PACIFIC JOURNAL OF MATHEMATICS
Vol. 157, No. 1, 1993

A STATE MODEL FOR THE MULTI-VARIABLE ALEXANDER POLYNOMIAL

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We construct a vertex type state model in Turaev's sense for the multi-variable (non-reduced) Alexander polynomial. Our model is a colored version of the 6-vertex free fermion model. To show the correspondence of our model and the multi-variable Alexander polynomial, we introduce colored braid groups and their Magnus representations. By using this model, a new set of axioms for the multi-variable Alexander polynomial is obtained.

1. Introduction. In [1], the Jones polynomial V in [9] and its higher spin versions are directly constructed from some solutions of Yang-Baxter equations. Let P be the HOMFLY polynomial in [5], [16] and F be the Kauffman polynomial in [12]. Then these invariants are both two-variable extensions of the Jones polynomial V. In [19], Turaev constructs P and F from vertex type state models. Turaev introduced an enhanced Yang-Baxter operator, from which we get an invariant of links. He constructed enhanced Yang-Baxter operators from the R-matrices in [7] and showed that the related invariants are specializations of P and F. But this family does not contain the Alexander polynomial, which is the most famous link invariant. Deguchi and Akutsu [4] propose enhanced Yang-Baxter operators associated with a family of link invariants, which includes Turaev's family corresponding to P and also includes the reduced Alexander polynomial. We construct an enhanced Yang-Baxter operator for the Conway potential function ∇. The potential function ∇ is a version of the non-reduced Alexander polynomial. As is shown in [6], ∇ of a link is defined uniquely as a Laurent polynomial in variables associated with the connected components of the link. Kauffman gives an interpretation of the multi-variable Alexander polynomial by using a state model in §6 of [11]. In his model, there is no corresponding model in statistical mechanics. On the other hand, as is shown in Remark 2.4, our model comes from a solution of the Yang-Baxter equation, which assures the solvability of a lattice model in statistical mechanics.

In §2, we introduce an enhanced colored Yang-Baxter operator. This operator was introduced by Turaev [19] for non-colored links. From an enhanced colored Yang-Baxter operator, we get an invariant of links with colored component. In Example 2.3, we give a colored Yang-Baxter operator. The main interest of this paper is to investigate this operator and related link invariants. This operator is a colored version of the solution in [**4**].

In §§3–5, we construct a link invariant from a colored enhanced Yang-Baxter operator by using Turaev's idea. We can apply Turaev's method in [**19**] for our operator to get a link invariant. But the resulting invariant is constantly equal to zero. To construct a non-trivial invariant, we introduce a notion of *redundant* enhanced colored Yang-Baxter operator. In §6, we prove that our invariant is equal to the Conway potential function. To show this fact, we need Magnus representation of a colored braid group. Our invariant and the Conway potential function are both related to the Magnus representation. They are linear combinations of traces of exterior product representations of the Magnus representation.

In §7, we give an "axiomatic determination" for the Conway potential function ∇. The Jones polynomial has a very simple, well-known axiomatic determination. It is determined by the skein relation. Turaev gave a set of axioms for ∇ in §4.2 of [**18**]. But the Doubling Axiom 4.2.6 in [**18**] is not a local relation. Local axioms for ∇ are discussed in [**6**] and [**15**]. But they did not succeed in getting a complete set of relations for links with more than 3 colors. Instead of Turaev's Doubling Axiom, a new local relation is added to the known relations. This relation is much more complicated in comparison with the other relations and a simpler local relation is still needed.

2. Enhanced color Yang-Baxter operator S_0**.** Let K be a field. We extend the contents of [**19**], §2 for enhanced colored Yang-Baxter operators. Let $d(1), d(2), \dots, d(c), \dots$ be non-negative integers and $V^{(1)}, V^{(2)}, \dots, V^{(c)}, \dots$ be $d(1), d(2), \dots, d(c), \dots$-dimensional K-vector spaces. Let $R^{(c_1, c_2)}\colon V^{(c_1)} \otimes V^{(c_2)} \to V^{(c_2)} \otimes V^{(c_1)}$ $(c_1, c_2 = 1, 2, \dots)$ be a (K-linear) isomorphism. The set of operators $\{R^{(c_1, c_2)}\}$ is called a *colored Yang-Baxter operator* (or, briefly, a CYB-operator) if it satisfies the equality

$$(2.1)\qquad (R^{(c_1, c_2)} \otimes \mathrm{id})(\mathrm{id} \otimes R^{(c_1, c_3)})(R^{(c_2, c_3)} \otimes \mathrm{id}) = (\mathrm{id} \otimes R^{(c_2, c_3)})(R^{(c_1, c_3)} \otimes \mathrm{id})(\mathrm{id} \otimes R^{(c_1, c_2)}).$$

This corresponds to the braid relation with colored strings.

For $f \in \mathrm{End}(V^{(c_1)} \otimes \cdots \otimes V^{(c_{n-1})} \otimes V^{(c_n)})$, we define an *operator trace* $\mathrm{Sp}_n^{(c_1,\dots,c_{n-1},c_n)}(f) \in \mathrm{End}(V^{(c_1)} \otimes \cdots \otimes V^{(c_{n-1})})$ by the following. Let $\{v_1^{(c)}, \dots, v_{d(c)}^{(c)}\}$ be a basis for $V^{(c)}$ for $c \in \{c_1, \dots, c_n\}$ and let $f_{i_1,\dots,i_{n-1},i_n}^{j_1,\dots,j_{n-1},j_n}$ denote the matrix element of f with respect to the above basis, i.e.

$$f(v_{i_1}^{(c_1)} \otimes \cdots \otimes v_{i_n}^{(c_n)}) = \sum_{\substack{1 \le j_1 \le d(c_1) \\ 1 \le j_n \le d(c_n)}} f_{i_1,\dots,i_{n-1},i_n}^{j_1,\dots,j_{n-1},j_n} v_{j_1}^{(c_1)} \otimes \cdots \otimes v_{j_n}^{(c_n)}.$$

For $1 \le i_1 \le d(c_1), \dots, 1 \le i_n \le d(c_n)$, we put

$$\begin{aligned}(2.2)\qquad & \mathrm{Sp}_n^{(c_1,\dots,c_{n-1},c_n)}(f)(v_{i_1}^{(c_1)} \otimes \cdots \otimes v_{i_{n-1}}^{(c_{n-1})}) \\ &= \sum_{\substack{1 \le j_1 \le d(c_1) \\ 1 \le j_{n-1} \le d(c_{n-1}) \\ 1 \le j \le d(c_n)}} f_{i_1,\dots,i_{n-1},j}^{j_1,\dots,j_{n-1},j} v_{j_1}^{(c_1)} \otimes \cdots \otimes v_{j_{n-1}}^{(c_{n-1})}.\end{aligned}$$

EXAMPLE 2.1. If $n = 1$, then the operator trace $\mathrm{Sp}_1^{(c)}: \mathrm{End}(V^{(c)}) \to K$ is the ordinary trace. Let $n=2$, $d(c)=2$ and $f \in \mathrm{End}(V^{(c)} \otimes V^{(c)})$; then we have

$$\mathrm{Sp}_2^{(c,c)}(f) = \begin{pmatrix} f_{11}^{11} + f_{12}^{12} & f_{11}^{21} + f_{12}^{22} \\ f_{21}^{11} + f_{22}^{12} & f_{21}^{21} + f_{22}^{22} \end{pmatrix}.$$

DEFINITION 2.2. Let S be a collection of a set of CYB-operators $\{R^{(c_1,c_2)}\}$ $(c_1, c_2 = 1, 2, \dots)$, K-homomorphisms $\mu^{(c)}: V^{(c)} \to V^{(c)}$ and non-zero elements $\alpha^{(c)}$ and $\beta^{(c)}$ in K $(c = 1, 2, \dots)$. Then S is called an *enhanced colored Yang-Baxter operator* (briefly, ECYB-operator) if the elements of S satisfying the following:

(1) $R^{(c_1,c_2)} \circ (\mu^{(c_1)} \otimes \mu^{(c_2)}) = (\mu^{(c_2)} \otimes \mu^{(c_1)}) \circ R^{(c_1,c_2)}$;

(2) $\mathrm{Sp}_2^{(c,c)}(R^{(c,c)} \circ (\mathrm{id} \otimes \mu^{(c)})) = \alpha^{(c)}\beta^{(c)}\mathrm{id}$; $\mathrm{Sp}_2^{(c,c)}((R^{(c,c)})^{-1} \circ (\mathrm{id} \otimes \mu^{(c)})) = (\alpha^{(c)})^{-1}\beta^{(c)}\mathrm{id}$.

The collection S is denoted by $S = (R^{(c_1,c_2)}; \mu^{(c)}, \alpha^{(c)}, \beta^{(c)})$.

EXAMPLE 2.3. Let $t_1, t_2, \dots$ be indeterminants and $K = \mathbf{C}(t_1, t_2, \dots)$ be the field of rational functions in $t_1, t_2, \dots$. Let $d(c) = 2$ for all positive integers c. Fix a basis $\{v_1^{(c)}, v_2^{(c)}\}$ for $V^{(c)}$.

Let

$$(2.3)\quad R^{(c_1,c_2)} = t_{c_1} E_{1,1}^{(c_1,c_2)} \otimes E_{1,1}^{(c_2,c_1)} + \frac{t_{c_1}^2 - 1}{t_{c_2}} E_{1,1}^{(c_1,c_2)} \otimes E_{2,2}^{(c_2,c_1)}$$
$$+ \frac{t_{c_1}}{t_{c_2}} E_{2,1}^{(c_1,c_2)} \otimes E_{1,2}^{(c_2,c_1)} + E_{1,2}^{(c_1,c_2)} \otimes E_{2,1}^{(c_2,c_1)}$$
$$- \frac{1}{t_{c_2}} E_{2,2}^{(c_1,c_2)} \otimes E_{2,2}^{(c_2,c_1)},$$

where the symbol $E_{i,k}^{(c_1,c_2)}$ denotes the homomorphism $V^{(c_1)} \to V^{(c_2)}$ which transforms $v_i^{(c_1)}$ to $v_k^{(c_2)}$ and transforms $v_j^{(c_1)}$, with $j \neq i$, into 0. The inverse of $R^{(c_1,c_2)}$ is given by

$$(2.4)\quad (R^{(c_1,c_2)})^{-1} = \frac{1}{t_{c_1}} E_{1,1}^{(c_2,c_1)} \otimes E_{1,1}^{(c_1,c_2)} + E_{2,1}^{(c_2,c_1)} \otimes E_{1,2}^{(c_1,c_2)}$$
$$+ \frac{t_{c_2}}{t_{c_1}} E_{1,2}^{(c_2,c_1)} \otimes E_{2,1}^{(c_1,c_2)}$$
$$- \frac{t_{c_1}^2 - 1}{t_{c_1}} E_{2,2}^{(c_2,c_1)} \otimes E_{1,1}^{(c_1,c_2)} - t_{c_2} E_{2,2}^{(c_2,c_1)} \otimes E_{2,2}^{(c_1,c_2)}.$$

Let $\mu^{(c)} = E_{1,1}^{(c,c)} - E_{2,2}^{(c,c)}$, $\alpha^{(c)} = 1$, $\beta^{(c)} = t_c^{-1}$ and $S_0 = (R^{(c_1,c_2)};\ \mu^{(c)}, \alpha^{(c)}, \beta^{(c)})$. A simple computation shows that

(1) the set of operators $\{R^{(c_1,c_2)}\}$ is a CYB-operator,

(2) S_0 is an ECYB-operator.

REMARK 2.4. Let $R^{(c_1,c_2)}$ be as above and $R^{(c_1,c_2)}(x) = R^{(c_1,c_2)}x - (R^{(c_2,c_1)})^{-1}x^{-1}$ for $x \in \mathbf{C}\backslash\{0\}$. Then $R^{(c_1,c_2)}(x)$ satisfies the Yang-Baxter equation with spectral parameters

$$(2.5)\quad (R^{(c_1,c_2)}(x) \otimes \mathrm{id})(\mathrm{id} \otimes R^{(c_1,c_3)}(xy))(R^{(c_2,c_3)}(y) \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes R^{(c_2,c_3)}(y))(R^{(c_1,c_3)}(xy) \otimes \mathrm{id})(\mathrm{id} \otimes R^{(c_1,c_2)}(x)).$$

This solution is a colored version of the free-fermion 6-vertex model (see, for example, **[17]**).

The main purpose of this paper is to investigate some properties of the ECYB-operator S_0 given in the above example.

3. Markov trace of colored links and colored braids.

DEFINITION 3.1 (colored links). A *colored link* is a pair of an oriented link and a mapping from the connected components of the link to $\mathbf{N}$.

