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STRONG INTEGRAL SUMMABILITY AND THE STONE-ČECH COMPACTIFICATION OF THE HALF-LINE

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An f-measure is a finitely additive nonnegative set function defined on a collection of subsets of $[0, \infty)$ which vanishes on bounded Lebesgue measurable sets. We define statistical convergence and convergence in density relative to an f-measure and use nonnegative regular integral summability methods to generate f-measures. We observe that, for a large class of regular integral summability methods, the notions of strong integral summability, convergence in density and statistical convergence (relative to the f-measure generated by the method) coincide for bounded functions.

The support set of an f-measure is a subset of the Stone-Čech compactification of $[0, \infty)$ that is generated by the measure. We characterize f-measures that generate nowhere dense support sets and f-measures which have P-sets for support sets. The support set of a nonnegative regular integral summability method is used to introduce some summability invariants for bounded strong integral summability. We show that the support sets of f-measures generated by some summability methods are compact zero-dimensional F-spaces of weight $\mathfrak{c}$ without isolated points, but that they need not be P'-spaces.

0. Introduction. Over the years a number of authors have discussed bounded strong summability, convergence in density and statistical convergence, where each of these notions is defined relative to a nonnegative regular matrix summability method. Each of these notions of convergence extends the usual definition of the limit and, it turns out, they are nicely related to one another. The pivot of most of these discussions is either the finitely additive measure generated by the matrix or the support set of the matrix. In this paper we adopt corresponding definitions for regular integral summability methods and show that, under necessary restrictions, many of the results known for matrix summability carry over to integral summability and can be used to establish summability invariants for bounded strong integral summability.

Curiously, although used in harmonic analysis **[20]** and differential equations **[19]**, there does not seem to be a standard introduction to integral summability. Most discussions of integral summability are centered on a collection of particular methods (cf. **[4]**, **[12]**, **[16]**) and, beyond the basics, little is known (to the authors, at least) of a general theory. In this paper we attempt to provide a framework for the study of strong integral summability and its relation to the Stone-Čech compactification of $[0, \infty)$.

The necessary definitions and results from integral summability theory are given in the first section of the paper. This section is also used to introduce f-measures and some properties of integral summability methods which will figure in the discussion of the support set. We also establish some sufficient conditions for an integral method to exhibit these properties and we give some examples of regular integral methods.

In the second section we establish a result in strong integral summability and, in the third, we introduce the support set of a regular integral summability method. In the fourth section the support set is used to introduce some summability invariants for bounded strong integral summability, and, in the fifth, we discuss some topological aspects of the support set. These last two sections of the paper can be read independently of one another.

Most of the technical difficulties encountered in establishing results in integral summability analogous to those of matrix summability theory are due to the different topological properties of $\mathbb{N}$ and $[0, \infty)$. In particular $[0, \infty)$ is connected, $\mathbb{N}$ is extremally disconnected and these properties carry over to their Stone-Čech compactifications. We overcome these difficulties by introducing the appropriate definitions, which, as it turns out, are not excessively restrictive and therefore our results include a large collection of regular integral summability methods.

1. Strong integral summability and f-measures. In this section we introduce f-measures and some of their properties while paying particular attention to f-measures generated by regular integral summability methods. The aim in this section is to show how a regular integral summability method gives rise to an f-measure and how properties of summability methods give rise to properties of f-measures.

We let $\mathbb{R}$ and $\mathbb{H}$ denote, respectively, the real numbers and the half-line $[0, \infty)$ and let m denote Lebesgue measure. We also let $\mathbb{N}$ denote the natural numbers and let ω denote the nonnegative integers.

To move on to the substance of this section, we recall the definition of a regular integral summability method. If $K: \mathbb{H} \times \mathbb{H} \to \mathbb{R}$

is Lebesgue measurable, if $K(s, t)$ is Lebesgue integrable for every $s \in \mathbb{H}$ and if K has the property that, for a Lebesgue measurable function f and a real number L,

$$\lim_{t\to\infty} f(t) = L \quad \text{implies} \quad \lim_{s\to\infty} \int_0^\infty K(s, t) f(t)\, dt = L,$$

then we say that K is a *regular integral summability method*. Some examples are given in 1.8. If $K(s, t) \geq 0$ for all $(s, t) \in \mathbb{H}\times\mathbb{H}$, we say that K is *nonnegative*. It can be shown that K is a nonnegative regular integral summability method if and only if K is nonnegative and $\lim_{s\to\infty} \int_T^\infty K(s, t)\, dt = 1$ for all $T \geq 0$ [7, page 351]. In this paper, we henceforth assume that all regular integral summability methods are nonnegative.

