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PERMUTATION ENUMERATION SYMMETRIC FUNCTIONS, AND UNIMODALITY

FRANCESCO BRENTI

We study the polynomials obtained by enumerating a set of permutations with respect to the number of excedances. We prove that these polynomials have only real zeros and are unimodal for many interesting classes of permutations. We then show how these polynomials also arise naturally from the theory of symmetric functions.

1. Introduction. Log-concave and unimodal sequences arise often in combinatorics, algebra, geometry and computer science, as well as in probability and statistics where these concepts were first defined and studied (see [**1**] for further information and references about the origin of the concept of a unimodal sequence). Even though log-concavity and unimodality have one-line definitions, it has now become apparent that to *prove* the unimodality or log-concavity of a sequence can sometimes be a very difficult task requiring the use of intricate combinatorial constructions ([**15**], [**19**], [**32**], [**33**]) or of refined mathematical tools. The number and variety of these tools has been constantly increasing and is quite bewildering and surprising. They include, for example, classical analysis ([**5**], [**29**], [**30**], [**31**]), linear algebra ([**17**]), the representation theory of Lie algebras and superalgebras ([**16**], [**21**], [**22**]), the theory of total positivity ([**2**], [**4**]), the theory of symmetric functions ([**3**], [**6**], [**20**]), and algebraic geometry ([**24**]). We refer the interested reader to [**25**] for an excellent survey of many of these techniques, problems, and results.

In this paper, motivated by a conjecture of R. Stanley, we study the unimodality of some polynomials obtained by enumerating a set of permutations with respect to the number of excedances. We prove that these polynomials are unimodal for many general classes of permutations including conjugacy classes, thus generalizing Stanley's conjecture. We then show how these polynomials also arise, in a natural though unexpected way, from the theory of symmetric functions.

The organization of the paper is as follows. In the next section we collect some notation, definitions and results that will be needed in the rest of the paper. In §3 we prove that the polynomials obtained by enumerating a conjugacy class of a symmetric group with respect to the number of excedances are symmetric and unimodal and have only real zeros. We then derive some consequences of this result, among which is Stanley's original conjecture. In §4 we show how the polynomials studied in §3 also arise naturally from the theory of symmetric functions. This approach also leads to the consideration of some related (but more mysterious) polynomials whose study yields some interesting identities for inverse Kostka numbers and characters of the symmetric group (Corollaries 4.15, 4.16, and 4.18). Finally, in §5, we discuss some of the main open problems arising from our work, some conjectures and possible directions for further research.

2. Notation and preliminaries. In this section we collect some definitions, notation and results that will be used in the rest of this paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}$ and $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \dots, a\}$ (where $[0] \stackrel{\text{def}}{=} \varnothing$). The cardinality of a set A will be denoted by $|A|$. A sequence $\{a_0, a_1, \dots, a_d\}$ (of real numbers) is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $i = 1, \dots, d-1$. It is said to be *unimodal* if there exists an index $0 \leq j \leq d$ such that $a_i \leq a_{i+1}$ for $i = 0, \dots, j-1$ and $a_i \geq a_{i+1}$ for $i = j, \dots, d-1$. It is said to have no *internal zeros* if there are not three indices $0 \leq i < j < k \leq d$ such that $a_i, a_k \neq 0$ and $a_j = 0$. It is said to be *symmetric* if $a_i = a_{d-i}$ for $i = 0, \dots, [\frac{d}{2}]$. We say that a polynomial $\sum_{i=0}^{d} a_i x^i$ is *log-concave* (respectively, *unimodal*, with no *internal zeros*, *symmetric*) if the sequence $\{a_0, a_1, \dots, a_d\}$ has the corresponding property. It is well known that if $\sum_{i=0}^{d} a_i t^i$ is a polynomial with nonnegative coefficients and with only real zeros, then the sequence $\{a_0, a_1, \dots, a_d\}$ is log-concave and unimodal, with no internal zeros (see, e.g., **[2]** or **[7]**, Theorem B, p. 270). If $p(x)$ is a symmetric unimodal polynomial then there is a unique $n \in \mathbf{N}$ such that $x^n p(\frac{1}{x}) = p(x)$. We call the number $\frac{n}{2}$ the *center of symmetry* of $p(x)$, and we write $C(p) = \frac{n}{2}$. So, for example, $C(x^2 + 3x^3 + x^4) = 3$ and $C(1 + x) = \frac{1}{2}$. An elementary, though crucial property of symmetric unimodal polynomials, which will be used repeatedly in this paper, is the following.

PROPOSITION 2.1. *Let $p(x)$ and $q(x)$ be two symmetric unimodal polynomials. Then $p(x)q(x)$ is a symmetric unimodal polynomial and $C(pq) = C(p) + C(q)$.*

Proposition 2.1 is well known and a proof of it can be found, e.g., in **[25]**, Proposition 1.2.

We follow [**14**, Chapter 1, §1] for the basic definitions, notation and terminology about (integer) partitions. In particular, given a partition λ, we let $m_i(\lambda)$ be the multiplicity of i as a part of λ and $z_\lambda \stackrel{\text{def}}{=} \prod_{i\geq 1} i^{m_i(\lambda)} m_i(\lambda)!$. We also denote by $d(\lambda)$ the length of the side of the Durfee square of λ, (i.e. $d(\lambda) \stackrel{\text{def}}{=} \max\{i \in [n] : \lambda_i \geq i\}$ if $\lambda = (\lambda_1, \dots, \lambda_n)$).

A *partition* of a (finite) set S is a collection of subsets $\mathscr{S} = \{S_1, \dots, S_p\}$ of S (which we may assume indexed so that $|S_1| \geq \cdots \geq |S_p|$) such that $\bigcup_{i=1}^{p} S_i = S$ and $S_i \cap S_j = \varnothing$ for $i \neq j$. The *type* of the partition $\mathscr{S}$ is the integer partition $\mathscr{T}(\mathscr{S}) \stackrel{\text{def}}{=} (|S_1|, |S_2|, \dots, |S_p|)$. Given an integer partition λ of n, we denote by $P_\lambda([n])$ the collection of all partitions of $[n]$ such that $\mathscr{T}(\mathscr{S}) = \lambda$. For example, if $\lambda = (2, 1, 1)$, then $P_{(2,1,1)}([4])$ consists of the 6 partitions $12/3/4$, $13/2/4$, $14/2/3$, $23/1/4$, $24/1/3$, and $34/1/2$. We also let $\mathscr{P}_\lambda(n) \stackrel{\text{def}}{=} |P_\lambda([n])|$. The following result is well known (see, e.g., [**14**, p. 22], or [7, Theorem B, p. 205]) and is here recalled only for completeness.

PROPOSITION 2.2. *Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a partition of n. Then*

$$\mathscr{P}_\lambda(n) = \frac{\binom{n}{\lambda_1, \dots, \lambda_p}}{\prod_{i\geq 1}(m_i(\lambda)!)}.$$

Let π be a permutation on $[n]$ (considered as a bijection $\pi: [n] \to [n]$) and $i \in [n]$. We say that i is an *excedance* (respectively, a *descent*) of π if $\pi(i) > i$ (respectively, $i \neq n$ and $\pi(i) > \pi(i+1)$). We denote by $e(\pi)$ (respectively, $d(\pi), c(\pi)$) the number of excedances (respectively, descents, cycles) of π. So, for example, if $\pi = 291586347$ then $e(\pi) = 4$, $d(\pi) = 3$, and $c(\pi) = 3$. For $n \in \mathbf{P}$ the polynomial

$$A_n(t) \stackrel{\text{def}}{=} \sum_{\pi \in S_n} t^{d(\pi)+1}$$

(where S_n denotes the symmetric group on n elements) is called the nth *Eulerian polynomial* and has been widely studied (see, e.g., [**10**]), for convenience we will let $A_0(t) \stackrel{\text{def}}{=} 1$. Given a partition λ of n we denote by $S_n(\lambda)$ the set of all $\sigma \in S_n$ of cycle type λ, we also let $\mathscr{C}_n \stackrel{\text{def}}{=} S_n((n))$. The next result is known (see [**10**]) but is here recalled and proved for completeness.