Let B_n be the braid group on n-strings and let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the standard generators of B_n. Let $\mathfrak{S}_n$ be the symmetric group of degree n. Let $\theta: B_n \to \mathfrak{S}_n$ be the group homomorphism sending σ_i to the transposition $(i\, i+1) \in \mathfrak{S}_n$ for $1 \leq i \leq n-1$. Then B_n acts on $\{1, 2, \ldots, n\}$ by θ.

DEFINITION 3.2 (colored braids). A *colored braid* is $(b; c_1, c_2, \ldots, c_n)$ where $b \in B_n$ and $c_1, \ldots, c_n \in \mathbf{N}$ with $c_{b(i)} = c_i$ for $1 \leq i \leq n$.

We denote by $\hat{b}$ the link represented by the closure of b. Then the above condition for the colors $c_1, \ldots, c_n$ implies that the closure of b has a coloring coming from $c_1, \ldots, c_n$. The connected component of $\hat{b}$ containing the ith point at the top of b is colored by c_i. We denote by $(b; c_1, c_2, \ldots, c_n)^\frown$ the colored link represented by $\hat{b}$ with colors defined as above. We need Alexander's theorem and Markov's theorem (Theorem 2.1 and Theorem 2.3 in [2]) for colored links and colored braids.

THEOREM 3.3 (*Alexander's theorem for colored links*). *A colored link can be represented by the closure of a colored braid.*

Proof. For a colored link L, let b be a braid whose closure represents L as a non-colored link. For $i = 1, 2, \ldots,$ let C_i be the component of L such that the corresponding component of $\hat{b}$ contains the ith point at the top of b. Let c_i be the color of C_i. Then the closure $(b; c_1, c_2, \ldots, c_n)^\frown$ represents L. □

DEFINITION 3.4 (Markov equivalence). Let B be the set of colored braids and let $\sim$ be the equivalence relation generated by the following.

(1) Let $b_1, b_2 \in B_n$ and $(b_1 b_2; c_1, c_2, \ldots, c_n)$ be a colored braid. Then

$$(b_1 b_2; c_1, c_2, \ldots, c_n) \sim (b_2 b_1; c_{b_1(1)}, c_{b_1(2)}, \ldots, c_{b_1(n)}).$$

(2) For $b \in B_n$, let $(b; c_1, c_2, \ldots, c_n)$ be a colored braid. Then

$$(b; c_1, c_2, \ldots, c_n) \sim (b\sigma_n^{\pm 1}; c_1, c_2, \ldots, c_n, c_n).$$

An element of the set of the equivalence classes $B/\sim$ is called a *Markov class.*

THEOREM 3.5 (*Markov's theorem for colored links*). *The closures of two colored braids are equivalent as colored links if and only if the colored braids belong to the same Markov class.*

Proof. Every step of the proof of Theorem 2.3 in [2] is compatible with the coloring. □

Now, we define an invariant of colored links by using an ECYB-operator $S = (R^{(c_1, c_2)}, \mu^{(c)}, \alpha^{(c)}, \beta^{(c)})$.

DEFINITION 3.6 (colored braid group). Let

$$B_n^{(c_1, \dots, c_n)} = \{b \in B_n | (b; c_1, \dots, c_n) \text{ is a colored braid}\}.$$

In other words, $b \in B_n^{(c_1, \dots, c_n)}$ if $c_{b(i)} = c_i$ for $1 \le i \le n$. Then the set $B_n^{(c_1, \dots, c_n)}$ is a subgroup of B_n and is called the *colored braid group* with the colors $c_1, c_2, \dots, c_n$.

For $b_1 \in B_n^{(c_1, \dots, c_n)}$ and $b_2 \in B_n$, we have $b_2^{-1} b_1 b_2 \in B_n^{(c_{b_2^{-1}(1)}, \dots, c_{b_2^{-1}(n)})}$. For $b \in B_n^{(c_1, \dots, c_n)}$, we define an element $\rho_S^{(c_1, \dots, c_n)}(b)$ in $\mathrm{End}(V^{(c_1)} \otimes V^{(c_2)} \otimes \cdots \otimes V^{(c_n)})$ as follows. Let $b = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_r}$. Put $b^{(k)} = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{k-1}}$ and

$$R_k = \mathrm{id}^{\otimes(i_k - 1)} \otimes R^{(c_{b^{(k)}(i_k)}, c_{b^{(k)}(i_k+1)})} \otimes \mathrm{id}^{\otimes(n - i_k - 1)}. \tag{3.1}$$

Let $\rho_S^{(c_1, \dots, c_n)}(b) = R_1 R_2 \cdots R_r$. Then

$$\rho_S^{(c_1, \dots, c_n)}(b) \in \mathrm{End}(V^{(c_1)} \otimes V^{(c_2)} \otimes \cdots \otimes V^{(c_n)}).$$

Since $R^{(c_1, c_2)}$ satisfies the colored braid relation (2.2), the above definition of $\rho_S^{(c_1, \dots, c_n)}$ implies the following

PROPOSITION 3.7.

$$\rho_S^{(c_1, \dots, c_n)} : B_n^{(c_1, \dots, c_n)} \to \mathrm{End}(V^{(c_1)} \otimes V^{(c_2)} \otimes \cdots \otimes V^{(c_n)})$$

is a representation of the group $B_n^{(c_1, \dots, c_n)}$.

DEFINITION 3.8 (Markov trace). Let $\mathrm{Sp}_{i,j}^{(c_1, \dots, c_i)}$ denote the composition of operator traces of $\mathrm{Sp}_i^{(c_1, \dots, c_i)}$, $\mathrm{Sp}_{i-1}^{(c_1, \dots, c_{i-1})}$, $\dots$, $\mathrm{Sp}_{j+1}^{(c_1, \dots, c_{j+1})}$, i.e.

$$\mathrm{Sp}_{i,j}^{(c_1, \dots, c_i)} = \mathrm{Sp}_j^{(c_1, \dots, c_{j+1})} \cdots \mathrm{Sp}_{i-1}^{(c_1, \dots, c_{i-1})} \mathrm{Sp}_i^{(c_1, \dots, c_i)} \tag{3.2}$$

for $i \ge j > 0$ and put

$$\begin{aligned} T_S^{(c_1, \dots, c_n)}(b) &= \left(\prod_{c \in \{c_1, \dots, c_n\}} (\alpha^{(c)})^{-w^{(c)}(b)} \right) \left(\prod_{k=1}^{n} \beta^{(c_i)} \right)^{-1} \mathrm{Sp}_{n,0}^{(c_1, \dots, c_n)} \\ &\quad \cdot (\rho_S^{(c_1, \dots, c_n)}(b)(\mu^{(c_1)} \otimes \mu^{(c_2)} \otimes \cdots \otimes \mu^{(c_n)})), \end{aligned} \tag{3.3}$$

where $w^{(c)}(b)$ denotes the number of crossings of b such that the strings of the over path and the under path are both colored by c. Then $T_S^{(c_1, \dots, c_n)}$ is a function from the colored braid group $B_n^{(c_1, \dots, c_n)}$ to K. The function $T_S^{(c_1, \dots, c_n)}$ is called the *Markov trace* of S.

PROPOSITION 3.9. *The Markov trace* $T_S^{(c_1, \dots, c_n)}$ *of an ECYB-operator* S *satisifes the following.*

(1) *For* $b_1 \in B_n^{(c_1, \dots, c_n)}$ *and* $b_2 \in B_n$, *we have*

$$T_S^{(c_{b_2(1)}, \dots, c_{b_2(n)})}(b_2^{-1} b_1 b_2) = T_S^{(c_1, \dots, c_n)}(b_1).$$

(2) *For* $b \in B_n^{(c_1, \dots, c_n)}$, $b\sigma_n^{\pm 1} \in B_{n+1}^{(c_1, \dots, c_n, c_{n+1})}$ *with* $c_n = c_{n+1}$, *we have*

$$\begin{aligned} T_S^{(c_1, \dots, c_n, c_{n+1})}(b_1 \sigma_n) &= T_S^{(c_1, \dots, c_n, c_{n+1})}(b_1 \sigma_n^{-1}) \\ &= T_S^{(c_1, \dots, c_n)}(b_1) \qquad (c_n = c_{n+1}). \end{aligned}$$

Proof. The proof of this theorem is similar to that of Theorem 3.1.2 in **[19]** and so we omit it. □

With Alexander's theorem and Markov's theorem for colored links and colored braids (Theorem 3.3 and Theorem 3.5), the above proposition implies the following theorem.

THEOREM 3.10. *Let* S *be an ECYB-operator. Let* X_S: {*colored braid*} $\to K$ *be the mapping defined by* $X_S(b; c_1, \dots, c_n) = T_S^{(c_1, \dots, c_n)}(b)$. *Then* X_S *induces an isotopy invariant of colored oriented links.*

EXAMPLE 3.11. Let S_0 be the ECYB-operator in Example 2.3. Then T_{S_0} is an invariant of colored links. But this invariant is equal to 0 for all the colored oriented links because of Proposition 4.4 given later and $\mathrm{Trace}(\mu^{(c_1)}) = 0$ for $c_1 = 1, 2, \dots$. So we need a new technique to withdraw a non-trivial invariant from the ECYB-operator S_0.

4. Redundant ECYB-operator and modified Markov trace. To withdraw a non-trivial invariant from the ECYB-operator S_0, we focus on a special property of S_0.

Let $S = (R^{(c_1, c_2)}; \mu^{(c)}, \alpha^{(c)}, \beta^{(c)})$ be an ECYB-operator. Fix positive integers n and $c_1, \dots, c_n$. Let $A_{S,n}^{(c_1, \dots, c_n)}$ be the subalgebra of $\mathrm{End}(V^{(c_1)} \otimes \cdots \otimes V^{(c_n)})$ spanned by the image $\rho_S^{(c_1, \dots, c_n)}(B_n^{(c_1, \dots, c_n)})$. We regard $A_{S,1}^{(c_1)}$ as the one-dimensional subalgebra of $\mathrm{End}(V^{(c_1)})$ spanned by the identity element.

DEFINITION 4.1 (redundant ECYB-operator). The ECYB-operator S is called *redundant* if, for $x \in A_{S,n}^{(c_1,\dots,c_n)}$,

$$\mathrm{Sp}_n^{(c_1,\dots,c_{n-1},c_n)}(x(\mathrm{id}^{\otimes(n-1)} \otimes \mu^{(c_n)})) \in A_{S,n-1}^{(c_1,\dots,c_{n-1})} \tag{4.1}$$

for all $n > 1$, $c_1, \dots, c_n \in \mathbf{N}$. Let (R, μ, α, β) be an enhanced Yang-Baxter operator in the sense of §2.3 in **[19]**. We regard this as an ECYB-operator by putting $R^{(c_1,c_2)} = R$, $\mu^{(c)} = \mu$, $\alpha^{(c)} = \alpha$ and $\beta^{(c)} = \beta$. We call $(R; \mu, \alpha, \beta)$ *redundant* if the associated ECYB-operator is redundant.

EXAMPLES 4.2. (1) The enhanced Yang-Baxter operators associated with the Jones polynomial V and its two-variable extensions P, F in **[19]** are redundant.

(2) Let $S_0 = (R^{(c_1,c_2)}; \mu^{(c)}, \alpha^{(c)}, \beta^{(c)})$ be the ECYB-operator in Example 2.3. Fix a positive integer c_0 and let $S_0^{(c_0)} = (R^{(c_0,c_0)}; \mu^{(c_0)}, \alpha^{(c_0)}, \beta^{(c_0)})$. Then $S_0^{(c_0)}$ is a redundant enhanced Yang-Baxter operator and the associated algebra $A_{S_0,n}^{(c_0,c_0,\dots,c_0)}$ is a quotient of Iwahori's Hecke algebra. (See Proposition 5.1 and Lemma 6.11.)

DEFINITION 4.3 (modified Markov trace). Let $S = (R^{(c_1,c_2)}; \mu^{(c)}, \alpha^{(c)}, \beta^{(c)})$ be an ECYB-operator. With the notation in (3.2), put

$$\begin{aligned} T_{S,1}^{(c_1,\dots,c_n)}(b) &= \left(\prod_{c\in\{c_1,\dots,c_n\}} (\alpha^{(c)})^{-w^{(c)}(b)}\right)\left(\prod_{k=1}^{n} \beta^{(c_i)}\right)^{-1} \mathrm{Sp}_{n,1}^{(c_1,\dots,c_n)} \\ &\quad \cdot (\rho_S^{(c_1,\dots,c_n)}(b)(\mathrm{id} \otimes \mu^{(c_2)} \otimes \cdots \otimes \mu^{(c_n)})). \end{aligned} \tag{4.2}$$

Then $T_{S,1}^{(c_1,\dots,c_n)}(b) \in \mathrm{End}(V^{(c_1)})$.