An *f-measure* is a monotone nonnegative finitely additive set function defined on a collection of subsets Γ of $\mathbb{H}$ which has the following properties:

(1) $\mu(B) = 0$ for any bounded member B of Γ,
(2) $\mu(\mathbb{H}) = 1$,
(3) If A is Lebesgue measurable, $A \subset B$ and $\mu(B) = 0$, then $A \in \Gamma$ and $\mu(A) = 0$.

Note that the μ-null sets and their complements form an algebra of sets. We say that an f-measure μ is *collapsing* if, whenever $m(A) < \infty$, then $\mu(A) = 0$; μ is *fine* if every unbounded open subset of $\mathbb{H}$ contains an unbounded open set of μ-measure 0; μ is *separating* if every open set of μ-measure L contains a closed subset of μ-measure L; μ is *strongly separating* if for every closed subset $F \subset \mathbb{H}$ there is a closed subset $E \subset \mathbb{H}$ such that $E \cap F = \varnothing$ and $\mu(E \cup F) = 1$. It is clear that strongly separating f-measures are separating. We also note:

1.1. Lemma. *Strongly separating f-measures are fine.*

Proof. Let μ be a strongly separating f-measure, let $D = \bigcup_{n\in\mathbb{N}}[n, n + \frac{1}{2}]$ and let E be a closed set such that $E \cap D = \varnothing$ and $\mu(E\cup D) = 1$. Set $W = \mathbb{H}-(E\cup D)$ and note that W is an unbounded open μ-null set.

Now let U be an arbitrary unbounded open subset of $\mathbb{H}$ and select a closed subset T of U such that $\mu(T \cup (\mathbb{H} - U)) = 1$. Observe that either $U - T$ is unbounded or, since $\mathbb{H}$ is connected and U is unbounded, there is an $\alpha > 0$ such that $(\alpha, \infty) \subseteq U$. In the first case, $U - T$ is an unbounded open μ-null subset of U, and, in the second case, $W \cap (\alpha, \infty)$ is an unbounded open μ-null subset of U.

If K is a regular integral summability method, then the *f-measure associated with* K, denoted μ_K, is a partial function on $\mathscr{P}(\mathbb{H})$ defined by

$$\mu_K(A) = \lim_{s\to\infty} \int_0^\infty K(s, t)\chi_A(t)\, dt$$

whenever this limit exists, in which case we call A *μ_K-measurable.* Note that μ_K is an f-measure, that Lebesgue null sets are also μ_K-null sets and if K is regular and $T \in \mathbb{H}$, then $\lim_{s\to\infty} \int_0^T K(s, t)\, dt = 0$. We will sometimes say that K has a property (e.g. K is collapsing) if the associated measure has the property.

Recall that a collection $\mathscr{G}$ of sets in $\mathbb{H}$ is *discrete* if every point in $\mathbb{H}$ has a neighborhood which meets at most one member of $\mathscr{G}$.

1.2. LEMMA. *Let G be an open set in $\mathbb{H}$. Then there is an $F \subset G$ such that $F = \bigcup_{n=1}^\infty [c_n, d_n]$ where $\{[c_n, d_n] : n \in \omega\}$ is a discrete family of closed intervals such that $m(G - F) < \infty$.*

Proof. We assume that G is unbounded and for all $n \in \omega$, let $G_n = G \cap (n, n+1) = \bigcup_{j\in\omega}(a_{nj}, b_{nj})$ where the intervals are pairwise disjoint. For each $n \in \omega$, let J_n be a finite subset of ω such that $m(G_n) < m(\bigcup_{j\in J_n}(a_{nj}, b_{nj})) + 2^{-(n+1)}$. Let $\varepsilon_n = 1/|J_n|2^{n+2}$ and let $H_n = \bigcup_{j\in J_n}[a_{nj} + \varepsilon_n, b_{nj} - \varepsilon_n]$ and note that

$$m\left(\left(\bigcup_{j\in J_n}(a_{nj}, b_{nj})\right) - H_n\right) \le 2|J_n|\varepsilon_n = \frac{1}{2^{n+1}}.$$

