PACIFIC JOURNAL OF MATHEMATICS
Vol. 157, No. 2, 1993

AN ANALYTIC FAMILY OF UNIFORMLY BOUNDED REPRESENTATIONS OF A FREE PRODUCT OF DISCRETE GROUPS

JANUSZ WYSOCZAŃSKI

We construct for each $|z| < 1$ a uniformly bounded representation π_z of a free product group. The correspondence $z \mapsto \pi_z$ is proved to be analytic. The representations are irreducible if the free product factors are infinite groups. On free groups they have as coefficients block radial functions—gives thus a new series of representations. They can be made unitary iff $z \in (-\frac{1}{N-1}, 1)$.

This paper is devoted to the construction of a family $\{\pi_z : |z| < 1\}$ of uniformly bounded representations of a free product of infinite groups. The construction is based on the ideas of Pytlik and Szwarc, who considered free groups on countably many generators. We have investigated a family of block radial functions discovered by W. Młotkowski. The functions were defined as follows: for $|z| < 1$,

$$\varphi_z(x) = \begin{cases} 1 & \text{if } x = e, \\ \frac{(N-1)z+1}{Nz} z^{\|x\|} & \text{if } x \neq e. \end{cases}$$

Each of these functions turns out to be a matrix coefficient of one of our representations $\{A_z : |z| < 1\}$, namely:

$$\varphi_z(x) = \langle A_z(x)\xi, \xi\rangle$$

where ξ is the common cyclic vector. The constructed representations will be shown to be irreducible, except when $z = 0$ or $z = -\frac{1}{N-1}$, which independently follows from Szwarc's general theorem on the family $\{\varphi_z : |z| < 1\}$ (see **[Sz.2]** Theorem). In the two exceptional cases $z = 0$ and $z = -\frac{1}{N-1}$ we identify the representations with the regular and the quasi-regular representation, respectively.

Next we consider the problem of whether some of the representations $\{A_z\}$ can be made unitary. For this purpose we introduce a family of operators $\{V_z : z \in \Omega\}$ where $\Omega = \{|z| < 1\} \backslash (-1, -\frac{1}{N-1}]$ and intertwine each representation by a proper V_z. In this way we get (Theorem 11) an analytic family of uniformly bounded representations $\{\pi_z : z \in \Omega\}$ which are unitary if and only if $z \in (-\frac{1}{N-1}, 1)$. All the representations are irreducible if the free product factors are infinite. As a corollary we get the result of Młotkowski which characterizes those of his functions which are positive definite. For real z that belong to the segment $(-\frac{1}{N-1}, 1)$ the corresponding representation π_z is unitary and for other z from $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ π_z is uniformly bounded and cannot be unitarized (by the result of Młotkowski).

One way of dealing with free products of groups in the context of representations was presented by Iozzi and Picardello (**[I-P]**) and came from the theory of Figa-Talamanca and Picardello (**[F-P.1]**, **[F-P.2]**). It used the fact that the space of radial functions on the group $G = *_{i=1}^{N} \mathbb{Z}_k^{(i)}$ is an abelian algebra. Another approach was found by Pytlik and Szwarc (in **[P-S]**). By considering a special operation on free groups they constructed an analytic family of uniformly bounded representations. Later Bożejko gave a general construction of a regular free product of representations (**[B.2]**). This allowed him to treat both lengths (ordinary and block one) on free groups as special cases.

The operation of cutting the last letter from a word in a free group $(x_1 \cdot \dots \cdot x_n \mapsto x_1 \cdot \dots \cdot x_{n-1})$, introduced in **[P-S]**, may be thought of as the translation of a vertex of the tree of the group towards the vertex representing the group identity. This remark inspired us to look in a similar manner at free products. They act by automorphisms on the trees defined by J-P. Serre in **[S]**. Our idea was to consider the square of the translation "towards the identity" on these trees. In this way we got an operation which preserved the subsets $X_0 = \{\text{elements of the group}\}$ and $X_1 = \{\text{right cosets with respect to free product factors}\}$ of the set of vertices. In this paper we are dealing with its "restriction" to X_1.

Most important for us is the case of the free group $\mathbb{F}_N$ on N free generators. This group is a free product of N copies of the group $\mathbb{Z}$ of integers. The block length on $\mathbb{F}_N$ differs essentially from the ordinary length related to free generators. Therefore our family of uniformly bounded representations of the free group $\mathbb{F}_N$ seems to be new in comparison to those known so far (see **[F-P.1]**, **[M-Z]** and **[P-S]**). Actually, an open problem is to prove that our representations are not equivalent to those associated with the ordinary length on $\mathbb{F}_N$.

1. Preliminary notation. Our group G is a free product of a family $\{G_i : i \in I\}$ of groups. Here and subsequently we will assume that $I = \{1, \dots, N\}$, $2 \leq N < \infty$, and that each group G_i is countably infinite. The tree $\Gamma(G)$ of G defined by Serre (see **[S]**) consists of the set X of vertices and the set $E(G)$ of edges. The set X is a disjoint union of two subsets: $X_0 = G = \bigcup_{g\in G}\{g\}$—the set of elements of G, $X_1 = \{gG_i : i \in I,\ g \in G\}$—the set of cosets with respect to the subgroups G_i. The set of edges of $\Gamma(G)$ consists of pairs (g, gG_i) and (gG_i, g). In the sequel, $\mathbf{P}$ stands for the square of the translation towards the point $e = \gamma_0$, restricted to X_1. This means that $\mathbf{P}$ is defined on the linear span of X_1 by $\mathbf{P}(G_i) = 0$ for $i = 1, 2, \dots, N$ and if $g \in G\backslash\{e\}$ is of the form (called *the standard form* of g): $g = g_1 \cdot \cdots \cdot g_n$ where $g_k \in G_{i_k}\backslash\{e\}$ for $n \geq 1$ and $i_1 \neq i_2 \neq \cdots \neq i_n$, then $\mathbf{P}(gG_i) = (g_1 \cdot g_2 \cdot \cdots \cdot g_{n-1})G_{i_n}$ for $i \neq i_n$. The action of G on X_1 will be denoted by L: $L(g)(hG_i) = (gh)G_i$ for $g, h \in G$, $i \in \{1, 2, \dots, N\}$. This action is easily seen to be an isometry of $\Gamma(G)$ with respect to the natural distance on the tree.