The definition of redundant ECYB-operators implies the following.

PROPOSITION 4.4. *Let $(b; c_1, \dots, c_n)$ be a colored braid. If the ECYB-operator S is redundant, then $T_{S,1}^{(c_1,\dots,c_n)}(b) \in \mathrm{End}(V^{(c)})$ is a scalar matrix. Moreover,*

$$T_S^{(c_1,\dots,c_n)}(b) = \mathrm{Trace}(\mu^{(c_1)}) T_{S,1}^{(c_1,\dots,c_n)}(b). \tag{4.3}$$

DEFINITION 4.5 (modified Markov trace). For a redundant ECYB-operator S, the mapping $T_{S,1}^{(c_1,\dots,c_n)}$ sending $b \in B_n^{(c_1,\dots,c_n)}$ to the scalar $T_{S,1}^{(c_1,\dots,c_n)}(b) \in K$ is called the *modified Markov trace* of S.

THEOREM 4.6 (*invariant of non-colored links*). *Let* $S = (R^{(c_1,c_2)};\ \mu^{(c)}, \alpha^{(c)}, \beta^{(c)})$ *be a redundant ECYB-operator. Fix a positive integer* c_0 *and let* $S^{(c_0)} = (R^{(c_0,c_0)}; \mu^{(c_0)}, \alpha^{(c_0)}, \beta^{(c_0)})$. *For* $b \in B_n$, *we put* $X_{S,1}^{(c_0)}(\hat{b}) = T_{S,1}^{(c_0,c_0,\dots,c_0)}(b)$. *Then* $X_{S,1}^{(c_0)}$ *is an invariant of noncolored links and*

$$X_S^{(c_0)}(b) = \mathrm{Trace}(\mu^{(c_0)}) X_{S,1}^{(c_0)}(b). \tag{4.4}$$

Theorem 3.10 and (4.3) imply (4.4). The claim of the above theorem is that $X_{S,1}^{(c_0)}$ is still an invariant of links even in the case $\mathrm{Trace}(\mu^{(c_1)}) = 0$. The proof of this theorem is similar to that of Theorem 3.1.2 in **[19]** and we omit it.

EXAMPLE 4.7. Let S_0 be the ECYB-operator in Example 2.3 and fix a positive integer c_0. Then $X_{S_0,1}^{(c_0)}$ coincides with the reduced Alexander-Conway polynomial in variable t_{c_0}. For details, see **[4]**, **[13]** and **[14]**. In **[13]** and **[14]**, they use an argument about one-tangles instead of the redundancy of $S_0^{(c_0)}$.

5. The multi-variable Alexander-Conway potential function.

PROPOSITION 5.1. *Let* S_0 *be the ECYB-operator defined by Example* 2.3. *Then* S_0 *is redundant.*

The proof of this proposition is long and so is given in Appendix A. The next two theorems are the main results of this paper.

THEOREM 5.2. *Let* S_0 *be the ECYB-operator in Example* 2.3. *For a colored braid* $(b; c_1, \dots, c_n)$, *let*

$$\Delta_{S_0}(b; c_1, \dots, c_n) = (t_{c_1} - t_{c_1}^{-1})^{-1} T_{S_0,1}^{(c_1,\dots,c_n)}(b). \tag{5.1}$$

Then Δ_{S_0} *is an isotopy invariant of colored links.*

Proof. We show that Δ_{S_0} is invariant for all the elements of a Markov class of colored braids introduced in Definition 3.4. The defining condition (2) of ECYB-operator implies that

$$\Delta_{S_0}(b\sigma_n^{\pm 1}; c_1, c_2, \dots, c_n, c_n) = \Delta_{S_0}(b; c_1, c_2, \dots, c_n). \tag{5.2}$$

The defining condition (1) of ECYB-operator implies that

$$(5.3)\qquad \Delta_{S_0}(b; c_1, c_2, \dots, c_n) = \Delta_{S_0}(\sigma_k^{-1} b \sigma_k; c_{\sigma_k(1)}, \dots, c_{\sigma_k(n)})$$

for $k \geq 2$. We show that

$$(5.4)\qquad \Delta_{S_0}(b; c_1, c_2, \dots, c_n) = \Delta_{S_0}(\sigma_1^{-1} b \sigma_1; c_2, c_1, c_3, \dots, c_n).$$

Since

$$\begin{aligned}
&\mathrm{Sp}_{n,2}^{(c_2, c_1, c_3, \dots, c_n)}(\rho_S^{(c_2, c_1, c_3, \dots, c_n)}(\sigma_1^{-1} b \sigma_1)(\mathrm{id}^{\otimes 2} \otimes \mu^{(c_3)} \otimes \cdots \otimes \mu^{(c_n)})) \\
&\quad = (R^{(c_2, c_1)})^{-1} \mathrm{Sp}_{n,2}^{(c_1, c_2, c_3, \dots, c_n)} \\
&\qquad \cdot (\rho_S^{(c_1, c_2, c_3, \dots, c_n)}(b)(\mathrm{id}^{\otimes 2} \otimes \mu^{(c_3)} \otimes \cdots \otimes \mu^{(c_n)})) R^{(c_2, c_2)},
\end{aligned}$$

we have

$$\begin{aligned}
&\mathrm{Sp}_{n,1}^{(c_2, c_1, c_3, \dots, c_n)}(\rho_S^{(c_2, c_1, c_3, \dots, c_n)}(\sigma_1^{-1} b \sigma_1)(\mathrm{id} \otimes \mu^{(c_1)} \otimes \mu^{(c_3)} \otimes \cdots \otimes \mu^{(c_n)})) \\
&\quad = \mathrm{Sp}_2^{(c_2, c_1)}(R^{(c_2, c_1)})^{-1} \mathrm{Sp}_{n,2}^{(c_1, c_2, \dots, c_n)} \\
&\qquad \cdot (\rho_S^{(c_1, c_2, \dots, c_n)}(b)(\mathrm{id}^{\otimes 2} \otimes \mu^{(c_3)} \otimes \cdots \otimes \mu^{(c_n)})) R^{(c_2, c_1)}(\mathrm{id} \otimes \mu^{(c_1)}).
\end{aligned}$$

Because S is redundant,

$$\mathrm{Sp}_{n,2}(\rho_S^{(c_1, \dots, c_n)}(b)(\mathrm{id}^{\otimes 2} \otimes \mu^{(c_3)} \otimes \cdots \otimes \mu^{(c_n)})) \in A_2^{(c_1, c_2)}$$

and so there are $\alpha, \beta \in K$ such that

$$\begin{aligned}
&\mathrm{Sp}_{n,2}^{(c_1, c_2, \dots, c_n)}(\rho_S^{(c_1, c_2, \dots, c_n)}(b)(\mathrm{id}^{\otimes 2} \otimes \mu^{(c_3)} \otimes \cdots \otimes \mu^{(c_n)})) \\
&\quad = \alpha + \beta R^{(c_2, c_1)} R^{(c_1, c_2)}.
\end{aligned}$$

But actual computation shows that

$$\mathrm{Sp}_2^{(c_1, c_2)}(\mathrm{id} \otimes \mu^{(c_2)}) = 0,$$

$$\mathrm{Sp}_2^{(c_1, c_2)}(R^{(c_2, c_1)} R^{(c_1, c_2)}(\mathrm{id} \otimes \mu^{(c_2)})) = t_{c_1} - t_{c_1}^{-1}$$

and

$$\mathrm{Sp}_2^{(c_2, c_1)}((R^{(c_2, c_1)})^{-1} R^{(c_2, c_1)} R^{(c_1, c_2)} R^{(c_2, c_1)}(\mathrm{id} \otimes \mu^{(c_1)})) = t_{c_2} - t_{c_2}^{-1}.$$

Hence we have (5.4). □

THEOREM 5.3. *Let S_0 be the ECYB-operator in Example 2.3 and let Δ_{S_0} be the invariant of colored links in Theorem 5.2. Then Δ_{S_0} is equal to Conway's potential function, which is a version of the multi-variable Alexander polynomial.*

The next section is devoted to the proof of the above theorem.

6. Magnus representation of colored braid groups and the multivariable Alexander polynomial. To prove Theorem 5.3, we use the relation between the multivariable Alexander polynomial and the Magnus representation of the colored braid group $B_n^{(c_1,\dots,c_n)}$. In this section, we focus on this relation. In Chapter 3 of the book [2], the Magnus representations of braid groups and pure braid groups are discussed. We reformulate them for the colored braid group $B_n^{(c_1,\dots,c_n)}$.

Let F_n be a free group of rank n, with generators $\alpha_1, \dots, \alpha_n$. The braid group B_n acts on F_n by

$$(6.1)\qquad \begin{aligned}&\sigma_i\cdot\alpha_i=\alpha_i\alpha_{i+1}\alpha_i^{-1}, \quad \sigma_i\cdot\alpha_{i+1}=\alpha_i,\\ &\sigma_i\cdot\alpha_j=\alpha_j \quad \text{if } j\neq i,\, i+1.\end{aligned}$$

This induces an action of the colored braid group $B_n^{(c_1,\dots,c_n)}$ on F_n since $B_n^{(c_1,\dots,c_n)}$ is a subgroup of B_n.

DEFINITION 6.1 (Fox's free-differential calculus). Let KF_n denote the group rings of F_n over $\mathbf{C}$. For each $j = 1, \dots, n$ there is a linear mapping

$$\frac{\partial}{\partial\alpha_i}\colon KF_n \to KF_n$$

given by

$$(6.2)\qquad \frac{\partial}{\partial\alpha_i}(\alpha_{i_1}^{\varepsilon_1}\cdots\alpha_{i_r}^{\varepsilon_r})=\sum_{k=1}^{r}\varepsilon_k\delta_{i_k,j}\alpha_{i_1}^{\varepsilon_1}\cdots\alpha_{i_k}^{(\varepsilon_k-1)/2},$$

where $\varepsilon_k=\pm1$ and $\delta_{i_k,j}$ is the Kronecker δ, where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$.

Proposition 3.2 of [2] shows that the mapping $\partial/\partial\alpha_i$ is well-defined. This mapping is called Fox's free-differential calculus.

Let $B_n^{(c_1,\dots,c_n)}$ be the colored braid group. Let $s_{c_1}, \dots, s_{c_n}$ be indeterminates corresponding to $c_1, \dots, c_n$ and let

$$K=\mathbf{C}(s_{c_1}, \dots, s_{c_n})$$

be the field of rational functions in $s_{c_1}, \dots, s_{c_n}$ with coefficients in $\mathbf{C}$. Let $\pi^{(c_1,\dots,c_n)}$ be the $\mathbf{C}$-algebra homomorphism from F_n to K which sends $\alpha_i^{\pm1}$ to $s_{c_i}^{\pm1}$.

DEFINITION 6.2 (Magnus representation). For $b\in B_n^{(c_1,\dots,c_n)}$, let $\xi^{(c_1,\dots,c_n)}(b)$ be the $n\times n$ matrix defined by

$$\xi^{(c_1,\dots,c_n)}(b)_{ij}=\pi^{(c_1,\dots,c_n)}\left(\frac{\partial(b\cdot\alpha_i)}{\partial\alpha_j}\right)$$

with entries in K. This mapping is a group homomorphism according to the following theorem and is called the *Magnus representation* of $B_n^{(c_1, \dots, c_n)}$.

THEOREM 6.3. *The mapping* $\xi^{(c_1, \dots, c_n)}: b \to \xi^{(c_1, \dots, c_n)}(b)$ *defines a group homomorphism from* $B_n^{(c_1, \dots, c_n)}$ *into the multiplicative group of* $n \times n$ *matrices over* K.

Proof. For $b \in B_n^{(c_1, \dots, c_n)}$, we have $\pi^{(c_1, \dots, c_n)}(b \cdot \alpha_i) = s_{c_{\theta(b)(i)}} = s_{c_i} = \pi^{(c_1, \dots, c_n)}(\alpha_i)$ from the definition of $B_n^{(c_1, \dots, c_n)}$. This implies that $\pi^{(c_1, \dots, c_n)}(b \cdot \alpha) = \pi^{(c_1, \dots, c_n)}(\alpha)$ for $\alpha \in F_n$, which is the condition (3-18) in [2]. Therefore, we can apply Theorem 3.9 in [2] for our case and then we get Theorem 6.3. □

For later use, we need a set of generators of $B_n^{(c_1, \dots, c_n)}$ and their representation matrices.