Let $F = \bigcup_{n\in\omega} H_n$. Now F is the union of a discrete family of closed intervals and $m(G - F) \le 2$ since

$$\begin{aligned}\sum_{n=0}^\infty m(G_n - H_n) &\le \sum_{n=0}^\infty \left[m\left(\bigcup_{j\in J_n}(a_{nj}, b_{nj})\right) + \frac{1}{2^{n+1}}\right] - m(H_n)\\ &= \sum_{n=0}^\infty m\left(\bigcup_{j\in J_n}(a_{nj}, b_{nj}) - H_n\right) + \frac{1}{2^{n+1}} \le 2.\end{aligned}$$

1.3. THEOREM. (1) *If K is a bounded nonnegative regular integral summability method, then μ_K is collapsing.*

(2) *If* μ *is any collapsing* f*-measure, then* μ *is strongly separating and hence fine.*

(3) *If* μ *is any collapsing* f*-measure, then every Lebesgue measurable set* $A \subset \mathbb{H}$ *with* $\mu(A) = L$ *contains a closed set* F *with* $\mu(F) = L$.

Proof. First we establish (1). Suppose that K is bounded by M and let $A \subset \mathbb{H}$ with $m(A) < \infty$. Let $\varepsilon > 0$ be given. Since $m(A) < \infty$, there is a $T \in \mathbb{H}$ such that $m(A \cap [T, \infty)) < \varepsilon/(2M)$. There is also an $S \in \mathbb{H}$ such that $\int_0^T K(s, t)\,dt < \varepsilon/2$ whenever $s \geq S$. Now, for all $s \geq S$,

$$\int_0^\infty K(s, t)\chi_A(t)\,dt \leq \int_0^T K(s, t)\,dt + \int_T^\infty M\chi_A(t)\,dt < \varepsilon.$$

Since ε was arbitrary, it follows that $\mu_K(A) = 0$.

Next we establish (2). Let F be closed in $\mathbb{H}$ and let $E \subset \mathbb{H} - F$ be as in 1.2. Then $m((\mathbb{H} - E) - F) < \infty$; hence $\mu(\mathbb{H} - (E \cup F)) = 0$ and so $\mu(F \cup E) = 1$. Then, by 1.1, μ is fine.

(3) follows from the fact that any Lebesgue measurable set A contains a closed set F such that $m(A - F) < \infty$. Since μ is collapsing, $\mu(A) = \mu(F)$.

In the spirit of 1.3(2), we give a sufficient condition for μ_K to be fine and separating. If K is a regular integral summability method, we say K has *bounded columns* if $\sup_{s\in\mathbb{H}}\{K(s, t) : 0 \leq t \leq x\} < \infty$ for all $x \in \mathbb{H}$.

1.4. Theorem. *If* K *is a nonnegative regular integral summability method with bounded columns, then* μ_K *is fine and separating.*

Proof. First we establish that μ_K is fine. Set $z_n = \sup_{s\in\mathbb{H}}\{K(s, t) : 0 \leq t \leq n\}$ for each $n \in \mathbb{N}$, let U be an unbounded open set and let $U_n = U \cap (n-1, n)$. For each $n \in \mathbb{N}$, let $V_n = \varnothing$ if $U_n = \varnothing$ and $V_n = (a_n, b_n)$ where $(a_n, b_n) \subset U_n$ and $z_n(b_n - a_n) < 2^{-n}$ if $U_n \neq \varnothing$. Let $V = \bigcup_{n=1}^\infty V_n$. Note that, since U is unbounded, V is also an unbounded open set.

We claim that $\mu_K(V) = 0$. Let $\varepsilon > 0$ be given and select N such that $\sum_{n>N} 2^{-n} < \varepsilon/2$. Select $S \in \mathbb{H}$ such that $s \geq S$ implies that

$\int_0^N K(s, t)\,dt < \varepsilon/2$. Now, if $s \geq S$, then

$$\begin{aligned}\int_0^\infty K(s, t)\chi_V(t)\,dt &< \varepsilon/2 + \sum_{n>N} \int_{n-1}^n K(s, t)\chi_V(t)\,dt \\ &\leq \varepsilon/2 + \sum_{n>N} z_n(b_n - a_n) < \varepsilon\end{aligned}$$

and hence μ_K is fine.

