r) & \frac{a}{\pi}\frac{1}{r^2} \end{bmatrix} .$$

In inequality (9) we take such X which has the form $X = u(r)e_\theta$ where $u(r)$ is a non-zero smooth function with compact support in

(R_1, R_2). For such X, the inequality (9) gives

$$2|a| \int_{R_1}^{R_2} \frac{1}{r}(u(r))^2\, dr \le \varepsilon \|\nabla(u(r)e_\theta)\|^2$$

from which $a = 0$ follows immediately by the arbitrariness of ε.

Next we discuss the case $n \ge 3$. It goes almost the same way as the case $n = 2$ with a slight difference in that in the case $n \ge 3$ the action of $\mathrm{SO}(n)$ on spheres has isotropy subgroup $\mathrm{SO}(n-1)$ which is not trivial. Therefore, the $\mathrm{SO}(n)$-invariant vector field $M(B_\varepsilon)$ is found to have simpler form $M(B_\varepsilon) = b(r)e_r$. The solenoidalness of $M(B_\varepsilon)$ and (8) determine $b(r)$ to be $b(r) = \frac{a}{\gamma_n} r^{-n+1}$, where γ_n is the area of the unit $(n-1)$-sphere. The associated deformation tensor $\mathscr{D} = (\mathscr{D}_{ij})$ (with respect to the cartesian coordinate system) is calculated to be

$$\mathscr{D}_{ij} = \frac{a}{\gamma_n} r^{-n}(-nr^{-2}x_i x_j + \delta_{ij}).$$

This tensor has eigenvalues $\frac{a}{\gamma_n} r^{-n}$ along the spherical direction with multiplicity $n-1$ and a simple eigenvalue $\frac{a}{\gamma_n} r^{-n}(-n+1)$ along the radial direction. Taking an appropriate spherical test vector field $X \in C_{0,\sigma}^\infty(D)$ as in case $n = 2$, we can conclude that $a = 0$.

Generalizing slightly, we can give a little more general examples.

THEOREM 2. *Let D be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma = \bigcup_{i=1}^N \Gamma_i$, Γ_i being the connected component of Γ. Assume that for each $i = 1, \dots, N$ there exists a diffeomorphism φ_i of $S^{n-1} \times [0, 1]$ into $\overline{D}$ such that $\varphi_i(S^{n-1} \times \{0\}) = \Gamma_i$ and $\varphi_i(S^{n-1} \times \{1\})$ is a sphere contained in D. Then the necessary and sufficient condition for a pair $\{D; \mu_1, \dots, \mu_N\}$ to satisfy condition* (L) *is $\mu_i = 0$, $i = 1, \dots, N$.*

Conclusion. The examples given in this section are quite insufficient for a general answer to our conjecture but are sufficient to convince us that if we want to attack the problem of existence or nonexistence of stationary solutions to the Navier-Stokes equations in the case in which the flux condition is not satisfied, new ideas should be thought out.

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Received November 15, 1989.

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