PROPOSITION 6.4 (*generators*). *The colored braid group* $B_n^{(c_1, \dots, c_n)}$ *is generated by the following elements* σ_{ij} *of* B_n.

(6.3) $$\sigma_{ij} = \sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}\sigma_i^{\gamma_{ij}}\sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \qquad (1 \le i < j \le n),$$

where $\gamma_{ij} = 1$ *if* $c_i = c_j$ *and* $\gamma_{ij} = 2$ *if* $c_i \neq c_j$.

Proof. Let H be the group generated by σ_{ij} $(1 \le i < j \le n)$. Then H contains the pure braid group P_n. Let $\mathfrak{S}_n$ be the symmetric group of degree n and $\theta: B_n \to \mathfrak{S}_n$ be the group homomorphism introduced in Definition 3.1. Let $\mathfrak{S}_n^{(c_1, \dots, c_n)} = \theta(B_n^{(c_1, \dots, c_n)})$. Then

$$\mathfrak{S}_n^{(c_1, \dots, c_n)} = \{\tau \in \mathfrak{S}_n | c_{\tau(i)} = c_i \ (1 \le i \le n)\},$$

and $\theta(\sigma_{ij})$ $(1 \le i < j \le n)$ generate $\mathfrak{S}_n^{(c_1, \dots, c_n)}$. Hence $\theta(H) = \mathfrak{S}_n^{(c_1, \dots, c_n)}$. On the other hand, the kernel of θ coincides with P_n, which is a normal subgroup of $B_n^{(c_1, \dots, c_n)}$. Hence $B_n^{(c_1, \dots, c_n)}$ is generated by σ_{ij} $(1 \le i < j \le n)$ since $B_n^{(c_1, \dots, c_n)} = P_n H = H$. □

To get the representation matrix of the generators, we have to compute $\partial(b \cdot \alpha_i)/\partial \alpha_j$. Let $\alpha_{pq} = \alpha_p \alpha_{p+1} \cdots \alpha_q$ for $1 \le p \le q \le n$. The definition (6.1) of the action of B_n on F_n implies that

(6.4) $$\sigma_{pq} \cdot \alpha_i = \begin{cases} \alpha_i & (\text{if } i \neq p, q), \\ \alpha_{pq}\alpha_{p,q-1}^{-1} & (\text{if } i = p) \quad , \\ \alpha_{p+1,q-1}^{-1}\alpha_{p,q-1} & (\text{if } i = q), \end{cases} \qquad \text{if } c_p = c_q,$$

and

$$(6.5)\quad \sigma_{pq}\cdot\alpha_i = \begin{cases} \alpha_i & (\text{if } i \neq p, q), \\ \alpha_{pq}\alpha_{p+1,q-1}^{-1}\alpha_{p,q-1}\alpha_{p,q}^{-1} & (\text{if } i = p), \\ \alpha_{p+1,q-1}^{-1}\alpha_{pq}\alpha_{p,q-1}^{-1}\alpha_{p+1,q-1} & (\text{if } i = q), \end{cases} \qquad \text{if } c_p \neq c_q .$$

Therefore, the representation matrices are given as follows. Let $s_{pq} = s_{c_p}s_{c_{p+1}}\cdots s_{c_q}$ and $s'_{c_i} = 1 - s_{c_i}$. If $c_p = c_q$,

$$(6.6)\quad \xi^{(c_1,\dots,c_n)}(\sigma_{pq}) = \begin{pmatrix} 1 & & p-1 & p & p+1 & & q-1 & q & q+1 & & n \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s'_{c_q} & s_{pp}s'_{c_q} & \cdots & s_{p,q-2}s'_{c_q} & s_{p,q-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s_{p+1,q-1}^{-1} & s_{p+2,q-1}^{-1}s'_{c_p} & \cdots & s_{q-1,q-1}^{-1}s'_{c_p} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

If $c_p \neq c_q$,

$$(6.7)\quad \xi^{(c_1,\dots,c_n)}(\sigma_{pq}) = \begin{pmatrix} 1 & & p-1 & p & p+1 & & q-1 & q & q+1 & & n \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s'_{c_p} + s_{c_p}s_{c_q} & -s_{pp}s_{c_q}s'_{c_p} & \cdots & -s_{p,q-2}s_{c_q}s'_{c_p} & s_{p,q-1}s'_{c_p} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s_{p+1,q-1}^{-1}s'_{c_q} & -s_{p+2,q-1}^{-1}s'_{c_p}s'_{c_q} & \cdots & -s_{q-1,q-1}s'_{c_p}s'_{c_q} & s_{c_p} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

As Lemma 3.11.1 of [2], we have

LEMMA 6.5. *The Magnus representation $\xi^{(c_1,\dots,c_n)}$ of $B_n^{(c_1,\dots,c_n)}$ is reducible to an $(n-1)$-dimensional representation.*

We denote the image of $b \in B_n^{(c_1,\dots,c_n)}$ in this $(n-1)$-dimensional representation by $\zeta^{(c_1,\dots,c_n)}(b)$. The representation $\zeta^{(c_1,\dots,c_n)}$ is irreducible. But we do not use this fact. As (3-28) of [2], we have

PROPOSITION 6.6. *Let* $b \in B_n^{(c_1,\dots,c_n)}$, *and* $\Delta(\hat{b})$ *be the Alexander polynomial of the closure* $\hat{b}$. *Assume that* $c_i \neq c_j$ *for some* $i \neq j$; *then*

$$(6.8)\qquad (s_{c_1}s_{c_2}\cdots s_{c_n}-1)\Delta(\hat{b}) = \det(\zeta^{(c_1,\dots,c_n)}(b)-\mathrm{id}).$$

Let $\iota\colon B_n^{(c_1,\dots,c_n)} \to B_n^{(c_n,\dots,c_1)}$ be the group isomorphism defined by

$$\iota(\sigma_{ij}) = \sigma_{n-j+1}\sigma_{n-j+2}\cdots\sigma_{n-i-1}\sigma_{n-i}^{\gamma_{ij}}\sigma_{n-i-1}^{-1}\cdots\sigma_{n-j+2}^{-1}\sigma_{n-j+1}^{-1}.$$

Let $\phi = \xi^{(c_n,\dots,c_1)}\circ\iota$ and $\psi = \zeta\circ\iota$. Then (6.8) implies that

$$(6.9)\qquad (s_{c_1}s_{c_2}\cdots s_{c_n}-1)\Delta(\hat{b}) = \det(\psi^{(c_1,\dots,c_n)}(b)-\mathrm{id}).$$

For $b \in B_n^{(c_1,\dots,c_n)}$, let $w_i(b)$ denote the sum of the signatures of crossing points of b for which the undercrossing arc has color i. Note that $w_i(b)$ is equal to the sum of the signatures of crossing points of b for which the overcrossing arc has color i. In fact, $w_i(b)$ is equal to the sum of the linking number $\mathrm{lk}(L_i, \hat{b}\backslash L_i)$ and the writhe of the sublink L_i of $\hat{b}$ consist of the components colored by i. Then (2.4) of [**6**] shows that the Conway potential function ∇ is given by

$$(6.10)\qquad \nabla(\hat{b}) = \left.\frac{\det(\psi^{(c_1,\dots,c_n)}(b)-\mathrm{id})\displaystyle\prod_{i\in\{c_1,\dots,c_n\}} t_i^{w_i(b)}}{(t_{c_1}t_{c_2}\cdots t_{c_n} - t_{c_1}^{-1}t_{c_2}^{-1}\cdots t_{c_n}^{-1})}\right|_{s_1=t_1^{-2},\, s_2=t_2^{-2},\,\dots}$$

Let U be the representation space of $\psi^{(c_1,\dots,c_n)}$. Let $\psi_k^{(c_1,\dots,c_n)}$ be the representation of $B_n^{(c_1,\dots,c_n)}$ on the space of k-fold exterior product $\bigwedge^k U$ defined by

$$\begin{aligned}&\psi_k^{(c_1,\dots,c_n)}(b)(v_1\wedge\cdots\wedge v_n)\\ &\quad = \psi^{(c_1,\dots,c_n)}(b)(v_1)\wedge\cdots\wedge\psi^{(c_1,\dots,c_n)}(b)(v_k).\end{aligned}$$

Similarly let $\phi_k^{(c_1,\dots,c_n)}$ be the representation of $B_n^{(c_1,\dots,c_n)}$ defined by the k-fold exterior product of $\phi^{(c_1,\dots,c_n)}$. By taking the eigenvalues of $\psi^{(c_1,\dots,c_n)}(b)$ into account, we have

PROPOSITION 6.7. *For* $b \in B_n^{(c_1,\dots,c_n)}$, *we have*

$$(6.11)\qquad \det(\psi^{(c_1,\dots,c_n)}(b)-\mathrm{id}) = \sum_{k=0}^{n-1}(-1)^{n-k-1}\mathrm{Trace}(\psi_k^{(c_1,\dots,c_n)}(b)).$$

Let $V^{(c)}$ be the 2-dimensional vector space introduced in Example 2.3 and v_1^c, v_2^c its basis. Let $V^{(c_1,\dots,c_n)} = V^{(c_1)} \otimes \cdots \otimes V^{(c_n)}$ and $V_k^{(c_1,\dots,c_n)}$ be the subspace of $V^{(c_1,\dots,c_n)}$ spanned by the elements $v_{i_1}^{c_1} \otimes \cdots \otimes v_{i_n}^{c_n}$ with $\#\{j|i_j = 2\} = k$. Then $V_k^{(c_1,\dots,c_n)}$ is invariant under the action of $\rho_S^{(c_1,\dots,c_n)}(B_n^{(c_1,\dots,c_n)})$, where $\rho_S^{(c_1,\dots,c_n)}$ is the representation of $B_n^{(c_1,\dots,c_n)}$ introduced in Proposition 3.6. Let $\rho_{S,k}^{(c_1,\dots,c_n)}$ denote the representation of $B_n^{(c_1,\dots,c_n)}$ on $V_k^{(c_1,\dots,c_n)}$. Let $\rho'^{(c_1,\dots,c_n)}_{S,k}$ be the representation of $B_n^{(c_1,\dots,c_n)}$ defined by

$$\rho'^{(c_1,\dots,c_n)}_{S,k}(b) = \left(\prod_{i\in\{c_1,\dots,c_n\}} (t_i)^{(k-1)w_i(b)}\right) \rho_{S,k}^{(c_1,\dots,c_n)}(b).$$

Let $\bigwedge^k \rho_{S,1}^{(c_1,\dots,c_n)}$ denote the representation of $B_n^{(c_1,\dots,c_n)}$ obtained from the natural action to $\bigwedge^k V_1^{(n)}$ induced by $\rho_{S,1}^{(c_1,\dots,c_n)}$.

Lemma 6.8. *Two representations $\rho'^{(c_1,\dots,c_n)}_{S,k}$ and $\bigwedge^k \rho_{S,1}^{(c_1,\dots,c_n)}$ are equivalent.*

Proof. The linear isomorphism

$$R_i = \mathrm{id}^{\otimes(i-1)} \otimes R^{(c_i,c_{i+1})} \otimes \mathrm{id}^{\otimes(n-i-1)}\colon V^{(c_1,\dots,c_n)} \to V^{(c_1,\dots,c_{i+1},c_i,\dots,c_n)}$$

implies $\bigwedge^k R^{(c,d)}\colon \bigwedge^k(V_1^{(c_1,\dots,c_n)}) \to \bigwedge^k(V_1^{(c_1,\dots,c_{i+1},c_i,\dots,c_n)})$. Let $f_j = v_1^{c_1}\otimes\cdots\otimes v_1^{c_{j-1}}\otimes v_2^{c_j}\otimes v_1^{c_{j+1}}\otimes\cdots\otimes v_1^{c_n} \in V_1^{(c_1,\dots,c_i,c_{i+1},\dots,c_n)}$ and $g_j = v_1^{c_2}\otimes\cdots\otimes v_1^{c_{j-1}}\otimes v_2^{c_j}\otimes v_1^{c_{j+1}}\otimes\cdots\otimes v_1^{c_n} \in V_1^{(c_1,\dots,c_{i+1},c_i,\dots,c_n)}$. Then $\{f_1, \dots, f_n\}$ and $\{g_1, \dots, g_n\}$ are bases of $V_1^{(c_1,\dots,c_n)}$ and $V_1^{(c_1,\dots,c_{i+1},c_i,\dots,c_n)}$ respectively. The matrix $\bigwedge^k R_i$ with respect to the basis $\{f_{i_1} \wedge \cdots \wedge f_{i_k} | i_1 < i_2 < \cdots < i_k\}$ and $\{g_{i_1} \wedge \cdots \wedge g_{i_k} | i_1 < i_2 < \cdots < i_k\}$ is equal to the matrix $t_{c_i}^{k-1}\mathrm{id}^{\otimes(i-1)}R^{(c_i,c_{i+1})} \otimes \mathrm{id}^{\otimes(n-i-1)}\colon V_k^{(c_1,\dots,c_n)} \to V_k^{(c_1,\dots,c_{i+1},c_i,\dots,c_n)}$, where the bases $f_{i_1} \wedge \cdots \wedge f_{i_k}$ and $g_{i_1} \wedge \cdots \wedge g_{i_k}$ correspond to $v_{j_1}^{c_1} \otimes \cdots \otimes v_{j_n}^{c_n}$ with $j_p = 1$ if $j_p \notin \{i_1, \dots, i_k\}$ and $j_p = 2$ if $j_p \in \{i_1, \dots, i_k\}$. This implies the statement of the above lemma. □

Lemma 6.9. *Let $\phi'^{(c_1,\dots,c_n)}$ be the representation of $B_n^{(c_1,\dots,c_n)}$ defined by*

$$(6.12)\quad \phi'^{(c_1,\dots,c_n)}(b) = \phi^{(c_1,\dots,c_n)}(b) \prod_{i\in\{c_1,\dots,c_n\}} t_i^{w_i(b)}\Bigg|_{s_1=t_1^{-2},\,s_2=t_2^{-2},\,\dots}.$$

Then the representation $\rho_{S,1}^{(c_1,\dots,c_n)}(b)$ *is equivalent to the representation* $\phi'^{(c_1,\dots,c_n)}$.