PROPOSITION 2.3. *Let* $n \in \mathbf{N}$. *Then*

$$\sum_{\sigma \in \mathscr{C}_{n+1}} t^{e(\sigma)} = A_n(t).$$

Proof. Each $\sigma \in \mathscr{C}_{n+1}$ can be written uniquely as an $n+1$ cycle of the form $\sigma = (n+1 a_1 \dots a_n)$ (so that $\sigma(n+1) = a_1$, $\sigma(a_i) = a_{i+1}$ for $i = 1, \dots, n-1$, and $\sigma(a_n) = n+1$), we then let $\overline{\sigma} \stackrel{\text{def}}{=} a_n \cdots a_1$ (i.e. $\overline{\sigma}(i) \stackrel{\text{def}}{=} a_{n+1-i}$, for $i = 1, \dots, n$). The correspondence $\sigma \mapsto \overline{\sigma}$ is clearly a bijection between $\mathscr{C}_{n+1}$ and S_n. Furthermore, we have that

$$\begin{aligned} e(\sigma) &= \sharp\{i \in [n+1] : \sigma(i) > i\} \\ &= \sharp\{i \in [n-1] : a_{i+1} > a_i\} + 1 = d(\overline{\sigma}) + 1, \end{aligned}$$

which establishes the result. □

Let $\mu = (\mu_1, \dots, \mu_l) \in \mathscr{P}$ (where $\mathscr{P}$ denotes the set of all integer partitions). We denote the diagram of μ by $D(\mu)$. A *special border strip* of $D(\mu)$ is a border strip that is contained in $D(\mu)$ and which has at least 1 cell in the first column of $D(\mu)$, (see, e.g., [**14**, p. 31], for the definition of a border strip). The *sign* of a border strip is $(-1)^{r-1}$ where r is the number of rows it occupies. A *special border strip tabloid* T of shape μ is a partition of $D(\mu)$ into special border strips. The *type* of T is its type as a (set) partition. The *sign* of T, denoted $\mathrm{sgn}(T)$, is the product of the signs of the special border strips of T. The following beautiful result appears in [**9**, Theorem 1], (we refer the reader to [**14**, Chapter 1, §6] for the definition of the Kostka matrix).

THEOREM 2.4. *Let* $\lambda, \mu \in \mathscr{P}$, *and* K *be the Kostka matrix. Then*

$$(K^{-1})_{\lambda, \mu} = \sum_T \mathrm{sgn}(T)$$

where the sum is over all special border strip tabloids of shape μ *and type* λ.

Given $\lambda, \mu \in \mathscr{P}$ with $D(\mu) \subseteq D(\lambda)$ we denote by $f^{\lambda \backslash \mu}$ the number of standard tableaux of shape $\lambda \backslash \mu$.

3. Permutation enumeration and unimodality. In this section we prove that the polynomials obtained by enumerating a conjugacy class of a symmetric group with respect to the number of excedances are symmetric and unimodal and have only real zeros. In order to prove this result in a clear and concise way it is convenient to establish a canonical way of writing partitions and permutations. We say that a partition $\{S_1, \dots, S_p\}$ of $[n]$ is indexed in *canonical form* if $|S_1| \geq \cdots \geq |S_p|$ and for $|S_i| = |S_j|$ we have that $i > j$ if and only if $\max(S_i) > \max(S_j)$. So, for example, the canonical indexing of the partition $157/23/9/4/86$ is $S_1 = \{1, 5, 7\}$, $S_2 = \{2, 3\}$, $S_3 = \{8, 6\}$, $S_4 = \{4\}$, $S_5 = \{9\}$. Analogously, we say that the cycles $C_1, \dots, C_p$ appearing in the disjoint cycle decomposition of a permutation $\pi \in S_n$ are indexed in canonical form if $|C_1| \geq \cdots \geq |C_p|$ and for $|C_i| = |C_j|$ we have that $i > j$ if and only if $\max(C_i) > \max(C_j)$. We can now prove one of the main results of this section.

THEOREM 3.1. *Let $n \in \mathbf{P}$ and $\lambda = (\lambda_1, \dots, \lambda_p)$ be a partition of n. Then*

$$\sum_{\pi \in S_n(\lambda)} t^{e(\pi)} = \frac{n!}{z_\lambda} \prod_{i=1}^{p} \frac{A_{\lambda_i - 1}(t)}{(\lambda_i - 1)!}. \tag{1}$$

Proof. Let $(S_i)_{i=1,\dots,p}$ be a partition of $[n]$ of type λ, written in canonical form. We define a map

$$\Phi\colon P_\lambda([n]) \times \prod_{i=1}^{p} (\mathscr{C}_{[\lambda_i]}) \to S_n(\lambda)$$

by

$$\Phi((S_i)_{i=1,\dots,p}, (\sigma_i)_{i=1,\dots,p}) \stackrel{\text{def}}{=} \prod_{i=1}^{p} T_{S_i}(\sigma_i),$$

where, for a subset $S \stackrel{\text{def}}{=} \{a_1, \dots, a_r\}_< \subseteq [n]$ (where $S = \{a_1, \dots, a_r\}_<$ means that $S = \{a_1, \dots, a_r\}$ and $a_1 < \cdots < a_r$), and $\sigma \in S_r$, $T_S(\sigma)$ is the permutation of S defined by

$$T_S(\sigma)(a_j) \stackrel{\text{def}}{=} a_{\sigma(j)},$$

for $j = 1, \dots, r$. Clearly, the map $\sigma \mapsto T_S(\sigma)$ is a bijection. Furthermore, we have that

$$e\left(\prod_{i=1}^{p} T_{S_i}(\sigma_i)\right) = \sum_{i=1}^{p} e(\sigma_i). \tag{2}$$

For $\pi \in S_n(\lambda)$ let $C_1 \dots C_p$ be the disjoint cycle decomposition of π, written in canonical form. We then let $S_i(\pi)$ be the subset of $[n]$

consisting of the elements appearing in C_i and $C_i(\pi) \stackrel{\text{def}}{=} (T_{S_i(\pi)})^{-1}(C_i)$ for $i = 1, \ldots, p$. Then the map

$$\theta : S_n(\lambda) \to P_\lambda([n]) \times \prod_{i=1}^{p} (\mathscr{C}_{[\lambda_i]})$$

defined by

$$\theta(\pi) \stackrel{\text{def}}{=} ((S_i(\pi))_{i=1,\ldots,p}, (C_i(\pi))_{i=1,\ldots,p})$$

is such that $(\Phi \circ \Theta)(\pi) = \pi$ for all $\pi \in S_n(\lambda)$. This proves that Φ is a bijection and hence, by (2), that

$$\sum_{\pi \in S_n(\lambda)} t^{e(\pi)} = \frac{\binom{n}{\lambda_1, \ldots, \lambda_p}}{\prod_{i \geq 1}(m_i(\lambda)!)} \prod_{i=1}^{p} \left(\sum_{\sigma \in \mathscr{C}_{\lambda_i}} t^{e(\sigma)} \right)$$

and (1) follows from Proposition 2.3. □

In what follows we will let, for simplicity,

$$E_D(t) \stackrel{\text{def}}{=} \sum_{\pi \in D} t^{e(\pi)},$$

for any $D \subseteq S_n$, and $E_\lambda(t) \stackrel{\text{def}}{=} E_{S_n(\lambda)}(t)$ if λ is a partition of n. From the preceding theorem we immediately deduce the following result.

THEOREM 3.2. *Let $n \in \mathbf{P}$ and λ be a partition of n. Then the polynomial $E_\lambda(t)$ is symmetric and unimodal with center of symmetry at $(|\lambda| - m_1(\lambda))/2$, and has only real zeros. In particular, $E_\lambda(t)$ is log-concave with no internal zeros.*

Proof. It is well known (see, e.g., [7, p. 241, eq. [5c], and p. 292, Ex. 3]) that, for $n \geq 1$, $A_n(t)$ is a symmetric and unimodal polynomial with $C(A_n) = \frac{n+1}{2}$. Therefore, from (1) and Proposition 2.1 we conclude that $E_\mu(t)$ is a symmetric and unimodal polynomial and that

$$C(E_\mu) = C\left(\prod_{i=1}^{p} A_{\mu_i - 1}\right) = C\left(\prod_{i \geq 2} (A_{i-1})^{m_i(\mu)}\right)$$
$$= \sum_{i \geq 2} m_i(\mu) \frac{i}{2} = \frac{|\mu| - m_1(\mu)}{2}.$$

Furthermore, it is well known (see, e.g., [7, p. 292]) that, for $n \geq 1$, the polynomials $A_n(t)$ have only real zeros; hence, by (1), the same is true for $E_\mu(t)$. □

Theorem 3.2 has some interesting consequences; we begin with the following one.

THEOREM 3.3. *Let $n, k, r \in \mathbf{P}$ and let $D_{r,k}(n)$ be the set of all $\pi \in S_n$ having exactly k cycles each of length $> r$. Then $E_{D_{r,k}(n)}(t)$ is a symmetric and unimodal polynomial with $C(E_{D_{r,k}(n)}) = \frac{n}{2}$.*

Proof. Let $\mu \stackrel{\text{def}}{=} (\mu_1, \ldots, \mu_k)$ be a partition of n of length k such that $\mu_i > r$ for $i = 1, \ldots, k$. Then, by Theorem 3.2, $E_\mu(t)$ is a symmetric and unimodal polynomial with $C(E_\mu) = (|\mu| - m_1(\mu))/2 = |\mu|/2 = n/2$. But, by our definitions

$$E_{D_{r,k}(n)} = \sum_{\mu} E_\mu,$$

where the sum is over all partitions $\mu = (\mu_1, \ldots, \mu_k)$ of length k such that $\mu_i > r$ for $i = 1, \ldots, k$, and the thesis follows. □

One consequence of the preceding result is particularly interesting (see [7, p. 257] for the definition of r-derangements).