Next we establish that μ_K is separating. Let U be an open set and suppose that $\mu_K(U) = L$. Set $U_n = U \cap (n-1, n)$ for all $n \in \mathbb{N}$. Using a procedure similar to the one used in 1.2, for each $n \in \mathbb{N}$ we can construct $W_n \subset U_n$ such that W_n is the union of a finite number of disjoint closed intervals and such that $m(U_n - W_n) < 1/(z_n 2^n)$. Set $W = \bigcup_{n=1}^\infty W_n$. Then $W \subset U$, W is closed and we claim that $\mu_K(W) = \mu_K(U)$. In order to see this last claim, let $\varepsilon > 0$ be given and select $N \in \mathbb{N}$ and $S \in \mathbb{H}$ such that $\sum_{n>N} 2^{-n} < \varepsilon/2$ and such that $s \geq S$ implies $\int_0^N K(s, t)\,dt < \varepsilon/2$. Now, if $s \geq S$,

$$\int_0^\infty K(s, t)\chi_{U-W}(t)\,dt < \frac{\varepsilon}{2} + \sum_{n>N} z_n \frac{1}{z_n 2^n} < \varepsilon.$$

Since ε was arbitrary, $\mu_K(U - W) = 0$ and hence $\mu_K(W) = \mu_K(U) - \mu_K(U - W) = \mu_K(U)$.

We say that an f-measure has the APO if, whenever $\langle A_n : n \in \omega \rangle$ is a sequence of closed subsets of $\mathbb{H}$ such that $\mu(A_n) = 0$ for each $n \in \omega$, there is a sequence $\langle B_n : n \in \omega \rangle$ of closed sets such that $A_n \Delta B_n$ is bounded for each n and $\mu(\mathrm{cl} \bigcup_{n\in\omega} B_n) = 0$. A regular integral summability method K is said to be *thinning* if for every $\varepsilon > 0$ and $S \in \mathbb{H}$ there is a $T \in \mathbb{H}$ such that $|\int_T^\infty K(s, t)\,dt| < \varepsilon$ for every $s \leq S$.

The above definition of the APO property is a modification of a similar definition for the density generated by a nonnegative regular summability matrix which was introduced by Freedman and Sember in **[10]** and which has been further developed in [2]. Our definition generalizes that of Freedman and Sember, but, in practice, the following characterization is sometimes more convenient.

1.5. LEMMA. *Let μ be an f-measure. The following are equivalent*:

(1) *μ has the APO.*

(2) *If* $\langle G_n : n \in \omega \rangle$ *is a decreasing sequence of open subsets of* $\mathbb{H}$ *with* $\mu(G_n) = 1$ *for all* $n \in \omega$, *then there is an open set* G *with* $\mu(G) = 1$ *and such that* $G - G_n$ *is bounded for each* $n \in \omega$.

Proof. First we establish that (1) implies (2). Suppose that $\langle G_n : n \in \omega \rangle$ is a collection of open sets that meet the hypothesis of (2). Then $\langle \mathbb{H} - G_n : n \in \omega \rangle$ is a family of closed null sets and hence there is a family $\langle B_n : n \in \omega \rangle$ and a sequence $\langle j_n : n \in \omega \rangle$ with $(\mathbb{H} - G_n)\Delta B_n \subset [0, j_n]$ and $\mu(\operatorname{cl} \bigcup_{n\in\omega} B_n) = 0$. Let $G = \mathbb{H} - \operatorname{cl} \bigcup_{n\in\omega} B_n$. Clearly $\mu(G) = 1$ and for $n \in \omega$, $G - G_n \subset (\mathbb{H} - G_n) - B_n \subset (\mathbb{H} - G_n)\Delta B_n \subset [0, j_n]$.

Now suppose that (2) holds and let $\langle A_n : n \in \omega \rangle$ be a family of closed null sets. For each $n \in \omega$, let $G_n = \mathbb{H} - \bigcup_{j\leq n} A_j$. Then $\langle G_n : n \in \omega \rangle$ is a decreasing sequence of open sets and $\mu(G_n) = 1$ for each $n \in \omega$, and so there is an open subset $G \subset \mathbb{H}$ and there is $j_n \in \mathbb{H}$ such that $\mu(G) = 1$ and $G - G_n \subset [0, j_n]$ for each $n \in \omega$.