Proof. This lemma is proved by comparing the representation matrices of generators of $B_n^{(c_1,\dots,c_n)}$. In fact, the matrices $\rho_{S,1}^{(c_1,\dots,c_n)}(\sigma_{ij})$ and $\phi'^{(c_1,\dots,c_n)}(\sigma_{ij})$ are intertwined by a diagonal matrix with diagonal elements $d_1 = 1$, $d_2 = t_{c_2}^2 t_{c_1}^{-1}$, $d_3 = t_{c_3}^2 t_{c_2}^{-1} d_2, \dots, d_n = t_{c_n}^2 t_{c_{n-1}}^{-1} d_{n-1}$. □

Combining above two lemmas, we know that the two representations $\rho_{S,k}'^{(c_1,\dots,c_n)}$ and $\bigwedge^k \phi'^{(c_1,\dots,c_n)}$ are equivalent. On the other hand, $\phi^{(c_1,\dots,c_n)} = \psi^{(c_1,\dots,c_n)} \oplus \psi_0$ where ψ_0 is the trivial representation of $B_n^{(c_1,\dots,c_n)}$ sending every element to 1. Hence we have

LEMMA 6.10. *Let* $\psi_k'^{(c_1,\dots,c_n)}$ *be the representation of* $B_n^{(c_1,\dots,c_n)}$ *defined by*

$$(6.13)\quad \psi_k'^{(c_1,\dots,c_n)}(b) = \psi_k^{(c_1,\dots,c_n)}(b) \prod_{i\in\{c_1,\dots,c_n\}} t_i^{w_i(b)}\Bigg|_{s_1=t_1^{-2},\,s_2=t_2^{-2},\,\dots}.$$

Then the representation $\rho_{S,k}^{(c_1,\dots,c_n)}(b)$ *is equivalent to the representation* $\psi_{k-1}'^{(c_1,\dots,c_n)} \oplus \psi_k'^{(c_1,\dots,c_n)}$.

Let $q \in \mathbf{C}\backslash\{0\}$ such that $q^k \neq 1$ for any integer k. Let $H_{n-1}(q)$ be Iwahori's Hecke algebra defined by

$$(6.14)\quad \begin{aligned} H_{n-1}(q) = \langle T_1, \dots, T_{n-1} | T_iT_j = T_jT_i (|i-j| \geq 2),\\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\ T_i^2 - (q-q^{-1})T_i - 1 = 0\rangle \end{aligned}$$

as a $\mathbf{C}$-algebra. Let I be the two-sided ideal of $H_{n-1}(q)$ generated by the elements $(T_i + q^{-1})(T_j + q^{-1})$ $(1 \leq j < i-1 \leq n-2)$.

LEMMA 6.11. *The algebra* $A_{S,n}^{(c_1,\dots,c_n)}$ *is isomorphic to* $(H_{n-1}(q)/I) \otimes K$ *as an abstract* K*-algebra.*

Proof. Lemmas 6.8–6.10 show that the algebra $A_{S,n}^{(c_1,\dots,c_n)}$ is isomorphic to $\bigoplus_{i=0}^{n-1} M_{{}_{n-1}C_i}(K)$ where $M_{{}_{n-1}C_i}(K)$ is the full-matrix algebra over K of size ${}_{n-1}C_i$ and ${}_{n-1}C_i = (n-1)!/i!(n-1-i)!$. By using the representation theory of $H_{n-1}(q)$, which is isomorphic to $\mathbf{C}\mathfrak{S}_n$, $(H_{n-1}(q)/I)$ is isomorphic to $\bigoplus_{i=0}^{n-1} M_{{}_{n-1}C_i}(\mathbf{C})$. Hence $(H_{n-1}(q)/I)\otimes K$ is isomorphic to $A_{S,n}^{(c_1,\dots,c_n)}$. □

Proof of Theorem 5.3. Since $\psi_k'^{(c_1,\dots,c_n)}$ is an irreducible representation, Lemma 6.10 implies that the invariant $T'_{S,1}$ is a linear combination of traces of representations $\psi_k'^{(c_1,\dots,c_n)}(b)$ $(0 \le k \le n-1)$. On the other hand, (6.10) and Proposition 6.7 imply that the Conway potential function is a linear combination of traces of representations $\psi_k'^{(c_1,\dots,c_n)}$ $(0 \le k \le n-1)$. Both invariants are equal to 0 for split links; we have

$$(6.15)\quad T'_{S,1}(\hat{1}) = T'_{S,1}((\sigma_1^2)^\frown) = \cdots = T'_{S,1}((\sigma_1^2\cdots\sigma_{n-2}^2)^\frown) = 0,$$
$$T'_{S,1}((\sigma_1^2\cdots\sigma_{n-1}^2)^\frown) = 1,$$
$$\Delta(\hat{1}) = \Delta((\sigma_1^2)^\frown) = \cdots = \Delta((\sigma_1^2\cdots\sigma_{n-2}^2)^\frown) = 0,$$
$$\Delta((\sigma_1^2\cdots\sigma_{n-1}^2)^\frown) = 1.$$

Hence the following proposition shows that $T'_{S,1}(\hat{b}) = \Delta(\hat{b})$. □

Let $\eta_n^{(c_1,\dots,c_n)}$ be a linear combination of traces of $\psi_k'^{(c_1,\dots,c_n)}$ with coefficients $\alpha_k \in K$, where $\psi_k'^{(c_1,\dots,c_n)}$ is the representation of $B_n^{(c_1,\dots,c_n)}$ introduced in Lemma 6.10;

$$\eta_n^{(c_1,\dots,c_n)}(b) = \sum_{k=0}^{n-1} \alpha_k \mathrm{Trace}\, \psi_k'^{(c_1,\dots,c_n)}(b) \quad \text{for } b \in B_n^{(c_1,\dots,c_n)}.$$

PROPOSITION 6.12. *The coefficients* α_k *are determined by the values of* $T_n^{(c_1,\dots,c_n)}(1)$, $T_n^{(c_1,\dots,c_n)}(\sigma_1^2), \dots, T_n^{(c_1,\dots,c_n)}(\sigma_1^2\sigma_2^2\cdots\sigma_{n-1}^2)$.

Proof. Let $H_{n-1}(q)$ be Iwahori's Hecke algebra defined by (6.14) and I the two-sided ideal of $H_{n-1}(q)$ generated by the elements $(T_i+q^{-1})(T_j+q^{-1})$ $(1 \le j < i-1 \le n-2)$. Then, for $x \in H_{n-1}(q)$, there are $b_0, \dots, b_{n-1} \in K$ and $g_1, \dots, g_{n-1} \in H_{n-1}(q)$ such that

$$(6.16)\qquad x \equiv b_0 + b_1 g_1^{-1} T_1^2 g_1 + b_2 g_2^{-1} T_1^2 T_2^2 g_2 + \cdots + b_{n-1} g_{n-1}^{-1} T_1^2 \cdots T_{n-1}^2 g_{n-1} \mod I.$$

Let $\tau\colon H_{n-1}(q)/I \to K$ be a linear function such that $\tau(xy) = \tau(yx)$. Then τ is a linear combination of the traces of irreducible representations of S_n corresponding to a hook type partition of n. A hook type partition is a partition of the form $(m, 1^k)$. Moreover, (6.16) implies that τ is determined by the values $\tau(1), \tau(T_1^2), \dots, \tau(T_1^2 \cdots T_{n-1}^2)$.

Let $\eta_n^{(t_{c_1}, \dots, t_{c_n})} = T_n^{(c_1, \dots, c_n)}$. Lemma 6.11 shows that $A_{s,n}^{(c_1, \dots, c_n)}$ is isomorphic to $(H_{n-1}(q)/I) \otimes K$ as an abstract K-algebra. Hence we can apply the above argument to $\eta_n^{(q, \dots, q)}$ and we know that $\eta_n^{(q, \dots, q)}$ is determined by the values at 1, $\sigma_1^2, \sigma_1^2\sigma_2^2, \dots, \sigma_1^2\sigma_2^2 \cdots \sigma_{n-1}^2$. This implies that $\eta_n^{(t_{c_1}, \dots, t_{c_n})}$ is determined by the values at 1, $\sigma_1^2, \sigma_1^2\sigma_2^2, \dots, \sigma_1^2\sigma_2^2 \cdots \sigma_{n_1}^2$ if $t_{c_1}, \dots, t_{c_n}$ are in a neighborhood of q, and so this statement is also true for generic $t_{c_1}, \dots, t_{c_n}$. □

7. Axioms for the Conway potential function. Hartley proposes axioms to determine the potential functions of bi-colored links in [**6**]. Nakanishi gives a complete set of axioms to determine the potential functions for colored links with up to 3 colors. In the following, we give axioms for the potential function of colored links. The potential function has the following characters.

(1) Let L_+, L_- and L_0 be three links which are identical except within a ball where they are shown as in Figure 1. Then the potential function ∇ satisfies

$$\nabla(L_+) - (t_c - t_c^{-1})\nabla(L_0) - \nabla(L_-) = 0.$$

(2) Let L_{++}, L_{--} and L_{00} be three links which are identical except within a ball where they are shown as in Figure 1. Then the potential function ∇ satisfies

$$\nabla(L_{++}) - (t_c t_d + t_c^{-1} t_d^{-1})\nabla(L_{00}) + \nabla(L_{--}) = 0.$$

(3) Let L_{2112}, L_{1221}, L_{1122}, L_{2211}, L_{11}, L_{22} and L_{000} be seven links which are identical except within a ball where they are shown as in Figure 1. Let

$$g_+(x) = x + x^{-1}, \quad g_-(x) = x - x^{-1}.$$

Then ∇ satisfies

$$\begin{aligned} &g_+(t_{c_1})g_-(t_{c_2})\nabla(L_{2112}) - g_-(t_{c_2})g_+(t_{c_3})\nabla(L_{1221}) \\ &\quad - g_-(t_{c_1}^{-1}t_{c_3})(\nabla(L_{1122}) + \nabla(L_{2211})) + g_-(t_{c_1}^{-1}t_{c_2}t_{c_3})g_+(t_{c_3})\nabla(L_{11}) \\ &\quad - g_+(t_{c_1})g_-(t_{c_1}t_{c_2}t_{c_3}^{-1})\nabla(L_{22}) - g_-(t_{c_1}^{-2}t_{c_3}^2)\nabla(L_{000}) = 0. \end{aligned}$$

(4) For a trivial knot L with color c, $\nabla(L) = 1/(t_c - t_c^{-1})$.

(5) Let L_5 and L_6 be four links which are identical except within a ball where they are shown as in Figure 1. Then ∇ satisfies $(t_c - t_c^{-1})\nabla(L_5) - \nabla(L_6) = 0$.

(6) For a split union L of a link and a trivial knot, $\nabla(L) = 0$.