2. The unboundedness of the operator $\mathbf{P}$. Set $\mathscr{H} = l_2(X_1)$. For every $g \in G$ the operator $L(g)$ extends to a unitary operator on $\mathscr{H}$. The operation $\mathbf{P}$ does not extend to a bounded operator on $\mathscr{H}$. In spite of this it plays the key role in our construction.

For fixed $z \in \mathbb{C}$ and a function $f \in \mathscr{H}$ with finite support one may write the formal series $\sum_{k=0}^{\infty} z^k \cdot \mathbf{P}^k f$. Here we understand that $\mathbf{P}$ acts as a linear operator on the linear span of X_1, denoted by $\mathscr{K}(X_1)$. In the sequel, this space will be identified with the subspace of finitely supported functions. Let us observe that the series has only finitely many nonzero terms. This follows from the fact that for every $\gamma \in X_1$ there exists $r \in \mathbb{N}$ such that $\mathbf{P}^r\gamma = 0$. Therefore the operator $(I - z\mathbf{P})$ is invertible on $\mathscr{K}(X_1)$ and $(I - z\mathbf{P})^{-1} f = \sum_{k=0}^{\infty} z^k \cdot \mathbf{P}^k f$ for all $f \in \mathscr{K}(X_1)$. For fixed $g \in G$ we will compare the actions of $\mathbf{P}$ and $L(g)\mathbf{P}L(g^{-1})$ on $\mathscr{K}(X_1)$. Assume that g has the representation $g = g_1 \cdot g_2 \cdot \cdots \cdot g_n$ in standard form and write: $\mathscr{G}(e) = \{G_i : i = 1, 2, \dots, N\}$,

$$\mathscr{G}(g) = \{g_0 \cdot g_1 \cdot g_2 \cdot \cdots \cdot g_m G_1 \in X_1 : 0 \leq m \leq n,\ g_0 = e,\ i \neq i_m\}.$$

Then we have the following

Lemma 1. *For every $\gamma \in X_1 \backslash \mathscr{G}(g)$ we have $\mathbf{P}\gamma = L(g)\mathbf{P}L(g^{-1})\gamma$.*

The proof is easily seen on the picture of the tree and we omit it here.

3. The uniformly bounded representations A_z. For $z \in \mathbb{C}$ and $g \in G$ define $A_z(g) = (I - z\mathbf{P})^{-1} L(g)(I - z\mathbf{P})$ which, at the least, may be considered as a linear operator on the vector space $\mathscr{K}(X_1)$ spanned by X_1. Using the expansion of $(I - z\mathbf{P})^{-1}$ on $\mathscr{K}(X_1)$ one gets for $f \in \mathscr{K}(X_1)$ the following expression for $A_z(g)f$.

LEMMA 2. *For every* $z \in \mathbb{C}$, $g \in G$, $f \in \mathscr{K}(X_1)$ *we have*

$$A_z(g)f = L(g)f + \sum_{k=0}^{\infty} z^{k+1}\mathbf{P}^k[\mathbf{P} - L(g)\mathbf{P}L(g^{-1})] \circ L(g)f .$$

For $|z| < 1$ the mapping $G \ni g \mapsto A_z(g)$ turns out to be a uniformly bounded representation of the group G in the Hilbert space $\mathscr{H}$.

THEOREM 3. *For* $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, A_z *extends to a uniformly bounded representation of G on* $\mathscr{H} = l_2(X_1)$ *and for every* $g \in G$

$$\|A_z(g)\| \leq 1 + 2\sqrt{N-1} \cdot \frac{|z|}{1-|z|} .$$

Proof. For $g = e$ the assertion of the theorem is trivial, because $A_z(e) = I$. Fix $g \in G\backslash\{e\}$ and consider the space $\mathscr{H}_g = l_2(\mathscr{G}(g)) \subseteq \mathscr{H}$. By Lemmas 1 and 2 the operator $A_z(g)L(g^{-1})$ acts as I on a dense subspace of an orthogonal complement of $\mathscr{H}_g$. So it may be extended onto the whole subspace $\mathscr{H}_g^{\perp}$ and $\|A_z(g)L(g^{-1})|_{\mathscr{H}_g^{\perp}}\| = 1$. Therefore we only need to look at the action of $A_z(g)L(g^{-1})$ on $\mathscr{H}_g$. Suppose that $f' = L(g)f \in \mathscr{H}_g$ and $f = \sum_{L_g\gamma\in\mathscr{H}_g} f(\gamma)\cdot\gamma$. The space $\mathscr{H}_g$ is finite dimensional so we may apply $A_z(g)$ to f (in fact $f \in \mathscr{K}(X_1)$). Write $\gamma_m = g_0 \cdot g_1 \cdot \cdots \cdot g_m G_{i_{m+1}}$ for $0 \leq m \leq n-1$ (where $g_0 = e$). Then we get the following formulas for $f' \in \mathscr{H}_g$:

$$\mathbf{P}f'(\gamma_m) = \sum_{i\neq i_{m+1}} f'(g_1 \cdot \cdots \cdot g_{m+1}G_i),$$

$$L(g)\mathbf{P}L(g^{-1})f'(\gamma_0) = \sum_{i\neq i_1} f'(G_i),$$

$$L(g)\mathbf{P}L(g^{-1})f'(\gamma_m) = \sum_{i\neq i_m, i_{m+1}} f'(g_1 \cdot \cdots \cdot g_m G_i) + f'(g_1 \cdot \cdots \cdot g_{m-1}G_{i_m})$$

since $g_1 \cdot \cdots \cdot g_m G_{i_m} = g_1 \cdot \cdots \cdot g_{m-1}G_{i_m}$. From them it follows that $\|\mathbf{P}f'\|^2 \leq (N-1)\cdot\|f'\|^2$ and similarly $\|L(g)\mathbf{P}L(g^{-1})f'\|^2 \leq (N-1)\cdot\|f'\|^2$. Observe that for $k \geq 2$, $\|\mathbf{P}^k f\| \leq \|\mathbf{P}f\|$ whenever $f \in \mathscr{H}_g$. To see this assume that $g = g_1 \cdot g_2 \cdot \cdots \cdot g_n$ is the representation of g of the standard form. We will denote by $\mathscr{F}(g)$ the set

$$\{g_0 \cdot g_1 \cdot \cdots \cdot g_{m-1}G_{i_m} : 0 \leq m < n,\ g_0 = e\}.$$