THEOREM 3.4. *Let $n, r \in \mathbf{P}$ and let $D_r(n)$ be the set of all r-derangements of S_n. Then $E_{D_r(n)}(t)$ is a symmetric and unimodal polynomial with $C(E_{D_r(n)}) = \frac{n}{2}$.*

Proof. Clearly

$$E_{D_r(n)} = \sum_{k \geq 1} E_{D_{r,k}(n)},$$

so the thesis follows from Theorem 3.3. □

In the case $r = 1$ the preceding result had been conjectured by Stanley (see the comments preceding and following Proposition 7.8 in [**25**]) and first proved in [**3**, Corollary 1]. We should remark, however, that the proof given in [**3**] makes use of the theory of symmetric functions while the one given here is elementary and completely combinatorial.

4. Symmetric functions and excedances. In this section we show how the combinatorially defined polynomials studied in the preceding section arise naturally from the theory of symmetric functions. We will use standard notation and terminology for this theory from **[14]**. In particular, we will denote by s_λ (respectively, h_λ, e_λ, m_λ, f_λ, and p_λ) the *Schur* (respectively, *complete homogeneous, elementary, monomial, forgotten* and *power sum*) symmetric functions, associated to the partition λ. All the symmetric functions considered in this section are assumed to be in the variables $(x_1, x_2, \dots)$. We start by defining a ring homomorphism

$$\xi\colon \Lambda_{\mathbf{Q}}[[t]] \to \mathbf{Q}[x][[t]]$$

by letting

$$\xi(e_i) \stackrel{\text{def}}{=} \frac{(1-x)^{i-1}}{i!}, \tag{3}$$

for $i \in \mathbf{P}$, and $\xi(e_0) \stackrel{\text{def}}{=} 1$. Since the e_i's are algebraically independent and generate $\Lambda_{\mathbf{Q}}$ over $\mathbf{Q}$ (see, e.g., [**14**, p. 13]) (3) uniquely defines a ring homomorphism $\xi\colon \Lambda_{\mathbf{Q}} \to \mathbf{Q}[x]$; we then extend ξ to all of $\Lambda_{\mathbf{Q}}[[t]]$ by letting

$$\xi\left(\sum_{n\geq 0} a_n t^n\right) \stackrel{\text{def}}{=} \sum_{n\geq 0} \xi(a_n) t^n, \tag{4}$$

where $a_n \in \Lambda_{\mathbf{Q}}$ for $n \in \mathbf{N}$. It is then easy to check that the map ξ defined by (4) is still a ring homomorphism.

THEOREM 4.1. *Let $\xi\colon \Lambda_{\mathbf{Q}} \to \mathbf{Q}[x]$ be the ring homomorphism defined by* (3). *Then, for $n \in \mathbf{P}$, we have that*

$$\xi(h_n) = \frac{A_n(x)}{xn!}. \tag{5}$$

Proof. It is a well-known result in the theory of symmetric functions that

$$\left(\sum_{n\geq 0} e_n(-t)^n\right)^{-1} = \sum_{n\geq 0} h_n t^n$$

in $\Lambda_{\mathbf{Q}}[[t]]$ (see, e.g., [**14**, p. 14]). Therefore, by (3) and (4), we obtain that

$$\begin{aligned}\sum_{n\geq 0} \xi(h_n)t^n &= \left(\sum_{n\geq 0} \xi(e_n)(-t)^n\right)^{-1} \\ &= \left(\frac{e^{t(x-1)}-x}{1-x}\right)^{-1} = 1 + \sum_{n\geq 1} \frac{A_n(x)}{xn!} t^n,\end{aligned}$$

where we have used a well-known generating function for Eulerian polynomials (see, e.g., [7, eq. [5j], p. 244]), and the thesis follows. □

We now come to one of the main results of this section.

THEOREM 4.2. *Let* $\xi : \Lambda_{\mathbf{Q}} \to \mathbf{Q}[x]$ *be the ring homomorphism defined by* (3) *and* λ *be a partition of* n. *Then*

$$\xi(p_\lambda) = \frac{z_\lambda}{n!} \sum_{\pi \in S_n(\lambda)} x^{e(\pi)}.$$

Proof. Define a second ring homomorphism $\bar{\xi} : \Lambda_{\mathbf{Q}} \to \mathbf{Q}[x]$ by letting

$$\bar{\xi}(p_i) \stackrel{\text{def}}{=} \frac{A_{i-1}(x)}{(i-1)!},$$

for $i \in \mathbf{P}$, and $\bar{\xi}(p_0) \stackrel{\text{def}}{=} 1$ (note that $\bar{\xi}$ is a well-defined ring homomorphism since the p_i's are algebraically independent and generate $\Lambda_{\mathbf{Q}}$ over $\mathbf{Q}$ [**14**, p. 16]). Then, for any partition $\lambda \stackrel{\text{def}}{=} (\lambda_1, \dots, \lambda_l)$, we have, by Theorem 3.1, that

$$\bar{\xi}(p_\lambda) = \prod_{i=1}^{l} \bar{\xi}(p_{\lambda_i}) = \prod_{i=1}^{l} \frac{A_{\lambda_i - 1}(x)}{(\lambda_i - 1)!} = \frac{z_\lambda}{|\lambda|!} \sum_{\pi \in S_n(\lambda)} x^{e(\pi)}. \tag{6}$$

Therefore, using a well-known identity in the theory of symmetric functions (see, e.g., [**14**, (2.14′), p. 17]), we obtain that, for $n \in \mathbf{P}$

$$\begin{aligned}\bar{\xi}(h_n) &= \bar{\xi}\left(\sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda\right) = \sum_{\lambda \vdash n} z_\lambda^{-1} \bar{\xi}(p_\lambda) \\ &= \sum_{\lambda \vdash n} \frac{1}{|\lambda|!} \sum_{\pi \in S_n(\lambda)} x^{e(\pi)} = \frac{1}{n!} \sum_{\pi \in S_n} x^{e(\pi)} = \frac{A_n(x)}{x n!}\end{aligned}$$

(where we have used Proposition 1.3.12 of [**23**]). By Theorem 4.1 this shows that $\bar{\xi}(h_n) = \xi(h_n)$ for all $n \in \mathbf{N}$ and since the h_n's are algebraically independent and generate $\Lambda_{\mathbf{Q}}$ over $\mathbf{Q}$ (see, e.g., [**14**, p. 14]) this implies that $\bar{\xi} \equiv \xi$, which, by (6), gives the desired result. □

Note that even though our definition (3) of ξ had nothing to do with excedances, they have now come naturally into the picture.

The preceding results show that $\xi(e_\lambda)$, $\xi(h_\lambda)$ and $\xi(p_\lambda)$ all have simple combinatorial interpretations and (by Theorem 3.2) only real zeros. It is therefore natural to ask whether $\xi(s_\lambda)$, $\xi(f_\lambda)$, and $\xi(m_\lambda)$ have combinatorial interpretations and whether they also have only real zeros or are symmetric and unimodal. Before doing this we note two general results that follow easily from Theorems 4.1 and 4.2 and that will be used in the sequel (we denote by $\Lambda_{\mathbf{Q}}^n$ the set of all elements of $\Lambda_{\mathbf{Q}}$ that are homogeneous of degree n).

PROPOSITION 4.3. *Let $n \in \mathbf{P}$, $k \in [n]$, and $a \in \Lambda_{\mathbf{Q}}^n$ be such that*

$$a = \sum_{\lambda} c_\lambda h_\lambda,$$

where $c_\lambda = 0$ unless $l(\lambda) = k$. Then $\xi(a)$ is a symmetric polynomial with $C(\xi(a)) = \frac{n-k}{2}$. Furthermore, if the c_λ's are nonnegative, the $\xi(a)$ is a unimodal polynomial.

Proof. As was observed in the proof of Theorem 3.2, the polynomials $A_n(x)$ are symmetric and unimodal with $C(A_n) = \frac{n+1}{2}$, for $n \geq 1$. This, by Proposition 2.1 and Theorem 4.1, implies that $\xi(h_\lambda)$ is a symmetric and unimodal polynomial with $C(\xi(h_\lambda)) = \frac{|\lambda|-l(\lambda)}{2}$, for all partitions λ, and the result follows. □

The proof of the following result is similar to that of the preceding one, and is therefore omitted.