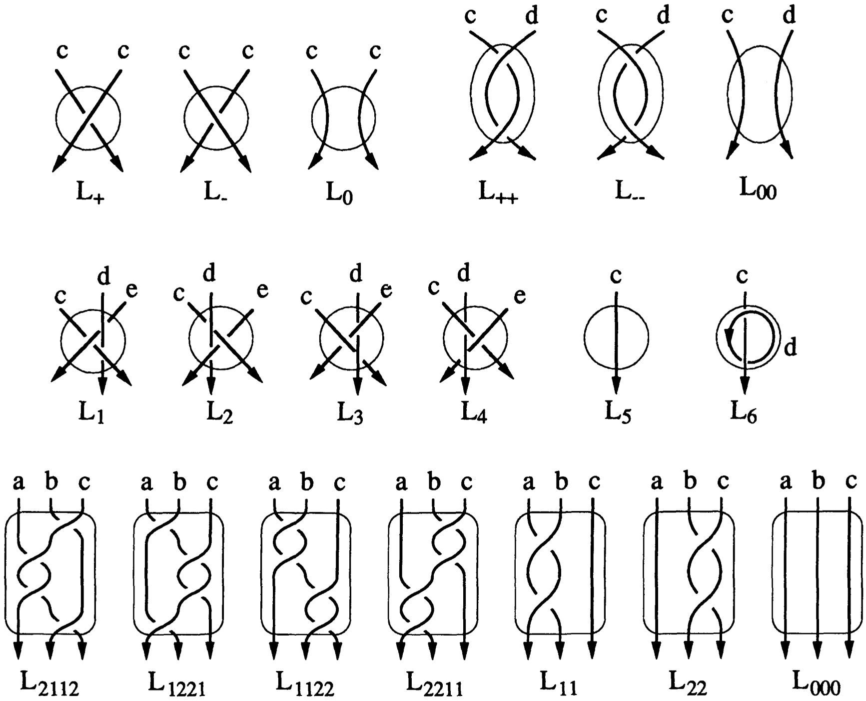

FIGURE 1

REMARK 7.1. (1) The 5th relation is a generalization of the relations (V) and (VII) in [15].

(2) The 3rd relation is not known before. But we can show this relation by a direct computation using the state model. This relation can be thought of as a generalization of (VIII) in [15]. We do not need the Doubling Axiom 4.2.6 in [18]. This is obtained by the following way. Let S_0 be the ECYB-operator in Example 2.3. Then the argument in §6 shows that the algebra $A_{S_0,3}^{(c_1,c_2,c_3)}$ is isomorphic to $H_2(q)$. Hence $A_{S_0,3}^{(c_1,c_2,c_3)}$ is 6-dimensional and so there must be a linear relation among seven elements $1, \sigma_1^2, \sigma_2^2, \sigma_1^2\sigma_2^2, \sigma_2^2, \sigma_1^2, \sigma_1\sigma_2^2\sigma_1, \sigma_2\sigma_1^2\sigma_2$. I actually computed this relation with MACSYMA by using the 8-dimensional representation of $A_{S_0,3}^{(c_1,c_2,c_3)}$ obtained by the ECYB-operator.

THEOREM 7.2. *The above relations* (1)–(6) *determine the potential function.*

REMARK 7.3. The first three relations are local relations. With these relations, we can reduce ∇ of a colored link to a linear combination of ∇'s of links which are split sums of trivial knots, Hopf links and connected sums of Hopf links. The last three relations determine ∇ of such reduced links.

Proof of Theorem 7.2. Let S be the ECYB-operator of Example 2.3. Then, with Theorem 5.3, we know that ∇ satisfies the relations (1)–(6) because a computation shows that $T'_{S,1}$ satisfies (1)–(6). So it remains to show that we can compute ∇ of any closed colored braid by using the relations (1)–(6).

Let θ be the group homomorphism from B_n to the symmetric group $\mathfrak{S}_n$ defined by $\theta(\sigma_i) = (i\, i{+}1)$. Then B_n acts on $\{1, 2, \dots, n\}$ by θ. Let $I_n^{(c_1, \dots, c_n)}$ be the two-sided ideal of $\mathbf{C}B_n^{(c_1, \dots, c_n)}$ generated by the elements $b^{-1}xb$, where $b \in B_n$ and x is one of the elements

$$\sigma_i - g_-(t_{c_{b(i)}}) - \sigma_i^{-1}, \qquad (\text{if } c_{b(i)} = c_{b(i+1)}),$$
$$\sigma_i^2 - g_+(t_{c_{b(i)}} t_{c_{b(i+1)}}) + \sigma_i^{-2} \qquad (\text{if } c_{b(i)} \neq c_{b(i+1)})$$

and

$$\begin{aligned} & g_+(t_{c_{b(i)}}) g_-(t_{c_{b(i+1)}}) \sigma_{i+1} \sigma_i^2 \sigma_{i+1} - g_-(t_{c_{b(i+1)}}) g_+(t_{c_{b(i+2)}}) \sigma_i \sigma_{i+1}^2 \sigma_i \\ & - g_-(t_{c_{b(i)}}^{-1} t_{c_{b(i+2)}})(\sigma_i^2 \sigma_{i+1}^2 + \sigma_{i+1}^2 \sigma_i^2) + g_-(t_{c_{b(i)}}^{-1} t_{c_{b(i+1)}} t_{c_{b(i+2)}}) g_+(t_{c_{b(i+2)}}) \sigma_i^2 \\ & - g_+(t_{c_{b(i)}}) g_-(t_{c_{b(i)}} t_{c_{b(i+1)}} t_{c_{b(i+2)}}^{-1}) \sigma_{i+1}^2 - g_-(t_{c_{b(i+1)}}^{-2} t_{c_{b(i+2)}}^2) \end{aligned}$$

of $\mathbf{C}B_n^{(c_{b(1)}, \dots, c_{b(n)})}$. Let $M_n^{(c_1, \dots, c_n)} = \mathbf{C}B_n^{(c_1, \dots, c_n)} / I_n^{(c_1, \dots, c_n)}$ and p_n the natural projection from $\mathbf{C}B_n^{(c_1, \dots, c_n)}$ to $M_n^{(c_1, \dots, c_n)}$ for $i = 1, 2, \dots, n-1$.

LEMMA 7.4. *The algebra* $M_3^{(c_1, c_2, c_3)}$ *is spanned by the images of* 1, σ_1^2, σ_2^2, $\sigma_1^2\sigma_2^2$, $\sigma_2^2\sigma_1^2$, $\sigma_1^2\sigma_2^2\sigma_1^2$ *as a* $\mathbf{C}$*-vector space.*

Proof of this lemma is given in Appendix B.

LEMMA 7.5. *The algebra* $M_n^{(c_1, \dots, c_n)}$ *is generated by* $p_n(\sigma_1^2)$, $p_n(\sigma_2^2)$, $\dots$, $p_n(\sigma_{n-1}^2) \in B_n^{(c_1, \dots, c_n)}$.

Proof. We claim that $(*)$ the mage of every generator $\sigma_{ij} \in B_n^{(c_1, \dots, c_n)}$ is written in terms of the images of $\sigma_1^2, \dots, \sigma_{n-1}^2$. This fact and Lemma 7.4 imply Lemma 7.5. To show $(*)$, we use the induction on n. If $n = 3$ then Lemma 7.4 implies $(*)$ and Lemma 7.5. For the $n = k > 3$ case, we assume that $(*)$ and Lemma 7.5 are proved for the case $n = k - 1$. Then the induction hypothesis implies that $p_k(\sigma_{ij})$ is written in terms of $p_k(\sigma_1^2), p_k(\sigma_2^2), \dots, p_k(\sigma_{k-2}^2)$ if $j < k$ or $i > 1$. It remains to show that $p_k(\sigma_{1k})$ is written in terms of $p_k(\sigma_1^2), p_k(\sigma_2^2), \dots, p_k(\sigma_{k-1}^2)$. Recall that

$$\sigma_{1k} = \sigma_{k-1} \cdots \sigma_2 \sigma_1^2 \sigma_2^{-1} \cdots \sigma_{k-1}^{-1}.$$

The middle part $\sigma_{k-2} \cdots \sigma_2 \sigma_1^2 \sigma_2^{-1} \cdots \sigma_{k-2}^{-1}$ can be considered as an element of $B_{k-1}^{(c_1, \dots, c_{k-2}, c_k)}$ and so we can apply the induction hypothesis to this part. Then Lemma 7.5 implies that

$$p_k(\sigma_{1k}) = \alpha p_k(y) + \beta p_k(z_1 \sigma_{k-1} \sigma_{k-2}^2 \sigma_{k-1}^{-1} z_2)$$

for some $\alpha, \beta \in \mathbf{C}$ and $y, z_1, z_2 \in \mathbf{C}B_{k-2}^{(c_1, \dots, c_{k-2})}$, where $\mathbf{C}B_{k-2}^{(c_1, \dots, c_{k-2})}$ is considered as a subalgebra of $\mathbf{C}B_k^{(c_1, \dots, c_k)}$. By Lemma 7.4, $p_k(\sigma_{k-1}\sigma_{k-2}^2\sigma_{k-1})$ is a linear combination of the images of 1, σ_{k-2}^2, σ_{k-1}^2, $\sigma_{k-2}^2\sigma_{k-1}^2$, $\sigma_{k-1}^2\sigma_{k-2}^2$ and $\sigma_{k-2}^2\sigma_{k-1}^2\sigma_{k-2}^2$. Hence we get Lemma 7.5 for the case n. □

Now prove Theorem 7.2 by an inducution on n. If $n = 1$, then the closure of a 1-braid is a trivial knot. Assume that $n \geq 2$. Note that the mapping ∇ from colored links to $\mathbf{C}$ can be considered as a mapping from $\mathbf{C}B_n^{(c_1, \dots, c_n)}$ to $\mathbf{C}$ by $\nabla(\alpha_1 b_1 + \cdots + \alpha_r b_r) = \alpha_1 \nabla(\hat{b}_1) + \cdots + \alpha_r \nabla(\hat{b}_r)$ for $\alpha_1, \dots, \alpha_r \in \mathbf{C}$ and $b_1, \dots, b_r \in B_n^{(c_1, \dots, c_n)}$. Since $\nabla(x) = 0$ for $x \in I_n^{(c_1, \dots, c_n)}$, ∇ is factored by $M_n^{(c_1, \dots, c_n)}$, we may consider ∇ as a linear mapping from $M_n^{(c_1, \dots, c_n)}$ to $\mathbf{C}$. Lemma 7.4 and Lemma 7.5 imply the following:

Lemma 7.6. *The algebra $M_n^{(c_1, \dots, c_n)}$ is a union of*

$$M_{n-1}^{(c_1, \dots, c_{n-1})} p_n(\sigma_{n-1}^2) M_{n-1}^{(c_1, \dots, c_{n-1})} \quad \textit{and} \quad M_{n-1}^{(c_1, \dots, c_{n-1})}.$$

This lemma implies that, for every $x \in \mathbf{C}B_n^{(c_1, \dots, c_n)}$, there are $\alpha, \beta \in \mathbf{C}$ and $y, z_1, z_2 \in M_{n-1}^{(c_1, \dots, c_{n-1})}$ such that $p_n(x) = \alpha y + \beta z_1 p_n(\sigma_{n-1}^2) z_2$. Hence $\nabla(x) = \alpha \nabla(y) + \beta \nabla(z_1 p_n(\sigma_{n-1}^2) z_2)$. But, by using the relation (5), we have $\nabla(x) = \alpha \nabla(y) + (t_{c_{n-1}} - t_{c_{n-1}}^{-1}) \beta \nabla(z_1 z_2)$. Hence the computation of ∇ for elements of $M_n^{(c_1, \dots, c_n)}$ is reduced to that of $M_{n-1}^{(c_1, \dots, c_{n-1})}$. This completes the proof of the theorem. □

Acknowledgment. I want to express my thanks to Y. Akutsu, who informed me of the state model in [**4**]. I should like to express my appreciation to K. Murasugi. To solve his problem, I got the notion of a redundant Yang-Baxter operator. I would like to thank Y. Nakanishi who gave me a lot of information about the Conway potential function, including the results of [**15**]. I greatly appreciate Symbolics MACSYMA on MacIvory II. It might be very hard to find the relation (7) in §7 without MACSYMA.

Appendix A. Proof of Proposition 4.8. To prove Proposition 4.8, we need the following two lemmas.

LEMMA A.1. *Let* $r^{(c_1,c_2)} = R^{(c_2,c_1)}R^{(c_1,c_2)} \in \mathrm{End}(V^{(c_1,c_2)})$. *Let* r_1, r_2 *be elements of* $\mathrm{End}(V^{(c_1,c_2,c_3)})$ *defined by* $r_1 = r^{(c_1,c_2)} \otimes \mathrm{id}$, $r_2 = \mathrm{id} \otimes r^{(c_2,c_3)}$. *Then* $\{1, r_1, r_2, r_1r_2, r_2r_1, r_1r_2r_1\}$ *is a basis of* $A_{S,3}^{(c_1,c_2,c_3)}$.