For every $k \geq 1$ and $f \in \mathscr{H}_g$, $\operatorname{supp}(\mathbf{P}^k f) \subseteq \mathscr{F}(g)$ because for each $\gamma \in \mathscr{G}(g)$, $\mathbf{P}\gamma \in \mathscr{F}(g)$. If we put $\gamma_m = g_0 \cdot g_1 \cdot \cdots \cdot g_m G_{i_{m+1}}$ $(g_0 = e)$ for $0 \leq m \leq n-1$ then

$$\mathbf{P}\gamma_m = \begin{cases} \gamma_{m-1} & \text{for } m = 1, 2, \ldots, n, \\ 0 & \text{for } m = 0. \end{cases}$$

This means that $\mathbf{P}$ acts as a shift on $\mathscr{F}(g)$. Similarly, one can see that

$$L(g)\mathbf{P}L(g^{-1})\gamma_m = \begin{cases} \gamma_{m+1} & \text{for } m = 0, 1, 2, \ldots, n-1, \\ 0 & \text{for } m = n, \end{cases}$$

and for $L(g)f \in \mathscr{H}_g$, $\operatorname{supp}(L(g)\mathbf{P}L(g^{-1})f) \subseteq \mathscr{F}(g)$. Therefore $\mathbf{P}$ and $L(g)\mathbf{P}L(g^{-1})$ have norms equal 1 on $l_2(\mathscr{F}(g))$. Consequently, for $\|g\| = n \geq 1$ and $f \in L(g^{-1})\mathscr{H}_g$, $\operatorname{supp}([\mathbf{P} - L(g)\mathbf{P}L(g^{-1})]L(g)f) \subseteq \mathscr{F}(g)$. Thus

$$\begin{aligned} \|A_z(g)f\| &\leq \|L(g)f\| \\ &\quad + \sum_{k=0}^{\infty} |z|^{k+1} \cdot \|\mathbf{P}^k \circ [\mathbf{P} - L(g)\mathbf{P}L(g^{-1})] \circ L(g)f\| \\ &\leq \|f\| + \sum_{k=0}^{\infty} |z|^{k+1} \cdot 2\sqrt{N-1} \cdot \|f\| \\ &= \|f\| \cdot \left\{1 + 2\sqrt{N-1} \cdot \frac{|z|}{1-|z|}\right\}. \end{aligned}$$

This is the desired conclusion. □

4. Some matrix coefficients of A_z. Let us adopt the following notation. Here and subsequently ξ_0 stands for the vector $G_1 + \cdots + G_N$ and $\xi = \frac{1}{\sqrt{N}} \cdot \xi_0$. Our next lemma exhibits a relation between φ_z and A_z.

LEMMA 4. *For every $g \in G$ we have $\varphi_z(g) = \langle A_z(g)\xi, \xi\rangle$.*

Proof. First we compute $A_z(g)\xi_0$. For this purpose we put $g = g_1 \cdot \cdots \cdot g_n$, $n \geq 1$ (the standard form), and $\gamma_m = g_0 \cdot g_1 \cdot \cdots \cdot g_m G_{i_{m+1}}$. Then

$$\begin{aligned} A_z(g)G_{i_n} &= \gamma_n + \sum_{k=0}^{\infty} z^{k+1}\mathbf{P}^{k+1}\gamma_n - \sum_{k=0}^{\infty} z^{k+1}\mathbf{P}^k L(g)\mathbf{P}L(g^{-1})\gamma_{n-1} \\ &= \sum_{k=0}^{\infty} z^k \mathbf{P}^k \gamma_n \end{aligned}$$

and, for $j \neq i_n$,

$$A_z(g)G_j = \sum_{k=0}^{\infty} z^k \mathbf{P}^k g G_j = gG_j + \sum_{k=0}^{\infty} z^{k+1}\mathbf{P}^k \gamma_n .$$

(Observe that $L(g)\mathbf{P}L(g^{-1})gG_i = 0$ for $i = 1, 2, \dots, N$.) In this way we obtain

$$A_z(g)\xi_0 = \sum_{i \neq i_n} gG_i + [1 + z(N-1)] \cdot \sum_{k=0}^{\infty} z^k \mathbf{P}^k \gamma_n .$$

Hence

$$\langle A_z(g)\xi_0 , \xi_0 \rangle = [1 + z(N-1)] \cdot z^{n-1} \cdot \langle G_{i_1} , \xi_0 \rangle = \frac{1 + z(N-1)}{z} \cdot z^n .$$

Therefore, for $g \neq e$,

$$\langle A_z(g)\xi , \xi \rangle = \frac{1 + z(N-1)}{Nz} \cdot z^n = \varphi_z(g) .$$

Obviously $A_z(e) = I$ and $\langle A_z(e)\xi , \xi \rangle = 1 = \varphi_z(e)$. This proves the lemma. □

5. The cyclic vector ξ for all A_z, $z \neq \frac{-1}{N-1}$. We will denote by $\mathscr{Z}$ the transformation from G into $\mathscr{H}(X_1)$ defined by the following formulas: $\mathscr{Z}(e) = \xi_0$ and $\mathscr{Z}(g) = \sum_{i \neq i_n} gG_i$ if g is of the standard form and $\|g\| = n \geq 1$. This transformation extends to a bounded operator from $l_2(G)$ into $\mathscr{H}$ which is injective.

REMARK. In our notation g stands for an element of G as well as for the characteristic function of $\{g\}$.