PROPOSITION 4.4. *Let $n \in \mathbf{P}$, $k \in \mathbf{N}$, $k \leq n$, and $a \in \Lambda_{\mathbf{Q}}^n$ be such that*

$$a = \sum_{\lambda} b_\lambda p_\lambda,$$

where $b_\lambda = 0$ unless $m_1(\lambda) = k$. Then $\xi(a)$ is a symmetric polynomial with $C(\xi(a)) = \frac{n-k}{2}$. Furthermore, if the b_λ's are nonnegative, then $\xi(a)$ is a unimodal polynomial. □

We will see later that the conditions of the preceding two propositions are not as restrictive as they may appear at first glance.

We now turn our attention to the investigation of the polynomials $\xi(s_\lambda)$. We begin with the following result.

PROPOSITION 4.5. *Let $n \in \mathbf{P}$ and λ be a partition of n. Then*

$$\xi(s_\lambda) = \frac{1}{n!} \sum_{\{w \in S_n \,:\, c(w) \geq d(\lambda)\}} \chi^\lambda(w) x^{e(w)}, \tag{7}$$

where χ^λ is the irreducible character of S_n associated to λ.

Proof. It is a well-known result (see, e.g., [**14**, p. 62]) that

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi_\mu^\lambda p_\mu .$$

Applying ξ to both sides and using Theorem 4.2 we then obtain

$$\xi(s_\lambda) = \frac{1}{n!} \sum_\mu \chi_\mu^\lambda \sum_{w \in S_n(\mu)} x^{e(w)} = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w) x^{e(w)} . \tag{8}$$

Furthermore, it follows immediately from [**14**, Ex. 5, p. 64], that $\chi^\lambda(w) = 0$ if $c(w) \le d(\lambda) - 1$, and (7) follows. □

We note the following curious consequence of (8).

COROLLARY 4.6. *Let $n \in \mathbf{P}$ and λ be a partition of n. Then $\xi(s_\lambda)(x)$ equals the immanant corresponding to λ of the $n \times n$ matrix $(a_{ij})_{1 \le i,j \le n}$ where*

$$a_{ij} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i \le j, \\ x, & \text{if } i > j. \end{cases}$$ □

Even though immanants have received some attention in recent years (see, e.g., [**12**], [**13**], [**26**], [**27**]) the preceding result seems to be of little use in the understanding of the polynomials $\xi(s_\lambda)$. An interesting consequence of Proposition 4.5 is the following.

PROPOSITION 4.7. *Let $n \in \mathbf{P}$ and λ be a partition of n. Then*

$$\xi(s_\lambda)(0) = \frac{f^\lambda}{n!} .$$

Proof. Since the only permutation of S_n having no excedances is the identity, letting $x = 0$ in (7) yields that

$$\xi(s_\lambda)(0) = \frac{\chi^\lambda(\mathrm{Id})}{n!} .$$

But it is well known (see, e.g., [**14**, p. 62]) that $\chi^\lambda(\mathrm{Id}) = f^\lambda$, and the thesis follows. □

It is also possible to obtain an expression for $\xi(s_\lambda)$ using the inverse Kostka matrix.

PROPOSITION 4.8. *Let $n \in \mathbf{P}$ and λ be a partition of n. Then*

$$\xi(s_\lambda) = \frac{1}{n!} \sum_{\mu} \binom{n}{\mu_1, \mu_2, \dots} (K^{-1})_{\mu, \lambda'} (1-x)^{n-l(\mu)}. \tag{9}$$

Proof. It is well known (see, e.g., [**14**, Chapter I, §6]) that

$$s_\lambda = \sum_{\mu} (K^{-1})_{\mu, \lambda} h_\mu. \tag{10}$$

Applying the automorphism ω (see [**14**, Chapter I, §2]) to both sides of (10) we obtain that

$$s_{\lambda'} = \sum_{\mu} (K^{-1})_{\mu, \lambda} e_\mu. \tag{11}$$

Applying ξ to both sides of (11) and using (3) we get

$$\xi(s_{\lambda'}) = \sum_{\mu} (K^{-1})_{\mu, \lambda} \frac{(1-x)^{|\mu|-l(\mu)}}{\prod_{i \geq 1} (i!)^{m_i(\mu)}},$$

and the thesis follows. □

An important consequence of the preceding proposition is the following result.

PROPOSITION 4.9. *Let $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathscr{P}$. Then $\xi(s_\lambda)$ is divisible by $(1-x)^{|\lambda|-\lambda_1}$.*

Proof. From Proposition 4.8 we see that, for $i = 1, 2, \dots, |\lambda|$, the coefficient of $(1-x)^{|\lambda|-i}$ in $\xi(s_\lambda)$ equals

$$\frac{1}{n!} \sum_{\{\mu \,:\, l(\mu) = i\}} \binom{|\lambda|}{\mu_1, \mu_2, \dots} (K^{-1})_{\mu, \lambda'}.$$

But, from Theorem 2.4, we know that $(K^{-1})_{\mu, \lambda'} = 0$ unless $l(\mu) \leq l(\lambda')$ (since if $l(\mu) > l(\lambda')$ then there are no special border strip tabloids of shape λ' and type μ) and the thesis follows. □

Note that the above result is not best possible since, for example, if $\lambda = (3, 3)$ then $\xi(s_{(3,3)}) = \frac{5}{6!}(1-x)^4$ even though $|\lambda| - \lambda_1 = 3$.

It seems to be difficult, in general, to give an *explicit* formula for $\xi(s_\lambda)$. However, as we will now show, this is possible if λ is a hook.

We begin with the following result which follows easily from Theorem 2.4.

Lemma 4.10. *Let* $\lambda \in \mathcal{P}$, $n \stackrel{\text{def}}{=} |\lambda|$, $r \in [n]$ *and* $\mu \stackrel{\text{def}}{=} (r, 1^{n-r})$. *Then*

$$(K^{-1})_{\lambda\mu} = \binom{l(\lambda)}{m_1(\lambda), \dots, m_n(\lambda)} (-1)^{n-l(\lambda)+r-1} \frac{m_{\geq r}(\lambda)}{l(\lambda)} \tag{12}$$

where $m_{\geq r}(\lambda) \stackrel{\text{def}}{=} \sum_{i=r}^{n} m_i(\lambda)$.

Proof. To choose a special border strip tabloid of shape $(r, 1^{n-r})$ and type λ we may first choose the special border strip H that contains the upper left square of $D(r, 1^{n-r})$ (note that this implies that $|H| \geq r$) and then choose a special border strip tabloid of shape $(1^{n-|H|})$ and type $\lambda \backslash \{H\}$. But all the border strips of $D(1^{n-|H|})$ are vertical; hence such a special border strip tabloid is equivalent to a permutation of the multiset $\{1^{m_1(\lambda)}, \dots, n^{m_n(\lambda)}\} \backslash \{H\}$. Therefore

$$\begin{aligned}(K^{-1})_{\lambda,(r,1^{n-r})} &= \sum_{\{r \leq i \leq n \,:\, m_i(\lambda) > 0\}} (-1)^{i-r} (K^{-1})_{\lambda\backslash\{i\},(1^{n-i})} \\ &= \sum_{i=r}^{n} (-1)^{i-r} \binom{l(\lambda) - 1}{m_1(\lambda), \dots, m_i(\lambda) - 1, \dots, m_n(\lambda)} (-1)^{n-l(\lambda)+1-i},\end{aligned}$$

and (12) follows. □

Using the preceding lemma we obtain the following result. We denote by $S(n, k)$, for $n, k \in \mathbf{N}$, the Stirling numbers of the second kind, i.e. $S(n, k)$ is the number of partitions of $[n]$ into k blocks.