Let $B_n = (\mathbb{H} - G) \cap A_n$. Since each $B_n \subset \mathbb{H} - G$, $\operatorname{cl} \bigcup_{n\in\omega} B_n \subset \mathbb{H} - G$ and hence $\mu(\operatorname{cl} \bigcup_{n\in\omega} B_n) = 0$. Now let $x \in B_n \Delta A_n$. Since $B_n \subset A_n$, $x \in A_n - B_n$, and so $x \in G$. Now $x \in A_n$ and so $x \notin G_n$ and we conclude $x \in G - G_n \subset [0, j_n]$.

1.6. THEOREM. *If* K *is a nonnegative thinning regular summability method, then* μ_K *has the APO.*

Proof. Let $\langle G(n) : n \in \mathbb{N} \rangle$ be a decreasing sequence of open subsets of $\mathbb{H}$ such that $\mu(G(n)) = 1$ for all $n \in \mathbb{N}$. We will establish that there is an open set G such that $\mu(G) = 1$ and $G - G(n)$ is bounded for each $n \in \mathbb{N}$.

First set $S_0 = T_0 = 1$. Suppose $S_0 < \cdots < S_{n-1}$ and $T_0 < \cdots < T_{n-1}$ have been selected. Now select $S_n > S_{n-1}$ such that $s \geq S_n$ implies that

$$\int_0^\infty K(s, t)\chi_{G(n)}(t)\, dt > 1 - \frac{1}{2^{n+1}} \quad \text{and}$$
$$\int_0^{T_{n-1}} K(s, t)\, dt < \frac{1}{2^{n+1}}$$

and select $T_n > T_{n-1}$ such that $\int_{T_n}^\infty K(s, t)\, dt < 1/2^n$ for all $s \leq S_n$. Set $G = \bigcup_{n=1}^\infty G(n) \cap (T_{n-1}, T_{n+1})$. Observe that G is an open subset of $\mathbb{H}$ and that if $S_j \le s \le S_{j+1}$, then

$$\begin{aligned}\int_{T_{j-1}}^{T_{j+1}} K(s, t)\chi_{G(j)}\, dt &= \int_0^{\infty} K(s, t)\chi_{G(j)}\, dt - \int_0^{T_{j-1}} K(s, t)\chi_{G(j)}\, dt \\ &\quad - \int_{T_{j+1}}^{\infty} K(s, t)\chi_{G(j)}\, dt \\ &\ge \left(1 - \frac{1}{2^{j+1}}\right) - \left(\frac{1}{2^{j+1}} + \frac{1}{2^{j+1}}\right)\end{aligned}$$

and hence, if $S_j \le s \le S_{j+1}$, then

$$\int_0^{\infty} K(s, t)\chi_G\, dt \ge \int_{T_{j-1}}^{T_{j+1}} K(s, t)\chi_{G(j)}\, dt \ge 1 - \frac{3}{2^{j+1}}.$$

We conclude that $\mu_K(G) = 1$.

Finally, observe that $G - G(n)$ is bounded by T_{n-1}: If $t > T_{n-1}$ and $t \in G$, then $t \in G(m) \cap (T_{m-1}, T_{m+1})$ for some $m \ge n$. Since $m \ge n$ implies that $G(m) \subset G(n)$, t must be in $G(n)$.

Using an approach suggested in **[20]**, we introduce a large class of integral summability methods. We say that K is a *γ-means* if there is a nonnegative integrable function $\gamma\colon (0, \infty) \to \mathbb{H}$ such that $K(s, t) = (cs)^{-1}\gamma(t/s)$ where $c = \int_0^{\infty} \gamma(u)\, du$ and $K(s, t) = 0$ if $s = 0$.