LEMMA 5. *The vector ξ is cyclic for A_z if and only if*

$$z \neq -\frac{1}{N-1} \qquad (|z| < 1) .$$

Proof. Let $g = g_0 \cdot g_1 \cdot \dots \cdot g_n$ be in the standard form and assume $n \geq 1$ and put $\overline{g} = g_0 \cdot g_1 \cdot \dots \cdot g_{n-1}$. Then we have formulas

$$A_z(g)\xi_0 = \mathscr{Z}(g) + [1 + z(N-1)] \cdot \sum_{k=0}^{\infty} z^k \mathbf{P}^k \gamma_n \quad \text{and}$$

$$A_z(\overline{g})\xi_0 = \mathscr{Z}(\overline{g}) + \frac{1}{z} \cdot [1 + z(N-1)] \cdot \sum_{k=1}^{\infty} z^k \mathbf{P}^k \gamma_n .$$

Subtraction of these formulas yields:

$$[A_z(g) - z \cdot A_z(\overline{g})]\xi_0 = \mathscr{Z}(g) - z \cdot \mathscr{Z}(\overline{g}) + [1 + z(N-1)] \cdot \gamma_n .$$

Suppose that we have $f \in \mathscr{H}$ such that for every $g \in G \langle A_z(g)\xi_0, f\rangle = 0$. Then $A_z(e)\xi_0 = \xi_0$ implies

$$\sum_{i=1}^{N} f(G_i) = 0 \quad \text{and} \quad \sum_{i \neq i_1} f(g_1 G_i) = z \cdot \sum_{i=1}^{N} f(G_i) - [1+z(N-1)] \cdot f(G_{i_1})$$

and in the general case

$$\sum_{i \neq i_n} f(gG_i) = z \cdot \sum_{i \neq i_{n-1}} f(\overline{g}G_i) - [1 + z(N-1)] \cdot f(\gamma_n).$$

Combining these we get $\sum_{i \neq i_1} f(g_1 G_i) = -[1 + z(N-1)] \cdot f(G_{i_1})$. In the same way one gets $\sum_{i \neq i_1} f(aG_i) = -[1 + z(N-1)] \cdot f(G_{i_1})$ for all $a \in G_{i_1}$. Since $f \in \mathscr{H}$ hence

$$\sum_{a \in G_{i_1}} \left| \sum_{i \neq i_1} f(aG_i) \right|^2 \leq N \cdot \sum_{a \in G_{i_1}} \sum_{i \neq i_1} |f(aG_i)|^2 < +\infty.$$

This means that $\sum_{a \in G_{i_1}} |1 + z(N-1)|^2 \cdot |f(G_{i_1})|^2 < +\infty$. However, it was assumed that each group G_i was infinite so $f(G_{i_1})$ must be zero. Since i_1 has been chosen arbitrary, we obtain $f(G_i) = 0$ for all $i \in \{1, 2, \ldots, N\}$. Assuming $f(gG_i) = 0$ holds for all $g \in G$ with $\|g\| \leq n-1$ we will prove it for $\|g\| = n$. Take $g \in G$ (of the standard form), $\|g\| = n$. Then, for every $a \in G_i$ and $i \neq i_n$,

$$\sum_{j \neq i} f(gaG_j) = z \cdot \sum_{j \neq i_n} f(gG_j) - [1 + z(N-1)] \cdot f(gG_i)$$

and by the assumption

$$\sum_{j \neq i_n} f(gG_j) = z \cdot \sum_{j \neq i_{n-1}} f(\overline{g}G_j) - [1 + z(N-1)] \cdot f(\overline{g}G_{i_n}) = 0.$$

Therefore $\sum_{j \neq i} f(gaG_j) = -[1+z(N-1)] \cdot f(gG_i)$. In the same manner as previously (by taking a sum over all $a \in G$) one gets $f(gG_i) = 0$ which completes the proof. □

REMARK. If one of the groups G_i (for example G_1) is finite then, for $i \neq 1$, $\sum_{j \neq i} f(gG_j) = 0 \Rightarrow f(gG_1) = -\sum_{j \neq i, 1} f(gG_j) = 0$ because each summand is equal zero. Thus the sufficient condition on the G_i's is that *all but one be infinite.*

6. The irreducibility of A_z. The representations A_z turn out to be irreducible if all the groups $G_1, \ldots, G_N$ are infinite.

THEOREM 8. *For all z of modulus less than 1 except $z = 0$ and $z = -\frac{1}{N-1}$ the representations A_z are irreducible meaning that there exists no nontrivial closed subspace of $\mathscr{H}$ invariant for the action of A_z.*

Proof. We are now going to prove that the operator S_z (constant multiple of the projection onto the cyclic vector ξ) defined as $S_z f = z \cdot N \cdot [1 + z(N-1)] \cdot \langle f, \xi\rangle \cdot \xi$ belongs to the von Neumann algebra $\mathscr{VN}(A_z)$ of each representation A_z. To do that we will produce a sequence S_n of operators in $\mathscr{VN}(A_z)$ which converge strongly to this projection. The assumption that each group G_i is infinite is necessary.

Before that we will make some remarks about the consequences of finding such a sequence. Therefore suppose for a moment that S_n is given with $S_n f \to c_z\langle f, \xi\rangle\xi = Sf$ for all $f \in \mathscr{H}$. If some nontrivial subspace $\mathscr{H}_1$ of $\mathscr{H}$ were invariant under the action of A_z we would choose a nonzero vector f in it. Then for all $n \in \mathbb{N}$, $S_n f$ would belong to the subspace and consequently so would $S_z f$. However, this means that $\langle f, \xi\rangle\xi \in \mathscr{H}_1$ which implies that $\xi \in \mathscr{H}_1$ if and only if $\langle f, \xi\rangle \neq 0$ or equivalently $\sum_{i=1}^N f(G_i) \neq 0$. Hence if we show that it is possible to find such f we will have $\xi \in \mathscr{H}_1$ which is possible only if $\mathscr{H}_1 = \mathscr{H}$. Thus we are reduced to proving the following:

LEMMA 9. *There exists $f \in \mathscr{H}_1$ such that $\langle f, \xi\rangle \neq 0$.*

Proof. Let us take any nonzero function h in $\mathscr{H}_1$ and choose gG_i in $\operatorname{supp}(h)$ such that g has the shortest length. We will assume that $g = g_1 \cdots g_n$ is in the standard form and that $i_n = 1$, if $g \neq e$. For each $x \in G$ we will write h_x for $h|_{\mathscr{M}(x)}$ (so $h = \sum_{x\in G} h_x$). There are two possible cases: (1) $\langle h, \mathscr{Z}(g)\rangle = c \neq 0$ (2) $\langle h, \mathscr{Z}(g)\rangle = 0$. Let us consider the first case. All is trivial if $g = e$. If not, we have $A_z(g^{-1})h = A_z(g^{-1})h_g + L(g^{-1})(\sum_{x\neq g} h_x)$ because $h_x \notin \mathscr{H}_{g^{-1}}$ if $x \in \operatorname{supp}(h)$, $x \neq g$. Hence $\langle A_z(g^{-1})h, \xi_0\rangle = \langle A_z(g^{-1})h_g, \xi_0\rangle$. Let us compute

$$A_z(g^{-1})h_g = \sum_{i=2}^{N} h(gG_i) \cdot A_z(g^{-1})gG_i = \sum_{i=2}^{N} h(gG_i) \cdot G_i - zc \cdot G_1 .$$

From this we get $\langle A_z(g^{-1})h, \xi_0\rangle = c(1-z) \neq 0$ if $c \neq 0$. Thus for $c \neq 0$ we put $f = A_z(g^{-1})h \in \mathscr{H}_1$.

Now let us consider the second case: $c = \langle h, \mathscr{Z}(g)\rangle = 0$. Since $h_g \neq 0$, there exists $i \in \{2, \ldots, N\}$ such that $h(gG_i) \neq 0$. To simplify the notation we put $i = 2$.

PROPOSITION. *For every nonzero constant $y \in \mathbb{C}$ there exists*

$$b \in G_2 \backslash \{e\} \quad \textit{such that } \sum_{j \neq 2} h(gbG_j) \neq y .$$

Proof of the proposition. Suppose, contrary to proposition's claim, that for every $b \in G_2\backslash\{e\}$ we have $\sum_{j\neq 2} h(gbG_j) = y$. Then for every $b \in G_2\backslash\{e\}$ there exists $j = j(b)$ such that $|h(gbG_j)| \geq |y|/(N-1)$ for otherwise $|\sum_{j\neq 2} h(gbG_j)| < |y|$. This would imply that $\sum_{j\neq 2} |h(gbG_j)|^2 \geq (|y|/(N-1))^2 > 0$ for all $b \in G_2$. However, summing up of both sides of the last inequality gives

$$\sum_b \sum_{j\neq 2} |h(gbG_j)|^2 \geq \sum_b \left(\frac{|y|}{(N-1)}\right)^2 = +\infty$$

which contradicts the assumption $h \in \mathscr{H}$. Thus the assertion of the proposition is true. □

Let us take $y = h(gG_2)$ and $b \in G_2$ as in the proposition and consider $A_z(b^{-1}g^{-1})(h_g + h_{gb})$. We have:

$$A_z(b^{-1}g^{-1})h_{gb} = \sum_{j\neq 2} h(gbG_j)G_j - zcG_2 \quad \text{where } c = \sum_{j\neq 2} h(gbG_j)$$

and

$$A_z(b^{-1}g^{-1})h_g = h(gG_2)\cdot(G_2 - zG_2) + \sum_{i=3}^{N} h(gG_i)\cdot b^{-1}G_i$$

because $\sum_{i=3}^{N} h(gG_i) = -h(gG_2)$. Therefore (putting $R = h - (h_b + h_{gb})$) one gets:

$$\begin{aligned}\langle A_z(b^{-1}g^{-1})h\,,\,\xi_0\rangle &= \langle A_z(b^{-1}g^{-1})(h_b + h_{gb} + R)\,,\,\xi_0\rangle \\ &= (1-z)(c + h(gG_2)) \neq 0 .\end{aligned}$$

Hence taking $f = A_z(b^{-1}g^{-1})h$ we get $f \in \mathscr{H}_1$ and $\langle f\,,\,\xi\rangle \neq 0$. This finishes the proof of Lemma 9. □

Now we are going to present a sequence S_n, with the desired property. Let us write each group $G_i\backslash\{e\}$ as a sequence $\{a_i^1\,,\,a_i^2\,,\,\ldots\}$ and define S_n as follows:

$$S_n = \frac{1}{n}\cdot\sum_{k=1}^{n}\left\{z\cdot\sum_{i=1}^{N} A_z(a_i^k) + \sum_{i=1}^{N}\sum_{j\neq i} A_z(a_j^k a_i^k)\right\} .$$

By a direct computation one can see that $\|S_n(gG_i)\| \to 0$ and $S_nG_t \to z[1 + z(N-1)]\cdot\xi_0 = zN[1 + z(N-1)]\cdot\langle G_t\,,\,\xi\rangle\cdot\xi$. This ends the proof of irreducibility of A_z. □

8. The problem of unitarization of the representations A_z. Let us consider the family $\{V_z : |z| < 1\}$ of operators on $\mathscr{H}$ defined by the formulas: $V_z G_i = G_i + \beta(z) \cdot \xi_0$, $V_z g G_i = gG_i + \delta(z) \cdot \mathscr{Z}(g)$ where $i = 1, 2, \ldots, N$, $g \in G\backslash\{e\}$, and

$$\beta(z) = -\frac{1}{N} = \frac{1}{N}\sqrt{\frac{1-z}{1+z(N-1)}},$$
$$\delta(z) = -\frac{1}{N-1} + \frac{1}{N-1}\sqrt{\frac{1}{1+z(N-1)}}.$$

Here we have taken the analytic branch of the square root in the domain $\Omega = \{z \in \mathbb{C}: |z| < 1\}\backslash(-1, \frac{1}{N-1}]$. The finite dimensional Hilbert space with an orthonormal basis $\{gG_i : i \neq i_n\}$ will be denoted by $\mathscr{M}(g)$ (g is thought to be of the standard form and of length $n \geq 1$; $\mathscr{M}(e)$ has the set $\{G_i : i = 1, 2, \ldots, N\}$ as its basis).