Theorem 4.11. *Let* $n \in \mathbf{P}$ *and* $r \in [n]$. *Then*

$$\begin{aligned}\xi(s_{(n-r+1,1^{r-1})}) = \frac{(-1)^{r-1}}{n!} &\sum_{k=1}^{n-r+1} (k-1)!(x-1)^{n-k} \\ &\cdot \sum_{i=r}^{n} \binom{n}{i} S(n-i, k-1).\end{aligned} \tag{13}$$

Proof. Let $\mu \stackrel{\text{def}}{=} (r, 1^{n-r})$. Then using Lemma 4.10, (9) becomes

$$\begin{aligned}(14)\quad &\xi(s_{(n-r+1,1^{r-1})})\\ &= \sum_{\lambda} \binom{l(\lambda)}{m_1(\lambda), \ldots, m_n(\lambda)} (-1)^{|\lambda|-l(\lambda)+r-1}\\ &\cdot \frac{(1-x)^{|\lambda|-l(\lambda)}}{\prod_{i=1}^{n}(i!)^{m_i(\lambda)}} \frac{m_{\geq r}(\lambda)}{l(\lambda)}\\ &= (-1)^{r-1} \sum_{\lambda} \frac{(l(\lambda)-1)!}{\prod_{i=1}^{n}(m_i(\lambda)!)(i!)^{m_i(\lambda)}} m_{\geq r}(\lambda)(x-1)^{|\lambda|-l(\lambda)}.\end{aligned}$$

But if $l(\lambda) > n-r+1$ then $m_i(\lambda) = 0$ for $i = r, \ldots, n$ (because if $m_i(\lambda) \geq 1$ for some $r \leq i \leq n$, then i is a part of λ and hence $l(\lambda) \leq n-i+1 \leq n-r+1$). Hence using Proposition 2.2 we deduce from (14) that

$$\begin{aligned}(15)\quad &\xi(s_{(n-r+1,1^{r-1})})\\ &= \frac{(-1)^{r-1}}{n!} \sum_{k=1}^{n-r+1} (k-1)!(x-1)^{n-k} \sum_{\{\lambda : l(\lambda)=k\}} \sum_{i=r}^{n} m_i(\lambda)\mathscr{P}_\lambda(n).\end{aligned}$$

To prove (13) it is therefore sufficient to show that

$$(16)\qquad \sum_{\{\lambda : l(\lambda)=k\}} m_i(\lambda)\mathscr{P}_\lambda(n) = \binom{n}{i} S(n-i, k-1).$$

But the LHS of (16) counts pairs (π, S) where π is a partition of $[n]$ into k blocks, having at least one block of size i, and S is a block of π of size i, while the RHS of (16) counts pairs (π', S) where S is a subset of $[n]$ of size i and π' is a partition of $[n]\backslash S$ into $k-1$ blocks. Since the map $(\pi, S) \mapsto (\pi\backslash S, S)$ is clearly a bijection between these sets of pairs, (16) follows and this concludes the proof. □

We should mention that the preceding theorem can also be derived from Ex. 9 on p. 30 of **[14]**. However, we thought a combinatorial proof to be more illuminating.

As an immediate consequence of (15), Theorems 4.1 and 4.11 we obtain the following expansion for Eulerian polynomials, when expressed in terms of powers of $(x-1)$, originally due to Frobenius (see [7, Theorem E, p. 244]).

COROLLARY 4.12. *Let* $n \in \mathbf{P}$. *Then*

$$A_n(x) = x \sum_{k=0}^{n} k! S(n, k)(x-1)^{n-k}. \tag{17}$$

Proof. Letting $r = 1$ in (15) we obtain that

$$\xi(s_{(n)}) = \frac{1}{n!} \sum_{\lambda} l(\lambda)! \mathscr{P}_\lambda(n)(x-1)^{n-l(\lambda)}. \tag{18}$$

But it is well known [**14**, p. 26] that $s_{(n)} = h_n$ in $\Lambda_{\mathbf{Q}}$. Hence (17) follows from (18) and Theorem 4.1. □

Besides giving a nice combinatorial interpretation to $\xi(s_\lambda)$ when λ is a hook, Theorem 4.11 can also be used to obtain a simple recurrence relation satisfied by these polynomials. In order to state the recurrence relation in a concise form it is convenient to normalize the $\xi(s_\lambda)$'s as follows. For $n \in \mathbf{P}$ and $r \in [n]$ we let

$$T_{n,r}(x) \stackrel{\text{def}}{=} (-1)^{r-1} x^{n-1} n! \xi(s_{(n-r+1, 1^{r-1})}) \left(\frac{x+1}{x}\right). \tag{19}$$

We then have the following result.

THEOREM 4.13. *Let* $n \in \mathbf{P}$ *and* $r \in [n]$. *Then the polynomials* $T_{n,r}(x)$ *defined by* (19) *satisfy the recurrence relation*

$$T_{n,r}(x) = T_{n-1,r-1}(x) + xT_{n-1,r}(x) + (x^2+x)T'_{n-1,r}(x) \tag{20}$$

for $r \geq 2$, *with the initial conditions*

$$T_{n,1}(x) = \frac{x^n}{x+1} A_n\left(\frac{x+1}{x}\right), \qquad T_{n,n}(x) = 1$$

for $n \in \mathbf{P}$.

Proof. Theorem 4.11 (and its proof) show that the coefficient of x^{k-1} in $T_{n,r}(x)$ counts pairs (π, S) where S is a subset of $[n]$ of size $\geq r$ and π is an ordered partition of $[n]\backslash S$ into $k-1$ blocks (let $T_{n,r}^{(k)}$ denote this number). If $n \in S$, then the map $(\pi, S) \mapsto (\pi, S\backslash\{n\})$ is a bijection with the pairs enumerated by $T_{n-1,r-1}^{(k)}$. If $n \notin S$, then there are two cases. If $\{n\}$ is a block of π, then $(\pi\backslash\{n\}, S)$ is a pair enumerated by $T_{n-1,r}^{(k-1)}$ and each such pair is obtained $k-1$ times (since π is *ordered*). If n is in a block of π of size ≥ 2, then removing it from this block yields a pair (π', S)

enumerated by $T^{(k)}_{n-1,r}$, and each such pair is obtained $k-1$ times (since π has $k-1$ blocks). Therefore

$$T^{(k)}_{n,r} = T^{(k)}_{n-1,r-1} + (k-1)(T^{(k-1)}_{n-1,r} + T^{(k)}_{n-1,r}) \tag{21}$$

for $n \geq 2$ and $k, r \in [n]$. Multiplying both sides of (21) by x^{k-1} and summing for $k = 1, \dots, n$ now yields (20); the initial conditions follow immediately from (3), (5), and (19), and the well-known facts [**14**, p. 26] that $s_{(n)} = h_n$, $s_{(1^n)} = e_n$, for $n \in \mathbf{P}$. $\square$

We now come to our main result about the polynomials $\xi(s_\lambda)$. Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ let $d \stackrel{\text{def}}{=} d(\lambda)$. We will find it convenient to associate to λ two partitions $\alpha(\lambda) \stackrel{\text{def}}{=} (\alpha_1, \dots, \alpha_d)$ and $\beta(\lambda) \stackrel{\text{def}}{=} (\beta_1, \dots, \beta_d)$ defined by

$$\alpha_i \stackrel{\text{def}}{=} \lambda_i - d(\lambda), \qquad \beta_i \stackrel{\text{def}}{=} \lambda'_i - d(\lambda),$$

for $i = 1, \dots, d(\lambda)$. Note that this implies that

$$\lambda = (\alpha_1+d-1, \alpha_2+d-2, \dots, \alpha_d | \beta_1+d-1, \beta_2+d-2, \dots, \beta_d) \tag{22}$$

if λ is written in Frobenius notation. For example, if $\lambda = (7, 6, 3, 3, 1, 1)$ then $d(\lambda) = 3$, $\alpha(\lambda) = (4, 2)$, and $\beta(\lambda) = (3, 1, 1)$. We also let $\tilde{\alpha} \stackrel{\text{def}}{=} (\alpha_1 - \alpha_d, \alpha_1 - \alpha_{d-1}, \dots, \alpha_1 - \alpha_2)$. Recall that the *sum* of two partitions $\lambda \stackrel{\text{def}}{=} (\lambda_1, \lambda_2, \dots)$ and $\mu \stackrel{\text{def}}{=} (\mu_1, \mu_2, \dots)$ is the partition $\lambda + \mu \stackrel{\text{def}}{=} (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$.

THEOREM 4.14. *Let $n \in \mathbf{P}$ and λ be a partition of n. Then the polynomial $\xi(s_\lambda)$ has degree $|\lambda| - d(\lambda)$ and leading coefficient*

$$(-1)^{|\beta(\lambda)|} \frac{f^{\gamma(\lambda)\backslash\delta(\lambda)}}{|\lambda|!}$$

where $\gamma(\lambda) \stackrel{\text{def}}{=} (\alpha_1 + d(\lambda))^{d(\lambda)} + \beta(\lambda)$, and $\delta(\lambda) = \tilde{\alpha}(\lambda)$.

Proof. Note first that if $\lambda = (n-r+1, 1^{r-1})$ for some $r \in [n]$ then $\tilde{\alpha}(\lambda) = \varnothing$ and $(\lambda_1)^{d(\lambda)} + \beta(\lambda) = (n)$ so that

$$(-1)^{|\beta(\lambda)|} \frac{f^{\gamma(\lambda)\backslash\delta(\lambda)}}{n!} = (-1)^{r-1} \frac{f^{(n)}}{n!} = \frac{(-1)^{r-1}}{n!}$$

and the result follows directly from Theorem 4.11. So assume that the same result holds for hooks and let $\lambda = (a_1, \dots, a_r | b_1, \dots, b_r)$, where $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$, and $r \stackrel{\text{def}}{=} d(\lambda)$.