1.7. Proposition. *If K is a γ-means, then K is a thinning nonnegative regular integral summability method.*

Proof. Suppose that $K(s, t) = (cs)^{-1}\gamma(t/s)$ where γ and c are as in the definition of γ-means. Observe that, for a fixed $s > 0$, setting $u = t/s$ yields that $\int_T^{\infty} K(s, t)\, dt = c^{-1}\int_{T/s}^{\infty} \gamma(u)\, du$ for all $T \ge 0$. Since

$$\lim_{s\to\infty} \int_T^{\infty} K(s, t)\, dt = \lim_{s\to\infty} \frac{1}{c} \int_{T/s}^{\infty} \gamma(u)\, du = 1$$

for all $T > 0$ and γ is nonnegative, K is nonnegative and regular. The proof that K is thinning is similar.

1.8. Examples. In this section we give a few examples to help distinguish between thinning, separating, strongly separating and fine f-measures.

(1) *The Cesàro, Abel and Gauss means.* These classical methods are all examples of γ-means and hence are regular and thinning. The Cesàro (K_C), Abel (K_A) and Gauss (K_G) means can be defined by

$$K_C(s, t) = s^{-1}\chi_{[0,s]}(t); \quad K_A(s, t) = s^{-1}e^{-t/s}; \quad \text{and}$$
$$K_G(s, t) = (s\sqrt{\pi})^{-1}2e^{-(t/s)^2}$$

respectively (with all being 0 if $s = 0$), and correspond to $\gamma_C(x) = \chi_{[0,1]}(x)$, $\gamma_A(x) = e^{-x}$ and $\gamma_G(x) = e^{-x^2}$ respectively. Observe that, since each of these methods corresponds to a bounded γ, 1.4 yields that they are fine and separating. Also note that the Cesàro method is collapsing.

(2) *A method that fails the APO.* This example indicates that a hypothesis similar to thinning is required for Theorem 1.6. For each $(n, k) \in \omega \times \omega$, let $I(n, k) = [2^n(k+1), 2^n(k+1)+1]$. Define $K(s, t)$ by:

$$\text{if } s \in \left[n + \frac{2^k - 1}{2^k}, n + \frac{2^{k+1}-1}{2^{k+1}}\right) \quad \text{then } K(s, t) = \chi_{I(n,k)}.$$

In order to see that the f-measure associated with K fails to have the APO, set $G(n) = \bigcup_{k=0}^{\infty} I(n, k)$. Observe that $G(n+1) \subset G(n)$ and $\mu_K(G(n)) = 1$ for all n. Now suppose that $\mu_K(G) = 1$. Select N such that $s > N$ implies that $\int_0^\infty K(s, t)\chi_G(t)\,dt > 3/4$. Note that $m(G \cap I(N, k)) > 3/4$ for each $k \in \omega$ and hence $G \cap I(N, k) \neq \varnothing$ for each $k \in \omega$. But now we can conclude that $G - G(N+1)$ is not bounded and hence μ_K cannot have the APO.

(3) *A thinning and nonseparating method.* For $(n, k) \in \omega \times \omega$ and $j \in \{0, \ldots, 2^k - 1\}$, set $I(n, k, j) = [n + j/2^{k+3}, n + (j+1)/2^{k+3})$. Define $K(s, t)$ by

$$K(s, t) = 2^{k+3}\chi_{I(n,k,j)} \quad \text{when}$$
$$s \in \left[n + \frac{2^k-1}{2^k} + \frac{j}{2^{k+3}}, n + \frac{2^k-1}{2^k} + \frac{j+1}{2^{k+3}}\right).$$

Observe that K is a thinning nonnegative regular integral summability method and, if U is an open set such that $\mu_K(U) = 0$, then U must be bounded.

We show that if μ_K is separating then every closed μ_K-null set is bounded. Since $\mu_K(\mathbb{N}) = 0$, this is clearly a contradiction. Suppose that A is closed and that $\mu_K(A) = 0$. Observe that $\mathbb{H} - A$ is open and that $\mu_K(\mathbb{H} - A) = 1$. If μ_K is separating, then there is a closed set F, $F \subset \mathbb{H} - A$, such that $\mu_K(F) = 1$. Now $\mathbb{H} - F$ is open and $\mu_K(\mathbb{H} - F) = 0$ and hence, from the construction of μ_K, $\mathbb{H} - F$ is bounded. Since $A \subset \mathbb{H} - F$, A is also bounded. Hence μ_K cannot be separating.