LEMMA 10. *For every $z \in \Omega$, V_z is bounded and invertible on $\mathscr{H}$. Both V_z and its inverse leave each subspace $\mathscr{M}(g)$ invariant.*

Proof. The invariance condition is an immediate consequence of the definition of V_z. Furthermore $V_z\xi_0 = [1 + \beta(z)N]\xi_0$ and, for $g \neq e$, $V_z\mathscr{Z}(g) = [1 + \delta(z)(N-1)]\mathscr{Z}(g)$. We define V_z^{-1} by setting

$$V_z^{-1}G_i = G_i - \frac{\beta(z)}{1+\beta(z)N} \cdot \xi_0$$

and

$$V_z^{-1}gG_i = gG_i - \frac{\delta(z)}{1+\delta(z)(N-1)} \cdot \mathscr{Z}(g) \quad \text{for } g \neq e.$$

It is evident that V_z^{-1} is an inverse of V_z. It remains to prove that both are bounded. We need only to estimate their norms on each $\mathscr{M}(g)$ separately (in fact, on $\mathscr{M}(e)$ and $\mathscr{M}(g)$ for some $g \in G$).

Let us state the following general remark.

Assume that an operator V acts on an m-dimensional Hilbert space $\mathfrak{H}$ with an orthonormal basis $e_1, \ldots, e_m$ in such a way that, for every $i \in \{1, \ldots, m\}$, $Ve_i = e_i + x \cdot \sum_{j=1}^m e_j$. Then for any vector $f \in \mathfrak{H}$ of the form $f = \sum_{j=1}^m f(j) \cdot e_j$ we have $Vf = \sum_{j=1}^m [f(j) + cx] \cdot e_j$

where $c = \sum_{j=1}^{m} f(j)$, and therefore

$$\begin{aligned}\|Vf\|^2 &\leq \sum_{j=1}^{m}\left(|f(j)|\cdot|(1+x)| + \left|x\cdot\sum_{i\neq j} f(i)\right|\right)^2 \\ &\leq \sum_{j=i}^{m}\left(\sum_{i=1}^{m}|f(i)|^2\right)\cdot(|(1+x)|^2+(m-1)\cdot|x|^2) \\ &= (|(1+x)|^2+(m-1)\cdot|x|^2)\cdot m\cdot\|f\|^2.\end{aligned}$$

Hence

$$\|V\| \leq [(|(1+x)|^2+(m-1)\cdot|x|^2)\cdot m]^{1/2}.$$

Using this remark one can get

$$\|V_z\| \leq \frac{3N}{|1+z(N-1)|^{1/2}} \quad \text{and} \quad \|V_z^{-1}\| \leq \frac{5N}{|1-z|^{1/2}}.$$

this finishes the proof of Lemma 10. □

The purpose of this section is to change (when possible) the representations A_z in such a way that we get a unitary family. This will be done by intertwining A_z with V_z.

THEOREM 11. *Let π_z be defined for $z \in \Omega$ by setting $\pi_z(g) = V_z^{-1}A_z(g)V_z$ for every $g \in G$. Then $\{\pi_z : z \in \Omega\}$ forms an analytic family of uniformly bounded representations of the group G on the Hilbert space $\mathscr{H}$. Moreover:*

(i) *π_z is unitary if and only if $z \in (-\frac{1}{N-1}, 1)$.*

(ii) $\|\pi_z(g)\| \leq \frac{15N^2}{|(1-z)(1+z(N-1))|^{1/2}}\cdot\left(1+\frac{2\sqrt{N-1}|z|}{|1-z|}\right)$

(iii) *$\pi_z(g) - V_z^{-1}L(g)V_z$ is a finite rank operator.*

(iv) *If all the groups G_i, $i = 1, 2, \ldots, N$, are infinite then no representation π_z has a nontrivial closed invariant subspace.*

REMARK. We should stress that the formulation of the theorem is almost the same as that of Pytlik and Szwarc (see **[P-S]**, Theorem 1) in the case of the free groups. All our constructions and results follow their ideas. However, our results apply to bigger class of groups and in the case of free groups our representations seem to differ in kind from all the other known.

As a simple consequence of the theorem we get a result of Młotkowski (see **[M.1]** Theorem 1).

COROLLARY. *The functions*

$$\varphi_z(x) = \begin{cases} 1 & \text{if } x = e, \\ \frac{(N-1)z+1}{Nz} z^{\|x\|} & \text{if } x \neq e \end{cases}$$

are positive definite on G *if* $z \in (-\frac{1}{N-1}, 1)$.

Proof. It has been proved that the functions are coefficients of the representations A_z. Using formulas $V_z\xi_0 = [1 + \beta(z)N]\xi_0$ and $V_z^{-1}\xi_0 = [1+\beta(z)N]^{-1} \cdot \xi_0$ we see at once that they are coefficients of π_z as well and $\varphi_z(g) = \langle \pi_z(g)\xi, \xi\rangle$. Hence, by (i) of Theorem 11, the corollary follows. □

Proof of Theorem 11. We need only to prove (i). One gets (ii) by a simple estimation $\|\pi_z(g)\| \leq \|V_z^{-1}\| \cdot \|A_z(g)\| \cdot \|V_z\|$. Also (iii) immediately follows from the fact that $A_z(g)L(g^{-1})$ is equal to the identity on the orthogonal complement of $\mathscr{H}_g$ and the space $V_z^{-1}\mathscr{H}_g$ is finite dimensional. The irreducibility of π_z follows at once from Theorem 9.