It is a well-known fact that in the theory of symmetric functions (see, e.g., [**14**, Ex. 9, p. 30]) that

$$s_\lambda = \det[s_{(a_i|b_j)}]_{1\le i,j\le r}\,. \tag{23}$$

Applying now the homomorphism ξ to both sides of (23) we obtain that

$$\begin{aligned}\xi(s_\lambda) &= \det[\xi(s_{(a_i|b_j)})]_{1\le i,j\le r}\\ &= \sum_{\sigma\in S_r}\operatorname{sgn}(\sigma)\xi(s_{(a_1|b_{\sigma(1)})})\cdots\xi(s_{(a_r|b_{\sigma(r)})})\,.\end{aligned} \tag{24}$$

Since the thesis holds for hooks, we know that $\xi(s_{(a_i|b_j)})$ is a polynomial of degree a_i+b_j with leading coefficient $(-1)^{b_j}/(a_i+b_j+1)!$. Therefore, from (24), we conclude that $\deg(\xi(s_\lambda))\le\sum_{i=1}^{d(\lambda)}(a_i+b_j)=|\lambda|-d(\lambda)$ and the coefficient of $x^{|\lambda|-d(\lambda)}$ in $\xi(s_\lambda)$ equals

$$\begin{aligned}&(-1)^{\sum_{i=1}^r b_i}\det\left[\frac{1}{(a_i+b_j+1)!}\right]_{1\le i,j\le r}\\ &=(-1)^{\sum_{i=1}^r b_i+\binom{r}{2}}\det\left[\frac{1}{(b_i+a_{r+1-j}+1)!}\right]_{1\le i,j\le r}\\ &=(-1)^{|\beta(\lambda)|+2\binom{r}{2}}\det\left[\frac{1}{(\gamma_i-i-\delta_j+j)!}\right]_{1\le i,j\le r}\end{aligned} \tag{25}$$

and since it is well known that this last determinant equals $f^{\gamma\backslash\delta}/|\lambda|!$ (see, e.g., [**11**, §5.4.8]), the thesis follows. □

Theorem 4.14 has several interesting consequences.

COROLLARY 4.15. *Let $n\in\mathbf{P}$ and λ be a partition of n. Then*

$$\begin{aligned}\sum_{\{\mu\,:\,l(\mu)=d(\lambda)\}}\binom{n}{\mu_1,\mu_2,\dots}(K^{-1})_{\mu,\lambda'}\\ =(-1)^{n-d(\lambda)+|\beta(\lambda)|}f^{\gamma(\lambda)\backslash\delta(\lambda)}\,.\end{aligned} \tag{26}$$

Proof. This follows immediately from Proposition 4.8 and Theorem 4.14. □

COROLLARY 4.16. *Let $n\in\mathbf{P}$ and λ be a partition of n. Then*

$$\sum_{\{w\in S_n\,:\,e(w)=n-d(\lambda)\}}\chi^\lambda(w)=(-1)^{|\beta(\lambda)|}f^{\gamma(\lambda)\backslash\delta(\lambda)}\,. \tag{27}$$

Proof. This follows immediately from Proposition 4.5 and Theorem 4.14. □

COROLLARY 4.17. *Let $\lambda \in \mathscr{P}$ be such that $\alpha(\lambda) = \varnothing$. Then*

$$\xi(s_\lambda) = \frac{f^\lambda}{|\lambda|!}(1-x)^{|\lambda|-d(\lambda)}.$$

Proof. If $\alpha(\lambda) = \varnothing$ then $\lambda_1 = d(\lambda)$ and the thesis follows from Propositions 4.7 and 4.9 and Theorem 4.14. □

From Proposition 4.5 and the preceding corollary we also obtain the following result.

COROLLARY 4.18. *Let $\lambda \in \mathscr{P}$ be such that $\alpha(\lambda) = \varnothing$ and $n \stackrel{\text{def}}{=} |\lambda|$. Then*

$$\sum_{\{w \in S_n \,:\, e(w)=i\}} \chi^\lambda(w) = (-1)^i \binom{n-d(\lambda)}{i} f^\lambda,$$

for $i = 0, 1, 2, \ldots$. □

It would be interesting to obtain combinatorial proofs of (26) and (27) as they might shed some light on the combinatorial interpretation of the coefficient of x^i in $\xi(s_\lambda)$ also for $i < |\lambda| - d(\lambda)$. In particular, we feel that a Schensted-type correspondence should exist that proves (26).

So far we have concentrated on the combinatorial properties of $\xi(s_\lambda)$. It seems to be difficult to say anything general about the unimodality properties of $\xi(s_\lambda)$. In this direction we are only able to give the following result, which follows easily from Proposition 4.3.

PROPOSITION 4.19. *Let $\lambda \in \mathscr{P}$ be such that $\beta(\lambda) = \varnothing$. Then $\xi(s_\lambda)$ is a symmetric polynomial with center of symmetry at $\frac{|\lambda|-d(\lambda)}{2}$.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$. Since $\beta(\lambda) = \varnothing$ this means that $\lambda_k \geq k$ and hence that $k = d(\lambda)$. Since $\lambda_k > k-1$, the Jacobi-Trudi identity implies that s_λ satisfies the conditions of Proposition 4.3, and the thesis follows. □

We now look at some properties of the polynomials $\xi(m_\lambda)$. To do this we will first need the following result, which is also of independent interest.

LEMMA 4.20. *Let $n \in \mathbf{P}$. Then*

$$p_n = \sum_{\mu \vdash n} (-1)^{n-l(\mu)} \binom{l(\mu)}{m_1(\mu), m_2(\mu), \ldots} \frac{n}{l(\mu)} e_\mu.$$

Proof. Let, for convenience, $P(t) \stackrel{\text{def}}{=} \sum_{r\geq 0} p_{r+1}t^r$ and $E(t) \stackrel{\text{def}}{=} \sum_{r\geq 0} e_r t^r$. Then it is well known (see, e.g., [**14**, p. 16]) that

$$P(-t) = E'(t)/E(t)$$

(as formal power series in $\Lambda_{\mathbf{Q}}[[t]]$). Therefore

$$\begin{aligned} P(-t) &= E'(t)\sum_{r\geq 0}(1-E(t))^r \\ &= E'(t)\sum_{r\geq 0}\left(-\sum_{i\geq 1} e_i t^i\right)^r \\ &= \sum_{r\geq 0}(r+1)e_{r+1}t^r\sum_{\lambda}(-1)^{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \dots} e_\lambda t^{|\lambda|}. \end{aligned}$$

Equating the coefficients of t^n on both sides of the equality we obtain

$$\begin{aligned} (-1)^n p_{n+1} &= \sum_{r=0}^{n}(r+1)e_{r+1}\sum_{\lambda\vdash n-r}(-1)^{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \dots} e_\lambda \\ &= \sum_{\mu\vdash n+1} e_\mu \sum_{r\geq 1}(-1)^{l(\mu)-1}\binom{l(\mu)}{m_1(\mu), m_2(\mu), \dots}\frac{r m_r(\mu)}{l(\mu)} \end{aligned}$$

for $n \in \mathbf{N}$, and the thesis follows. □

We can now prove our first main result on the polynomials $\xi(m_\lambda)$.

THEOREM 4.21. *Let* $n \in \mathbf{P}$ *and* $\lambda = (\lambda_1, \lambda_2, \dots)$ *be a partition of* n. *Then* $\xi(m_\lambda)$ *is a polynomial of degree* $|\lambda| - 1$ *and leading coefficient*

$$\frac{(-1)^{l(\lambda)-1}}{n!}\frac{|\lambda|}{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \dots}.$$

Furthermore, $\xi(m_\lambda)$ *is divisible by* $(1-x)^{|\lambda|-\lambda_1}$.

Proof. Since the e_λ's are a basis for Λ we have that

$$m_\lambda = \sum_{\mu} A_{\lambda,\mu}e_\mu \tag{28}$$

for some integers $A_{\lambda\mu}$, where the sum runs over all partitions of $|\lambda|$. Applying ξ to both sides of (28) yields

$$\xi(m_\lambda) = \sum_{\mu} A_{\lambda,\mu}\frac{(1-x)^{|\lambda|-l(\mu)}}{\prod_{i\geq 1}(\mu_i!)}. \tag{29}$$

But it is known (see, [**14**, Chapter I, §6]) that if $A_{\lambda,\mu} \neq 0$ then $\mu \geq \lambda'$ (in the reverse lexicographic ordering) and hence, in particular, $l(\mu) \leq l(\lambda') = \lambda_1$. Hence we conclude from (29) that $(1-x)^{|\lambda|-\lambda_1}$ divides $\xi(m_\lambda)$. Furthermore, there follows from (29) that $\deg(\xi(m_\lambda)) \leq |\lambda|-1$ and that the coefficient of $x^{|\lambda|-1}$ in $\xi(m_\lambda)$ equals

$$\frac{(-1)^{n-1}}{n!}A_{\lambda,(n)}.$$

But, using some well-known properties of the operator $\omega\colon \Lambda \to \Lambda$ and of the inner product $\langle\ ,\ \rangle$ on Λ (see [**14**, Chapter I, §4] for details) we obtain from (28) and Lemma 4.20 that

$$\begin{aligned} A_{\lambda,(n)} &= \langle m_\lambda, f_{(n)}\rangle = \langle \omega(m_\lambda), \omega(f_{(n)})\rangle \\ &= \langle f_\lambda, m_{(n)}\rangle = \langle f_\lambda, p_n\rangle \\ &= (-1)^{n-l(\lambda)}\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \dots}\frac{n}{l(\lambda)}, \end{aligned}$$

and the thesis follows. □

Note that the second part of the above result is not best possible since, for example, if $\lambda = (3, 3)$ then $\xi(m_{(3,3)}) = \frac{-1}{240}x(x-1)^4$ even though $|\lambda| - \lambda_1 = 3$.