Proof. Let $A'^{(c_1,c_2,c_3)}_{S,3}$ is the subalgebra of $A_{S,3}(c_1, c_2, c_3)$ generated by 1, r_1 and r_2. Let

$$g_+(x) = x + x^{-1}, \qquad g_-(x) = x - x^{-1}.$$

From the definition of $R^{(c_1,c_2)}$, we have

$$r^{(c_1,c_2)} + (r^{(c_1,c_2)})^{-1} = g_+(t_{c_1}t_{c_2}), \tag{A.1}$$

$$\begin{aligned} g_-(t_{c_1}t_{c_2})r_2r_1r_2 - g_-(t_{c_2}t_{c_3})r_1r_2r_1 &= g_-(t_{c_1}/t_{c_3})(r_1r_2 + r_2r_1) \\ &+ (-g_-(t_{c_1}t_{c_2}t_{c_3}^2) + g_-(t_{c_2}t_{c_3}^2/t_{c_1}) + g_-(t_{c_1}/t_{c_2}))r_1 \\ &- (-g_-(t_{c_1}^2t_{c_2}t_{c_3}) + g_-(t_{c_1}^2t_{c_2}/t_{c_3}) + g_-(t_{c_3}/t_{c_2}))r_2 \\ &+ g_-(t_{c_1})g_-(t_{c_3})g_-(t_{c_1}^{-1}t_{c_3}). \end{aligned} \tag{A.2}$$

From these two relations, we know that the algebra $A'^{(c_1,c_2,c_3)}_{S,3}$ is spanned by $\{1, r_1, r_2, r_1r_2, r_2r_1, r_1r_2r_1\}$ as a linear space. Actual computation shows that $\{1, r_1, r_2, r_1r_2, r_2r_1, r_1r_2r_1\}$ is linearly independent. Hence $\{1, r_1, r_2, r_1r_2, r_2r_1, r_1r_2r_1\}$ is a basis of $A'^{(c_1,c_2,c_3)}_{S,3}$. In the following, we show that $A_{S,3} = A'^{(c_1,c_2,c_3)}_{S,3}$ by showing that the generators of $A_{S,3}$ are written in terms of r_1 and r_2.

Case 1. First, we treat the case $c_1 = c_2 = c_3 = c$. In this case, $B_3^{(c,c,c)}$ is generated by σ_1 and σ_2. Hence $A_{S,3}^{(c,c,c)}$ is generated by

$R^{(c,c)} \otimes \mathrm{id}$ and $\mathrm{id} \otimes R^{(c,c)}$. But we have $R^{(c,c)} = g_-(t_c)^{-1}((R^{(c,c)})^2 - 1)$ and so we have $A_{S,3}^{(c,c,c)} = A_{S,3}'^{(c,c,c)}$.

Case 2. Assume that $c_1 = c_2 \neq c_3$. In this case, $B_3^{(c_1,c_1,c_3)}$ is generated by σ_1 and σ_2^2. Hence $A_{S,3}^{(c_1,c_1,c_3)}$ is generated by $R^{(c_1,c_1)} \otimes \mathrm{id}$ and r_2. But we have $R^{(c,c)} = ((R^{(c,c)})^2 - 1)/(q_c - q_c^{-1})$ and so we have $A_{S,3}^{(c_1,c_1,c_3)} = A_{S,3}'^{(c_1,c_1,c_3)}$.

Case 3. Assume that $c_1 \neq c_2 = c_3$. In this case, $B_3^{(c_1,c_1,c_3)}$ is generated by σ_1^2 and σ_2. Hence, as in Case 2, we get $A_{S,3}^{(c_1,c_2,c_2)} = A_{S,3}'^{(c_1,c_2,c_2)}$.

Case 4. Assume that $c_1 = c_3 \neq c_2$. In this case, $B_3^{(c_1,c_2,c_1)}$ is generated by σ_1^2, σ_2^2 and $\sigma_1^{-1}\sigma_2\sigma_1$. Hence $A_{S,3}^{(c_1,c_2,c_1)}$ is generated by r_1, r_2 and $((R^{(c_1,c_2)})^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes R^{(c_1,c_1)})(R^{(c_1,c_2)} \otimes \mathrm{id})$. But a computation shows that

$$
\begin{aligned}
\text{(A.3)} \qquad & ((R^{(c_1,c_2)})^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes R^{(c_1,c_1)})(R^{(c_1,c_2)} \otimes \mathrm{id}) \\
&= \frac{g_-(t_{c_2}^2)}{g_-(t_{c_1}t_{c_2})} + \frac{1}{g_-(t_{c_2})} r_1 \\
&+ \frac{g_-(t_{c_1}^2 t_{c_2})}{g_-(t_{c_1})g_-(t_{c_2})g_-(t_{c_1}t_{c_2})} r_2 \\
&- \frac{g_+(t_{c_1})}{g_-(t_{c_1})g_-(t_{c_2})g_-(t_{c_1}t_{c_2})} r_1 r_2 \\
&- \frac{g_-(t_{c_1}t_{c_2}^2)}{g_-(t_{c_1})g_-(t_{c_2})g_-(t_{c_1}t_{c_2})} r_2 r_1 \\
&+ \frac{g_+(t_{c_2})}{g_-(t_{c_1})g_-(t_{c_2})g_-(t_{c_1}t_{c_2})} r_1 r_2 r_1 .
\end{aligned}
$$

So we have $A_{S,3}^{(c_1,c_2,c_1)} = A_{S,3}'^{(c_1,c_2,c_1)}$.

Case 5. Assume that $c_1 \neq c_2$, $c_2 \neq c_3$ and $c_1 \neq c_3$. In this case, $B_3^{(c_1,c_2,c_3)}$ is generated by σ_1^2, σ_2^2 and $\sigma_1^{-1}\sigma_2^2\sigma_1$. Hence $A_{S,3}^{(c_1,c_2,c_1)}$ is generated by r_1, r_2 and $((R^{(c_1,c_2)})^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes R^{(c_3,c_1)})(\mathrm{id} \otimes R^{(c_1,c_3)}) \cdot (R^{(c_1,c_2)} \otimes \mathrm{id})$. But a computation shows that

(A.4) $((R^{(c_1,c_2)})^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes R^{(c_3,c_1)})(\mathrm{id} \otimes R^{(c_1,c_3)})(R^{(c_1,c_2)} \otimes \mathrm{id})$

$$= \frac{g_-(t_{c_2}/t_{c_1})g_-(t_{c_1}t_{c_2}/t_{c_3})}{g_-(t_{c_2})g_-(t_{c_1}t_{c_2})} + \frac{g_-(t_{c_2}/t_{c_1})g_-(t_{c_3})}{g_-(t_{c_2})g_-(t_{c_1}t_{c_2})}r_1$$
$$+ \frac{g_+(t_{c_1}^2 t_{c_2})}{g_-(t_{c_2})g_-(t_{c_1}t_{c_2})}r_2 - \frac{g_+(t_{c_1})}{g_-(t_{c_2})g_-(t_{c_1}t_{c_2})}r_1 r_2$$
$$- \frac{g_+(t_{c_1}t_{c_2}^2)}{g_-(t_{c_2})g_-(t_{c_1}t_{c_2})}r_2 r_1 + \frac{g_+(t_{c_2})}{g_-(t_{c_2})g_-(t_{c_1}t_{c_2})}r_1 r_2 r_1 .$$

So we have $A_{S,3}^{(c_1,c_2,c_1)} = A_{S,3}^{\prime(c_1,c_2,c_1)}$. □

Lemma A.2. *Let $A_{S,n}^{(c_1,\dots,c_n)}$ be the associated algebra of S. We regard $A_{S,n-1}^{(c_1,\dots,c_{n-1})}$ as a subalgebra of $A_{S,n}^{(c_1,\dots,c_n)}$ naturally. Then*

(A.5) $$A_{S,n}^{(c_1,\dots,c_n)} = A_{S,n-1}^{(c_1,\dots,c_{n-1})} + A_{S,n-1}^{(c_1,\dots,c_{n-1})} \cdot (\mathrm{id}^{\otimes(n-2)} \otimes (R^{(c_n,c_{n-1})}R^{(c_{n-1},c_n)}))A_{S,n-1}^{(c_1,\dots,c_{n-1})} .$$

Proof. We prove by an induction on n. First, we treat the case $n = 2$. The algebra $A_{S,2}^{(c_1,c_2)}$ is generated by 1 and $R^{(c_2,c_1)}R^{(c_1,c_2)}$ if $c_1 \neq c_2$. If $c_1 = c_2 = c$, then $A_{S,2}^{(c,c)}$ is generated by 1 and $R^{(c,c)}$. But $R^{(c,c)} = ((R^{(c,c)})^2 - 1)/(t_c - t_c^{-1})$ and so $A_{S,2}^{(c,c)}$ is generated by 1 and $(R^{(c,c)})^2$. Hence $A_{S,2}^{(c_1,c_2)}$ is generated by 1 and $R^{(c_2,c_1)}R^{(c_1,c_2)}$ for any c_1 and c_2. The quadratic relation (3.8) proves the lemma. Next, treat the case $n = 3$. In this case, Lemma 3.15 implies (A.5). Now, prove for $n > 3$. The group $B_n^{(c_1,\dots,c_n)}$ is generated by its subgroup $B_{n-1}^{(c_1,\dots,c_{n-1})}$ and the elements $\sigma_{n-1}^{-1}\cdots\sigma_{k+1}^{-1}\sigma_k^{\gamma_k}\sigma_{k+1}\cdots\sigma_{n-1}$ $(1 \le k \le n-1)$ where $\gamma_k = 1$ if $c_n = c_k$ and $\gamma_k = 2$ if otherwise. By the induction hypothesis, it is enough to show that $\rho_S^{(c_1,\dots,c_n)}(\sigma_{n-1}^{\gamma_k})$ and $\rho_S^{(c_1,\dots,c_n)}(\sigma_{n-1}^{-1}\sigma_{n-2}^{\gamma_{n-2}}\sigma_{n-1})$ are contained in

$$A_{S,n-1}^{(c_1,\dots,c_{n-1})} + A_{S,n-1}^{(c_1,\dots,c_{n-1})}(\mathrm{id}^{\otimes(n-2)} \otimes (R^{(c_n,c_{n-1})}R^{(c_{n-1},c_n)}))A_{S,n-1}^{(c_1,\dots,c_{n-1})} .$$

We know that $\rho_S^{(c_1,\dots,c_{n-1},c_n)}(\sigma_{n-1}) = \rho_S^{(c_1,\dots,c_{n-1},c_n)}(\sigma_{n-1}^2 - 1)/(t_{c_n} - t_{c_n}^{-1})$ if $c_{n-1} = c_n$. Hence, from the formula (3.10) and (3.11), we know that the element $\rho_S^{(c_1,\dots,c_n)}(\sigma_{n-1}^{-1}\sigma_{n-2}^{\gamma_{n-2}}\sigma_{n-1})$ can be written as a linear combination of the elements $\rho_S^{(c_1,\dots,c_n)}(1)$, $\rho_S^{(c_1,\dots,c_n)}(\sigma_{n-2}^2)$, $\rho_2^{(c_1,\dots,c_n)}(\sigma_{n-1}^2)$,

$\rho_S^{(c_1,\dots,c_n)}(\sigma_{n-2}^2\sigma_{n-1}^2)$, $\rho_S^{(c_1,\dots,c_n)}(\sigma_{n-1}^2\sigma_{n-2}^2)$ and $\rho_S^{(c_1,\dots,c_n)}(\sigma_{n-2}^2\sigma_{n-1}^2\sigma_{n-2}^2)$. Hence the lemma is proved. □