Proof of (i). It suffices to show that for $z \in (-\frac{1}{N-1}, 1)$ all the operators $\pi_z(a)$ for $a \in \bigcup_{i\in I} G_i^*$ (here $G_i^* = G_i\backslash\{e\}$) are unitary. This will be done by showing that for such z all $R_z(a) = \pi_z(a)L(a^{-1}) = V_z^{-1}A_z(a)V_zL(a^{-1})$ are unitary.

Let us fix $a \in G_i\backslash\{e\}$ and for simplicity assume $i = 1$. Then $\mathscr{H}_a = l_2(\{G_1, \dots, G_N, aG_2, \dots, aG_n\})$ and we have

LEMMA 12. *For every* $z \in \Omega$ *the operator* $R_z(a)$ *preserves* $\mathscr{H}_a$ *and its orthogonal complement* $\mathscr{H}_a^\perp$.

Proof. (a) $R_z(a)\colon \mathscr{H}_a \to \mathscr{H}_a$; it suffices to write the following sequence:

$$\mathscr{H}_a \xrightarrow{L(a^{-1})} \mathscr{H}_{a^{-1}} \xrightarrow{V_z} \mathscr{H}_{a^{-1}} \xrightarrow{A_z(a)} \mathscr{H}_a \xrightarrow{V_z^{-1}} \mathscr{H}_a.$$

This sequence follows immediately from the definition of V_z and the fact that the action of $A_z(a)$ on $\mathscr{H}_{a^{-1}}$ may be described by

$$A_z(a)G_1 = G_1, \quad A_z(a)G_i = aG_i + z\cdot G_1, \quad A_z(a)a^{-1}G_i = G_i - z\cdot G_1, \quad i = 2, \dots, N.$$

(b) $R_z(a)\colon \mathscr{H}_a^\perp \to \mathscr{H}_a^\perp$; write the following sequence

$$\mathscr{H}_a^\perp \xrightarrow{L(a^{-1})} \mathscr{H}_{a^{-1}}^\perp \xrightarrow{V_z} \mathscr{H}_{a^{-1}}^\perp \xrightarrow{A_z(a)=L(a)} \mathscr{H}_a^\perp \xrightarrow{V_z^{-1}} \mathscr{H}_a^\perp$$

which may be justified as follows: $L(a)$ is unitary so it maps $\mathscr{H}_{a^{-1}}^{\perp}$ into $\mathscr{H}_a^{\perp}$. For the same reason $L(a^{-1})$ maps $\mathscr{H}_a^{\perp}$ into $\mathscr{H}_{a^{-1}}^{\perp}$. Moreover, by Lemma 8.1 on the space $\mathscr{H}_{a^{-1}}^{\perp}$ we have $A_z(a) = L(a)$. □

LEMMA 13. *For every* $g \in G\backslash\{e, a\}$, $V_z L(a^{-1}) g G_i = L(a^{-1}) V_z g G_i$ *which means that* $V_z^{-1} L(a) V_z L(a^{-1}) = I$ *on* $\mathscr{H}_a^{\perp}$.

Proof. Under the above assumption $L(a^{-1}) g G_i = a^{-1} g G_i$. Hence $V_z L(a^{-1}) g G_i = a^{-1} g G_i + \beta(z) \cdot \mathscr{Z}(a^{-1} g) = L(a^{-1}) V_z g G_i$. □

COROLLARY. $R_z(a) = I$ *on the subspace* $\mathscr{H}_a^{\perp}$.

Proof. $R_z(a)|_{\mathscr{H}_a^{\perp}} = V_z^{-1} L(a) V_z L(a^{-1})|_{\mathscr{H}_a^{\perp}} = I|_{\mathscr{H}_a^{\perp}}$ by the above lemma.

Continuation of the proof of Theorem 11. In virtue of the above we need only to consider $R_z(a)$ on $\mathscr{H}_a$. The unitary condition $R_z(a) R_z(a)^* = R_z(z)^* R_z(a) = I$ on that subspace is equivalent to $A_z(a) W_z A_z(a)^* = W_z$, where $W_z = V_z V_z^*$. Let us write this as $[A_z(a) L(a^{-1})][L(a) W_z L(a^{-1})][L(a) A_z(a)^*] = W_z$ and for simplicity of notation put $W = W_z$, $B = B_z(a) = A_z(a) L(a^{-1})$, $\mathscr{W} = \mathscr{W}_z(a) = L(a) W_z L(a^{-1})$ on $\mathscr{H}_a$. Then we have the following formulas:

(I) $\quad BG_1 = G_1, \quad BG_i = G_i - z \cdot G_1, \quad BaG_i = aG_i + z \cdot G_1.$

(II)
$$B^* G_1 = G_1 - z \cdot \sum_{i=2}^{N} G_i + z \cdot \sum_{i=2}^{N} aG_1,$$
$$B^* G_i = G_i, \; B^* aG_i = aG_i.$$

(III)
$$WG_i = G_i + r \cdot \xi_0 \text{ for } i \geq 1,$$
$$WaG_i = aG_i + s \cdot \mathscr{Z}(a) \text{ for } i \geq 2.$$

(IV)
$$\mathscr{W} G_1 = (1 + r) \cdot G_1 + r \cdot \mathscr{Z}(a),$$
$$\mathscr{W} G_i = G_i - s \cdot G_1 + s \cdot \xi_0,$$
$$\mathscr{W} aG_i = aG_i + r \cdot G_1 + r \cdot \mathscr{Z}(a),$$