It is also possible to obtain an analogue of Proposition 4.5 for the polynomials $\xi(m_\lambda)$. Given $n \in \mathbf{P}$ and a partition λ of n we let P_λ be the digraph consisting of the disjoint union of $m_i(\lambda)$ directed paths of size i, for $i \geq 1$. We then define a function $I_\lambda\colon S_n \to \mathbf{N}$ by letting, for $w \in S_n$, $I_\lambda(w)$ be the number of (directed) subgraphs of the functional digraph of w (see, e.g., [7, p. 29]) isomorphic to P_λ.

PROPOSITION 4.22. *Let $n \in \mathbf{P}$ and λ be a partition of n. Then*

$$\xi(m_\lambda) = \frac{(-1)^{n-l(\lambda)}}{n!}\sum_{\{w\in S_n\,:\,c(w)\leq l(\lambda)\}} \operatorname{sgn}(w)I_\lambda(w)x^{e(w)}. \tag{30}$$

Proof. It follows from [**27**, Proposition 1.1] (see also [**26**, §3]) that

$$m_\lambda = \frac{(-1)^{n-l(\lambda)}}{n!}\sum_{w\in S_n}\operatorname{sgn}(w)I_\lambda(w)p_{\rho(w)}, \tag{31}$$

where $\rho(w)$ is the cycle type of w. But it is clear that I_λ is a class function and that $I_\lambda(w) = 0$ unless $c(w) \leq l(\lambda)$. Hence we may rewrite (31) as

$$m_\lambda = (-1)^{n-l(\lambda)}\sum_{\{\mu\,:\,l(\mu)\leq l(\lambda)\}}(-1)^{n-l(\mu)}I_\lambda^\mu\frac{p_\mu}{z_\mu}, \tag{32}$$

where I_λ^μ denotes the value of I_λ on permutations of cycle type μ. Now applying ξ to both sides of (32) and using Theorem 4.2 yields

$$\xi(m_\lambda) = \frac{(-1)^{n-l(\lambda)}}{n!} \sum_{\{\mu\,:\,l(\mu)\le l(\lambda)\}} (-1)^{n-l(\mu)} I_\lambda^\mu \sum_{w\in S_n(\mu)} x^{e(w)},$$

and (30) follows. □

Note that it is possible to derive the first part of Theorem 4.21 from Proposition 4.22. However, we thought a self-contained proof to be preferable. As an immediate consequence of the preceding proposition we obtain the following result.

COROLLARY 4.23. *Let $n \in \mathbf{P}$ and λ be a partition of n, $\lambda \neq (1^n)$. Then $\xi(m_\lambda)$ is divisible by x.* □

As is the case for the $\xi(s_\lambda)$'s, it seems to be difficult, in general, to give an explicit formula for $\xi(m_\lambda)$, except when λ is a hook. In order to do this we will first need the following result.

LEMMA 4.24. *Let $n \in \mathbf{P}$ and $r \in [n]$. Then*

$$m_{(n-r+1,\,1^{r-1})} = \sum_{i=0}^{r-1} (-1)^{r-1-i} p_{n-i} e_i. \tag{33}$$

Proof. Note that, for $r \in [n]$,

$$\begin{aligned} p_{n-r+1}e_{r-1} &= \sum_{i\ge1} x_i^{n-r+1} e_{r-1}(x_1, x_2, \ldots) \\ &= \sum_{i\ge1} x_i^{n-r+1}[e_{r-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots) \\ &\qquad\qquad + x_i e_{r-2}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots)] \\ &= m_{(n-r+1,\,1^{r-1})} + m_{(n-r+2,\,1^{r-2})}, \end{aligned}$$

(where $m_{(n-r+2,\,1^{r-2})} \stackrel{\text{def}}{=} 0$ if $r = 1$). Inverting this system of linear equations yields (33). □

We can now prove the following surprising result.

THEOREM 4.25. *Let $n \in \mathbf{P}$ and $r \in [n]$. Then*

$$\xi(m_{(n-r+2,\,1^{r-1})}) = x\xi(s_{(n-r+1,\,1^{r-1})}).$$

Proof. Note that it follows from Theorems 4.1 and 4.2 that

$$\xi(p_{n+1}) = x\xi(h_n)$$

for $n \in \mathbf{P}$. Hence from Lemma 4.24 we conclude that

$$\begin{aligned}\xi(m_{(n-r+2,1^{r-1})}) &= \sum_{i=0}^{r-1}(-1)^{r-1-i}\xi(p_{n+1-i})\xi(e_i)\\ &= x\sum_{i=0}^{r-1}(-1)^{r-1-i}\xi(h_{n-i})\xi(e_i)\\ &= x\xi(s_{(n-r+1,1^{r-1})}),\end{aligned}$$

where in the last equality we have used a well-known symmetric functions identity (see [**14**, Ex. 9, p. 30]). □

Since so many of the results that we have proved for the $\xi(s_\lambda)$'s have an analogue for the $\xi(m_\lambda)$'s it is natural to ask whether Corollary 4.17 and Proposition 4.19 also have an analogue for the latters. Unfortunately,

$$\xi(m_{(3,3,3)}) = \frac{3}{9!}x(x+1)(x-1)^6,$$

which shows that the condition $\alpha(\lambda) = \varnothing$ is not sufficient to imply that $\xi(m_\lambda)/x$ is a power of $(1-x)$. However, a much stronger version of Proposition 4.19 holds for the $\xi(m_\lambda)$'s, in fact we have the following result.

PROPOSITION 4.26. *Let $\lambda \in \mathscr{P}$ be such that $m_1(\lambda) = 0$. Then $\xi(m_\lambda)$ is a symmetric polynomial with center of symmetry at $\frac{|\lambda|}{2}$.*

Proof. Let $\lambda \in \mathscr{P}$ be such that $m_1(\lambda) = 0$. Then it follows immediately from their definition that (if $|\mu| = |\lambda|$) then $I_\lambda^\mu = 0$ unless $m_1(\mu) = 0$. Therefore from (32) we conclude that m_λ satisfies the conditions of Proposition 4.4, and the thesis follows. □

We conclude this section by investigating the polynomials $\xi(f_\lambda)$, where f_λ denotes the "forgotten" symmetric function associated to λ. These functions are defined by $f_\lambda \stackrel{\text{def}}{=} \omega(m_\lambda)$ and we refer the reader to [**14**, Chapter I, §2] and [**8**] for further information about them. The theory of the $\xi(f_\lambda)$'s is parallel to that of the $\xi(m_\lambda)$'s but simpler.

THEOREM 4.27. *Let $n \in \mathbf{P}$ and $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n. Then $\xi(f_\lambda)$ is a polynomial of degree $|\lambda| - 1$ and leading coefficient*

$$\frac{(-1)^{|\lambda|-l(\lambda)}}{n!} \frac{|\lambda|}{l(\lambda)} \binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \dots}.$$

Proof. Since the e_λ's are a basis for Λ we have that

$$f_\lambda = \sum_\mu C_{\lambda,\mu} e_\mu$$

for some integers $C_{\lambda,\mu}$, where the sum runs over all partitions of $|\lambda|$. Reasoning as in the proof of Theorem 4.21 we conclude that $\deg(\xi(f_\lambda)) \leq |\lambda| - 1$ and that the coefficient of $x^{|\lambda|-1}$ in $\xi(f_\lambda)$ equals

$$\frac{(-1)^{n-1}}{n!} C_{\lambda,(n)}.$$

But

$$f_{(n)} = \omega(m_{(n)}) = \omega(p_n) = (-1)^{n-1} p_n.$$

Hence

$$\begin{aligned} C_{\lambda,(n)} &= \langle f_\lambda, f_{(n)} \rangle = (-1)^{n-1} \langle f_\lambda, p_n \rangle \\ &= (-1)^{l(\lambda)-1} \binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \dots} \frac{n}{l(\lambda)}, \end{aligned}$$

by Lemma 4.20, and the thesis follows. □

PROPOSITION 4.28. *Let $n \in \mathbf{P}$ and λ be a partition of n. Then*

$$\xi(f_\lambda) = \frac{(-1)^{n-l(\lambda)}}{n!} \sum_{\{w \in S_n : c(w) \leq l(\lambda)\}} I_\lambda(w) x^{e(w)}.$$

Proof. Applying the automorphism ω to both sides of (32) yields

$$f_\lambda = (-1)^{n-l(\lambda)} \sum_{\{\mu : l(\mu) \leq l(\lambda)\}} I_\lambda^\mu \frac{p_\mu}{z_\mu};$$

reasoning now as in the proof of Proposition 4.22 gives the desired result. □

COROLLARY 4.29. *Let $n \in \mathbf{P}$ and λ be a partition of n, $\lambda \neq (1^n)$. Then $\xi(f_\lambda)$ is divisible by x.* □

PROPOSITION 4.30. *Let $n \in \mathbf{P}$ and $r \in [n]$. Then*

$$(-1)^{n-r} \xi(f_{(n-r+1, 1^{r-1})})$$

is a unimodal polynomial.