(4) *A separating, not strongly separating, f-measure.* Let $\Gamma = \{A \subset \mathbb{H} : A$ or $\mathbb{H} - A$ is bounded$\}$ and, for all $A \in \Gamma$, let $\mu(A) = 0$ if A is bounded and let $\mu(A) = 1$ if $\mathbb{H} - A$ is bounded. It is clear that μ is separating. Note, however, that if $F = \bigcup_{n \in \omega} [n, n + 1/2]$ and E is a closed subset of $\mathbb{H} - F$, then $E \cup F \notin \Gamma$ and hence μ is not strongly separating.

2. Convergence defined by f-measures. If μ is an f-measure, $f : \mathbb{H} \to \mathbb{R}$ is a function and $L \in \mathbb{R}$ we say that:

(1) f is *convergent in μ-density to* L if there is a subset $A \subset \mathbb{H}$ such that $\mu(A) = 1$ and $\lim_{t \to \infty} (f(t) - L)\chi_A(t) = 0$, and

(2) f is *μ-statistically convergent to* L if $\mu(\{t : |f(t) - L| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$.

If K is a nonnegative regular integral summability method we say that

(3) f is *strongly K-summable to* L if

$$\lim_{s \to \infty} \int_0^\infty K(s, t)|f(t) - L|\, dt = 0.$$

We note that strong integral summability has also been discussed in **[13]**.

These definitions are similar to definitions which appear in matrix summability theory (see **[2]**). It can be shown in the context of matrix summability theory that for measures generated by nonnegative regular summability methods and for bounded sequences, each of (1), (2) and (3) implies the others. The situation is similar for regular integral summability methods.

2.1. THEOREM. *Let f be a bounded measurable function on $\mathbb{H}$ and K be a nonnegative regular integral summability method. Consider the following statements.*

(1) *f is convergent in μ_K-density to L.*

(2) *f is strongly K-summable to L.*

(3) *f is μ_K-statistically convergent to L.*

Then (1) *implies* (2) *and* (2) *implies* (3). *If, in addition, K is thinning, then* (1), (2) *and* (3) *are equivalent.*

Proof. Without loss of generality, we assume that $L = 0$ and we let $\sup\{|f(x)| : x \in \mathbb{H}\} \leq M$.

First we establish that (1) implies (2). Let $A \subset \mathbb{H}$ be such that $\mu_K(A) = 1$ and $\lim_{t\to\infty} f(t)\chi_A(t) = 0$. Then

$$\int_0^\infty K(s, t)|f(t)|\,dt \le \int_A K(s, t)|f(t)|\,dt + M\int_{\mathbb{H}-A} K(s, t)\,dt$$

and, since $\mu_K(\mathbb{H} - A) = 0$, $\lim_{s\to\infty} M\int_{\mathbb{H}-A} K(s, t)\,dt = 0$. Now let $\varepsilon > 0$ be given and select $T > 0$ such that if $t > T$ and $t \in A$, then $|f(t)| < \varepsilon$. Note that

$$\begin{aligned} 0 &\le \int_0^\infty K(s, t)|f(t)|\chi_A(t)\,dt \\ &\le M\int_0^T K(s, t)\,dt + \varepsilon\int_T^\infty K(s, t)\chi_A(t)\,dt. \end{aligned}$$

Observe that, since K is regular, $\lim_{s\to\infty} M\int_0^T K(s, t)\,dt = 0$ and also that, since $\mu_K(A) = 1$, $\lim_{s\to\infty} \varepsilon\int_T^\infty K(s, t)\chi_A(t)\,dt = \varepsilon$. Hence $\lim_{s\to\infty}\int_0^\infty K(s, t)|f(t)|\,dt = 0$.

Next we establish that (2) implies (3). Suppose that f is strongly K-summable to 0 and let $\varepsilon > 0$ be given. Set $A(\varepsilon) = \{t : |f(t)| \ge \varepsilon\}$. Now,

$$\int_0^\infty K(s, t)|f(t)|\,dt \ge \varepsilon\int_0^\infty K(s, t)\chi_{A(\varepsilon)}\,dt \ge 0.$$

Since $\lim_{s\to\infty}\int_0^\infty K(s, t)|f(t)|\,dt = 0$, it follows that $\mu_K(A(\varepsilon)) = 0$ and so f is μ_K-statistically convergent to 0.