where $r = r(z) = \beta + \overline{\beta} + |\beta|^2 N$, $s = s(z) = \delta + \overline{\delta} + |\delta|^2 (N - 1)$ are functions of z. We need only to check the condition $B \mathscr{W} B^* = W$ on the orthonormal basis $\{G_1, \ldots, G_n, aG_2, \ldots, aG_N\}$ of $\mathscr{H}_a$. This will provide us with necessary and sufficient conditions for unitarity. For $i \geq 2$ $B \mathscr{W} B^* G_i = G_i + s \cdot \xi_0 - [s + z + zs(n - 1)] G_1$. Comparing this with formula (III) we get the necessary conditions on z: (i) $s + z + zs(N-1) = 0$ and (ii) $r = s$. By definition $1 + s \cdot (N-1) = |1 + z(N-1)|^{-1}$. Hence $s \in \mathbb{R}$ and $s = \frac{-z}{1+z(N-1)}$ or equivalently $z = \frac{-s}{1+s(N-1)}$ which means that also $z \in \mathbb{R}$. Moreover, from (i) we get $1+s\cdot(N-1) = (1+z(N-1))^{-1}$ thus $|1+z(N-1)| = 1+z(N-1)$. This forces $1+z(N-1)$ to be positive which implies that $z > -\frac{1}{N-1}$. Thus the necessary condition for $A_z(a)$ to be unitary is $-\frac{1}{N-1} < z < 1$. Now one can prove $r = s$ for such z. Similarly, for $i \geq 2$, $B\mathscr{W}B^*aG_i = aG_i + [r+z+zr(N-1)]\cdot G_1 + r\cdot\mathscr{Z}(a)$. This, compared to (III), gives the same conditions on z. By virtue of the former case, $B\mathscr{W}B^*aG_i = WaG_i$ for $i \geq 2$. By a straightforward computation one shows that $B\mathscr{W}B^*G_1 = WG_1$ for all z from the interval under consideration.

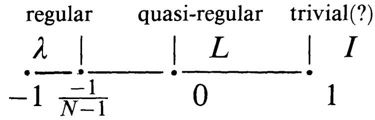
In this manner we have proved that on the space $\mathscr{H}_a$ the equality $B\mathscr{W}B^* = W$ (equivalent to $R_z(a)R_z(a)^* = I = R_z(a)^*R_z(a)$) holds if and only if $z \in (-\frac{1}{N-1}, 1)$. This finishes the proof of Theorem 11.

Final remarks. In all that has been done in this paper we studied groups with discrete topology. Our assumption was that the groups G were free products $G = *_{i=1}^{N} G_i$ of N groups of the same infinite cardinality. Geometrical properties of such groups are essential in our investigations—they act on semi-homogeneous trees (called also Bruhat-Tits trees by Ol'shanskii **[O]**). The construction of such trees related to free products of groups was presented by Serre in his book **[S]**.

Our representations all act on the same Hilbert space and are irreducible if all the free product factors are infinite. We believe that this does not happen when the factors are finite (as in the case of free groups studied by Szwarc in **[Sz.1]**). Actually, the construction can be done even if the free product factors are of different cardinality, so that the associated tree is no longer semi-homogenous. However the number N of the free product factors has to be finite.

As in **[P-S]** one can show that if $z, s \in \Omega$ and $z \neq s$, $z+s+\frac{1}{N-1} \neq 0$ then the representations π_z and π_s are non-equivalent. We do not know whether $z + s + \frac{1}{N-1} = 0$ implies the equivalence $\pi_z \approx \pi_s$.

It is worthwhile to see the "picture" of the family of the representations π_z (or A_z for $|z| < 1$) as a function of z:



One can recognize two representations A_z for $z = 0$ and $z = -\frac{1}{N-1}$: $A_0 = L$ (quasi-regular representation) and for $u = -\frac{1}{N-1} A_u \circ \mathscr{U} = \mathscr{U} \circ \lambda$ where $\mathscr{U} = \frac{1}{N-1} \cdot \mathscr{Z}$, so A_u is similar to the regular representation λ of the group G. Looking at Młotkowski's function φ_z for $z = 1$ one sees that $\varphi_1 \equiv 1$ is a constant function, which is a matrix coefficient of the trivial representation of the group. However it seems to us that the construction cannot work for $z = 1$.

The author wishes to thank Professor Marsk Bożejko for suggesting the problems and many stimulating conversations. The author appreciates also Professor Pytlik's reading the manuscript and pointing out some necessary improvements.

References

[B.1] M. Bożejko, *Remark on Herz Schur multipliers on free groups*, Math. Ann., **258** (1981), 11–15.

[F-P.1] A. Figa-Talamanca and M. A. Picardello, *Spherical functions and harmonic analysis on free groups*, J. Funct. Anal., **47** (1982), 281–293.

[F-P.2] ——, *Harmonic analysis on free groups*, Lecture Notes in Pure and Applied Math., **87** (1983), M. Dekker, New York.

[I-P] A. Iozzi and M. A. Picardello, *Spherical functions on symetrric graphs*, in: *Harmonic analysis*, Proceedings, Cortona, Italy 1982, Lecture Notes in Math., **992**.

[J-V.1] P. Julg and A. Valette, *$\mathscr{K}$-moyennabilite pour les groups operant sur un arbre*, C. R. Acad. Paris, Ser I, **296** (1983), 977–980.

[M-Z] A. M. Mantero and A. Zappa, *The Poisson transform on free groups and uniformly bounded representations*, J. Funct. Anal., **47** (1983), 372–400.

[M.1] W. Młotkowski, *Positive definite radial functions on free product of groups*, Boll. U. M. Ital., (7) **2-B** (1988), 53–66.

[M.2] ——, *Positive definite functions on free product of groups*, Boll. U. M. Ital., (7) **3-B** (1989), 343–355.

[O] G. I. Olshanskii, *Classification of irreducible representations of groups of automorphisms of Bruhat-Tits trees*, Funct. Anal. Appl., **11** (1), (1977), 26–34.

[P-S] T. Pytlik and R. Szwarc, *An analytic family of uniformly bounded representations of free groups*, Acta Math., **157** (1986), 287–309.

[S] J. P. Serre, *Trees*, Springer Verlag, New York, 1980.

[Sz.1] R. Szwarc, *An analytic series of irreducible representations of the free group*, Ann. Inst. Fourier (Grenoble), **38**, 1 (1988), 87–100.

[Sz.2] ——, *Matrix coefficients of irreducible representations of free product of groups*, Studia Math., **T.XCIV** (1989), 179–185.

[Sz.3] ——, *Groups acting on trees and approximation properties of the Fourier algebra*, to appear in J. Funct. Anal.

Received January 2, 1991 and in revised form July 22, 1991.

University of Wroclaw
pl. Grunwaldzki 2/4
Wroclaw, Poland