Proof. If $r = n$ then $f_{(1^n)} = \omega(m_{(1^n)}) = \omega(e_n) = h_n$ and the thesis follows from Proposition 4.3. If $r \leq n-1$ then applying the automorphism ω to both sides of (33) we obtain that

$$f_{(n-r+1, 1^{r-1})} = (-1)^{n-r} \sum_{i=0}^{r-1} p_{n-i} h_i .$$

Now applying ξ to both sides of this equality we get

$$\xi(f_{(n-r+1, 1^{r-1})}) = (-1)^{n-r} \sum_{i=0}^{r-1} \xi(p_{n-i}) \xi(h_i) . \tag{34}$$

But, from Propositions 4.3, 4.4, and 2.1 we deduce that $\xi(p_{n-i})\xi(h_i)$ is a symmetric unimodal polynomial with center of symmetry at $\frac{n-i}{2} + \frac{i-1}{2} = \frac{n-1}{2}$ if $1 \leq i \leq r-1$ and $\frac{n}{2}$ if $i=0$. Therefore $\sum_{i=0}^{r-1} \xi(p_{n-i})\xi(h_i)$ is a unimodal (but not necessarily symmetric) polynomial, and the thesis follows from (34). □

It is not hard to give a simple combinatorial interpretation of the polynomials considered in the previous result using Proposition 4.28. Given $n \in \mathbf{P}$ and $w \in S_n$ we let $c_i(w)$ be the number of cycles of w of length i, for $i = 1, \ldots, n$.

Proposition 4.31. *Let* $n \in \mathbf{P}$ *and* $r \in [n]$. *Then*

$$(-1)^{n-r} \xi(f_{(n-r+1, 1^{r-1})}) = \frac{1}{n!} \sum_{\{w \in S_n \,:\, c(w) \leq r\}} \sum_{i=n-r+1}^{n} i c_i(w) x^{e(w)} . \tag{35}$$

Proof. It follows immediately from its definition that if $\lambda = (n - r+1, 1^{r-1})$ and $w \in S_n$ then $I_\lambda(w)$ equals the sum of the lengths of the cycles of w having size $\geq n-r+1$, and (35) follows from Proposition 4.28. □

The proof of the following result is similar to that of Proposition 4.26 and is therefore omitted.

Proposition 4.32. *Let* $\lambda \in \mathscr{P}$ *be such that* $m_1(\lambda) = 0$. *Then* $(-1)^{|\lambda| - l(\lambda)} \xi(f_\lambda)$ *is a symmetric unimodal polynomial with center of symmetry at* $\frac{|\lambda|}{2}$. □

5. Conjectures and open problems. There are several problems that are suggested by the research presented in this paper. Most of them concern the ring homomorphism ξ defined by (3). In view of Theorem 4.21 and Proposition 4.9 it is natural to define, for $\lambda \in \mathscr{P}$, $S_\lambda(x)$

and $M_\lambda(x)$ to be the unique polynomials such that

$$\xi(s_\lambda) = \frac{(-1)^{|\beta(\lambda)|}}{|\lambda|!}(x-1)^{p(\lambda)}S_\lambda(x),$$

$$\xi(m_\lambda) = \frac{(-1)^{l(\lambda)-1}}{|\lambda|!}(x-1)^{q(\lambda)}M_\lambda(x),$$

and $S_\lambda(1)$, $M_\lambda(1) \neq 0$, for some (necessarily unique) $p(\lambda)$, $q(\lambda) \in \mathbf{N}$. Regarding these polynomials we feel that the following statements are true.

Conjecture 5.1. Let $\lambda \in \mathscr{P}$. Then $M_\lambda(x)$ has only real zeros.

Conjecture 5.2. Let $\lambda \in \mathscr{P}$. Then $S_\lambda(x)$ has only real zeros.

These conjectures have been verified for $|\lambda| \leq 8$. Unfortunately, the interest in the above conjectures is somewhat diminished by the fact that the polynomials $M_\lambda(x)$ and $S_\lambda(x)$ do not always have nonnegative coefficients. For example, one can compute that $S_{(4,4)}(x) = 14(x^2 - 6x + 1)$ and $M_{(4,4)}(x) = 4x(x^2 - 16x + 1)$. In particular, this shows that $S_\lambda(x)$ need not be unimodal even if $\beta(\lambda) = \varnothing$. However, it would be interesting to know for which partitions λ we have that $S_\lambda(x)$, $M_\lambda(x) \in \mathbf{N}[x]$. In this respect, we feel that the following statement holds.

Conjecture 5.3. Let λ be a hook. Then the polynomial $S_\lambda(x)$ (equivalently, $M_\lambda(x)$) has nonnegative coefficients and only real zeros. In particular, it is log-concave and unimodal.

This conjecture has been verified for $|\lambda| \leq 8$. It is not hard to show that it holds if $l(\lambda) \leq 2$ or if $l(\lambda') \leq 3$, we have also verified that it holds if $3 \leq l(\lambda) \leq 5$ and $|\lambda| \leq 12$. As further evidence that $S_\lambda(x) \in \mathbf{N}[x]$ if λ is a hook let us mention that it follows easily from our definitions and Theorem 4.11 that

$$S_{(n-r+1,1^{r-1})}(1) = (n)_{n-r},$$

for $n \in \mathbf{P}$ and $r \in [n]$. Thus it is possible that a simple combinatorial interpretation exists for the coefficients of $S_\lambda(x)$ when λ is a hook.

From the algebraic point of view the most interesting problem about ξ is that of characterizing its kernel. Theorems 4.1 and 4.2 imply that

$$\xi(p_{n+1} - p_2 h_n) = 0,$$

for $n \in \mathbf{P}$. This naturally suggests the following question.

Problem 5.4. Does $\{p_{n+1} - p_2 h_n : n \in \mathbf{P}\}$ generate $\ker(\xi)$?

From the point of view of permutation enumeration we feel that the results presented in §3 naturally suggest the investigation of the following general question.

Problem 5.5. For which subsets $T \subseteq S_n$ does the polynomial $E_T(x) \stackrel{\text{def}}{=} \sum_{\pi \in T} x^{e(\pi)}$ have only real zeros?

Theorem 3.2 shows that the answer to the above problem is affirmative if T is a conjugacy class of S_n. That the answer is affirmative for $T = S_n$ is a well-known, but nontrivial, classical result (see, e.g., [7, p. 292, Ex. 3]) while for $T = D_n$ (where D_n denotes the set of all derangements of S_n) it was conjectured in [**3**, p. 1140] and recently proved by R. Canfield (private communication). That there should be many more subsets T for which Problem 5.5 has an affirmative solution is also suggested, in view of the close relationship existing between excedances and descents (see, e.g., [**23**, Proposition 1.3.12]), by the following conjecture, which was made by R. Stanley in 1986.

Conjecture 5.6. Let (P, ω) be a finite labeled poset and $\mathscr{L}(P, \omega) \subseteq S_{|P|}$ be its Jordan Hölder set (see, e.g., [**23**, p. 131], for the definition of this terminology). Then the polynomial $\sum_{\pi \in \mathscr{L}(P, \omega)} x^{d(\pi)}$ has only real zeros.

Conjecture 5.6 can be stated in many equivalent ways (see, e.g., [**2**, §§1.2, 1.4, and 6.3]) and is known as the "Poset Conjecture" or as the "Neggers-Stanley Conjecture" (since J. Neggers conjectured it in the case that ω is a natural labeling). It is known to be true for many infinite classes of posets but is still open in general, even for natural labelings. We refer the reader to [**2**], [**18**], [**25**], and [**29**] for further information about it.

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INSTITUT MITTAG LEFFLER
AURAVÄGEN 17
S-182 62 DJURSHOLM, SWEDEN