Proof of Proposition 4.8. By Lemma 3.16, every element $x \in A_{S,n}^{(c_1,\dots,c_{n-1},c_n)}$ is written as $x = y_1 + y_2 R^{(c_n,c_{n-1})}R^{(c_{n-1},c_n)}y_3$ where $y_1, y_2, y_3 \in A_{S,n-1}^{(c_1,\dots,c_{n-1})}$. Hence we have

$$\begin{aligned}(\text{A.6})\quad \mathrm{Sp}_n^{(c_1,\dots,c_n)}&(x(\mathrm{id}^{\otimes(n-1)}\times \mu^{(c_n)}))\\ &= \mathrm{Sp}_n^{(c_1,\dots,c_n)}(y_1(\mathrm{id}^{\otimes(n-1)}\otimes \mu^{(c_n)}))\\ &\quad + \mathrm{Sp}_n^{(c_1,\dots,c_n)}(y_2R^{(c_n,c_{n-1})}R^{(c_{n-1},c_n)}y_3(\mathrm{id}^{\otimes(n-1)}\otimes \mu^{(c_n)}))\\ &= y_2y_3,\end{aligned}$$

which is contained in $A_{S,n-1}^{(c_1,\dots,c_{n-1})}$. □

Appendix B. Proof of Lemma 7.4. Let

$$g_+(x) = x + x^{-1}, \qquad g_-(x) = x - x^{-1}.$$

The definition of $I_3^{(c_1,c_2,c_3)}$ and the relations of B_3 imply that

LEMMA B.1. *Let* $b \in B_3^{(c_1,c_2,c_3)}$ *and* x *be one of*

$$\begin{aligned}& g_+(t_{c_{b(1)}})g_-(t_{c_{b(2)}})\sigma_1^2\sigma_2\sigma_1^2\sigma_2\\ & -g_-(t_{c_{b(2)}})g_+(t_{c_{b(3)}})(g_+(t_{c_{b(1)}}t_{c_{b(2)}})\sigma_1\sigma_2^2\sigma_1 - \sigma_1^{-1}\sigma_2^2\sigma_1)\\ & -g_-(t_{c_{b(1)}}^{-1}t_{c_{b(3)}})(\sigma_1^4\sigma_2^2 + \sigma_1^2\sigma_2^2\sigma_1^2) + g_-(t_{c_{b(1)}}^{-1}t_{c_{b(2)}}t_{c_{b(3)}})(t_{c_{b(3)}} + t_{c_{b(3)}}^{-1})\sigma_1^4\\ & -g_+(t_{c_{b(1)}})g_-(t_{c_{b(1)}}t_{c_{b(2)}}t_{c_{b(3)}}^{-1})\sigma_1^2\sigma_2^2 - g_-(t_{c_{b(2)}}^{-2}t_{c_{b(3)}}^2)\sigma_1^2,\end{aligned}$$

$$\begin{aligned}& g_+(t_{c_{b(1)}})g_-(t_{c_{b(2)}})\sigma_1^2\sigma_2\sigma_1^2\sigma_2 - g_-(t_{c_{b(2)}})g_+(t_{c_{b(3)}})\sigma_1^2\sigma_2^2\sigma_1^2\\ & -g_-(t_{c_{b(1)}}^{-1}t_{c_{b(3)}})(g_+(t_{c_{b(1)}}t_{c_{b(2)}})\sigma_1\sigma_2^2\sigma_1 - \sigma_1^{-1}\sigma_2^2\sigma_1\\ & \qquad + (g_+(t_{c_{b(1)}}t_{c_{b(2)}})\sigma_1\sigma_2^2\sigma_1 - \sigma_1\sigma_2^2\sigma_1^{-1})\\ & +g_-(t_{c_{b(1)}}^{-1}t_{c_{b(2)}}t_{c_{b(3)}})g_+(t_{c_{b(3)}})\sigma_1^4 - g_+(t_{c_{b(1)}})g_-(t_{c_{b(1)}}t_{c_{b(2)}}t_{c_{b(3)}}^{-1})\sigma_1\sigma_2^2\sigma_1\\ & -g_-(t_{c_{b(2)}}^{-2}t_{c_{b(3)}}^2)\sigma_1^2,\end{aligned}$$

$$\begin{aligned}& g_+(t_{c_{b(1)}})g_-(t_{c_{b(2)}})\sigma_1^2\sigma_2\sigma_1^2\sigma_2\\ & -g_-(t_{c_{b(2)}})g_+(t_{c_{b(3)}})(g_+(t_{c_{b(1)}}t_{c_{b(2)}})\sigma_1\sigma_2^2\sigma_1 - \sigma_1\sigma_2^2\sigma_1^{-1})\\ & -g_-(t_{c_{b(1)}}^{-1}t_{c_{b(3)}})(\sigma_1^2\sigma_2^2\sigma_1^2 + \sigma_2^2\sigma_1^4) + g_-(t_{c_{b(1)}}^{-1}t_{c_{b(2)}}t_{c_{b(3)}})g_+(t_{c_{b(3)}})\sigma_1^4\\ & -g_+(t_{c_{b(1)}})g_-(t_{c_{b(1)}}t_{c_{b(2)}}t_{c_{b(3)}}^{-1})\sigma_2^2\sigma_1^2 - g_-(t_{c_{b(2)}}^{-2}t_{c_{b(3)}}^2)\sigma_1^2,\end{aligned}$$

$$g_+(t_{c_{b(3)}})g_-(t_{c_{b(2)}})(g_+(t_{c_{b(3)}}t_{c_{b(2)}})\sigma_2\sigma_1^2\sigma_2 - \sigma_1\sigma_2^2\sigma_1^{-1})$$
$$-g_-(t_{c_{b(2)}})g_+(t_{c_{b(3)}})\sigma_1^2\sigma_2\sigma_1^2\sigma_2 - g_-(t_{c_{b(1)}}^{-1}t_{c_{b(3)}})(\sigma_2^2\sigma_1^2\sigma_2^2 + \sigma_2^4\sigma_1^2)$$
$$+g_-(t_{c_{b(1)}}^{-1}t_{c_{b(2)}}t_{c_{b(3)}})g_+(t_{c_{b(3)}})\sigma_2^2\sigma_1^2 - g_+(t_{c_{b(1)}})g_-(t_{c_{b(1)}}t_{c_{b(2)}}t_{c_{b(3)}}^{-1})\sigma_2^4$$
$$-g_-(t_{c_{b(2)}}^{-2}t_{c_{b(3)}}^2)\sigma_2^2,$$

$$g_+(t_{c_{b(1)}})g_-(t_{c_{b(2)}})\sigma_2^2\sigma_1^2\sigma_2^2 - g_-(t_{c_{b(2)}})g_+(t_{c_{b(3)}})\sigma_1^2\sigma_2\sigma_1^2\sigma_2$$
$$-g_-(t_{c_{b(1)}}^{-1}t_{c_{b(3)}})(g_+(t_{c_{b(3)}}t_{c_{b(2)}})\sigma_2\sigma_1^2\sigma_2 - \sigma_1^{-1}\sigma_2^2\sigma_1)$$
$$+(g_+(t_{c_{b(3)}}t_{c_{b(2)}})\sigma_2\sigma_1^2\sigma_2 - \sigma_1\sigma_2^2\sigma_1^{-1})$$
$$+g_-(t_{c_{b(1)}}^{-1}t_{c_{b(2)}}t_{c_{b(3)}})g_+(t_{c_{b(3)}})\sigma_s\sigma_1^2\sigma_2$$
$$-g_+(t_{c_{b(1)}})g_-(t_{c_{b(1)}}t_{c_{b(2)}}t_{c_{b(3)}}^{-1})\sigma_2^4 - g_-(t_{c_{b(2)}}^{-2}t_{c_{b(3)}}^2)\sigma_2^4,$$

$$g_+(t_{c_{b(1)}})g_-(t_{c_{b(2)}})(g_+(t_{c_{b(3)}}t_{c_{b(2)}})\sigma_2\sigma_1^2\sigma_2 - \sigma_1^{-1}\sigma_2^2\sigma_1)$$
$$-g_-(t_{c_{b(2)}})g_+(t_{c_{b(3)}})\sigma_1^2\sigma_2\sigma_1^2\sigma_2 - g_-(t_{c_{b(1)}}^{-1}t_{c_{b(3)}})(\sigma_1^2\sigma_2^4 + \sigma_2^2\sigma_1^2\sigma_2^2)$$
$$+g_-(t_{c_{b(1)}}^{-1}t_{c_{b(2)}}t_{c_{b(3)}})g_+(t_{c_{b(3)}})\sigma_1^2\sigma_2^2 - g_+(t_{c_{b(1)}})g_-(t_{c_{b(1)}}t_{c_{b(2)}}t_{c_{b(3)}}^{-1})\sigma_2^4$$
$$-g_-(t_{c_{b(2)}}^{-2}t_{c_{b(3)}}^2)\sigma_2^2.$$

Then $p_n(b^{-1}xb) = 0$.

Proof of Lemma 7.4. We can solve the above six equations with respect to $\sigma_1\sigma_2^2\sigma_1$, $\sigma_2\sigma_1^2\sigma_2$, $\sigma_1^{-1}\sigma_2^2\sigma_1$, $\sigma_1\sigma_2^2\sigma_1^{-1}$, $\sigma_2^2\sigma_1^2\sigma_2^2$ and $\sigma_1^2\sigma_2\sigma_1^2\sigma_2$ if

$$\begin{aligned}
&-t_{c_{b(1)}}^{-6}t_{c_{b(2)}}^{-8}t_{c_{b(3)}}^{-5}(t_{c_{b(1)}}^2+1)^2(t_{c_{b(2)}}-1)^3(t_{c_{b(2)}}+1)^3(t_{c_{b(3)}}^2+1)\\
&\times(t_{c_{b(1)}}^3t_{c_{b(2)}}^4t_{c_{b(3)}}^2 - 2t_{c_{b(1)}}^2t_{c_{b(2)}}^3t_{c_{b(3)}}^2 - t_{c_{b(1)}}^3t_{c_{b(2)}}^2t_{c_{b(3)}}^2 + t_{c_{b(1)}}t_{c_{b(2)}}^2t_{c_{b(3)}}^2\\
&\quad + t_{c_{b(1)}}^2t_{c_{b(2)}}t_{c_{b(3)}}^2 - t_{c_{b(2)}}t_{c_{b(3)}}^2 - t_{c_{b(1)}}t_{c_{b(3)}}^2 + t_{c_{b(1)}}^3t_{c_{b(2)}}^4 + t_{c_{b(1)}}^4t_{c_{b(2)}}^3\\
&\qquad - t_{c_{b(1)}}^2t_{c_{b(2)}}^3 - t_{c_{b(1)}}^3t_{c_{b(2)}}^2 + t_{c_{b(1)}}t_{c_{b(2)}}^2 + 2t_{c_{b(1)}}^2t_{c_{b(2)}} - t_{c_{b(1)}})\\
&\times(t_{c_{b(2)}}^4t_{c_{b(3)}}^6 - t_{c_{b(1)}}^4t_{c_{b(2)}}^6t_{c_{b(3)}}^4 - 2t_{c_{b(1)}}^2t_{c_{b(2)}}^6t_{c_{b(3)}}^4 - t_{c_{b(2)}}^6t_{c_{b(3)}}^4\\
&\quad + 2t_{c_{b(1)}}^4t_{c_{b(2)}}^4t_{c_{b(3)}}^4 + t_{c_{b(1)}}^2t_{c_{b_2}}^4t_{c_{b(3)}}^4 + t_{c_{b(2)}}^4t_{c_{b(3)}}^4 - t_{c_{b(1)}}^4t_{c_{b(2)}}^2t_{c_{b(3)}}^4\\
&\quad - t_{c_{b(1)}}^2t_{c_{b(2)}}^2t_{c_{b(3)}}^4 + t_{c_{b(2)}}^2t_{c_{b(3)}}^4 + t_{c_{b(1)}}^4t_{c_{b(2)}}^4t_{c_{b(3)}}^2 - t_{c_{b(1)}}^2t_{c_{b(2)}}^4t_{c_{b(3)}}^2\\
&\quad - t_{c_{b(2)}}^4t_{c_{b(3)}}^2 + t_{c_{b(1)}}^4t_{c_{b(2)}}^2t_{c_{b(3)}}^2 + t_{c_{b(1)}}^2t_{c_{b(2)}}^2t_{c_{b(3)}}^2 + 2t_{c_{b(2)}}^2t_{c_{b(3)}}^2\\
&\qquad - t_{c_{b(1)}}^4t_{c_{b(3)}}^2 - 2t_{c_{b(1)}}^2t_{c_{b(3)}}^2 - t_{c_{b(3)}}^2 + t_{c_{b(1)}}^4t_{c_{b(2)}}^2) \neq 0.
\end{aligned}$$

This implies that all the elements in $M_3^{(c_1, c_2, c_3)}$ can be written in terms of 1, σ_1^2, σ_2^2, $\sigma_1\sigma_2$, $\sigma_2\sigma_1$ and $\sigma_1^2\sigma_2^2\sigma_1^2$ if the parameters t_{c_1}, t_{c_2} and

t_{c_3} are generic. Note that the above condition is also satisfied in the case $c_1 = c_2$, $c_1 = c_3$, $c_2 = c_3$ or $c_1 = c_2 = c_3$ if the parameters $t_1, t_2, \cdots$ are generic. This implies Lemma 7.4. □

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Received August 19, 1990 and in revised form August 31, 1991.

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