If K is thinning, then (3) implies (1) follows from the proof of 1.6. Suppose that f is μ_K-statistically convergent to 0 and let $G(n) = \{t : |f(t)| < n^{-1}\}$ for each $n \in \mathbb{N}$. Note that $\mu_K(G(n)) = 1$ for each n. The proof of 1.6 yields that there is a μ_K-measurable set G such that $\mu_K(G) = 1$ and $G - G(n)$ is bounded for each n. It is now straightforward to verify that $\lim_{t\to\infty} f(t)\chi_G(t) = 0$ and hence f is convergent to 0 in μ_K-density.

In §4 we will also establish that generating a fine f-measure is a summability invariant for bounded strong integral summability. In particular, we will show that if K and K' are two regular integral summability methods such that a bounded function is strongly K-summable if and only if it is strongly K'-summable and μ_K is fine, then $\mu_{K'}$ is also fine. We will also establish that having the APO is a bounded strong summability invariant for separating f-measures.

3. The support set of a measure. We turn now to questions about the relation between f-measures on $\mathbb{H}$ and certain subsets of $\beta\mathbb{H}$.

We will show that in many cases the f-measure will be associated with a large zero-dimensional nowhere dense P-set in $\beta\mathbb{H}-\mathbb{H}$. First, however, we remind the reader of a few definitions and facts.

For a topological space X, $C^*(X) = \{f: X \to \mathbb{R}: f \text{ is bounded and continuous}\}$. A subset A of X is *C^*-embedded* in X if for all $f \in C^*(A)$ there is $\hat{f} \in C^*(X)$ such that $\hat{f} \restriction A = f$. A *zero-set* $Z = Z(f)$ is a set of the form $f^{\leftarrow}\{0\}$ where $f \in C^*(X)$, and a *cozero-set* is the complement of a zero-set. A *z-ultrafilter* on X is a maximal filter in the collection of zero-sets of X.

The Stone-Čech compactification βX of a Tychonoff space X can be thought of as the collection of all z-ultrafilters on X, with the fixed ultrafilters being identified with the points of X (and hence $X \subset \beta X$), topologized so that X is dense and C^*-embedded in βX. In the case of $\mathbb{H}$ (or any space in which closed sets are zero-sets) we have, for closed A, $p \in \mathrm{cl}_{\beta\mathbb{H}} A$ if and only if $A \in p$ (where in the first occurrence p is a point in $\beta\mathbb{H}$ and in the second, p is a z-ultrafilter on $\mathbb{H}$), and hence $p \in \mathbb{H}^*$ if and only if for all $T \in \mathbb{H}$, $[T, \infty) \in p$. For A closed in $\mathbb{H}$, we set $A^* = \mathrm{cl}_{\beta\mathbb{H}} A - A$. If A and B are closed in $\mathbb{H}$, then $A^* \cap B^* = \varnothing$ if and only if $A \cap B$ is bounded. Recall also that if $\mathscr{F}$ is any collection of closed subsets of $\mathbb{H}$ with the finite intersection property, then there is $p \in \bigcap\{\mathrm{cl}_{\beta\mathbb{H}} F: F \in \mathscr{F}\}$; that is, there is a z-ultrafilter on $\mathbb{H}$ containing $\mathscr{F}$.

For a space X we define the *weight of* X, $w(X)$, by

$$w(X) = \min\{|\mathscr{B}|: \mathscr{B} \text{ is an open base for the topology on } X\}.$$

Returning now to f-measures, let μ be any f-measure on $\mathbb{H}$. We define

$$\mathscr{F}_\mu = \{A \subset \mathbb{H}: A \text{ is closed and } \mu(A) = 1\}$$

and we define S_μ, the support set of the f-measure μ by

$$S_\mu = \bigcap\{\mathrm{cl}_{\beta\mathbb{H}} A: A \in \mathscr{F}_\mu\}.$$

Note that $S_\mu = \{p \in \beta\mathbb{H}: \mathscr{F}_\mu \subseteq p\}$. (See **[15]**, **[14]** and **[1]** for a discussion of the support set of a nonnegative regular matrix summability method.)

Since $\mu([T, \infty)) = 1$ for all $T \in \mathbb{H}$, $S_\mu \subseteq \mathbb{H}^*$. In this section we shall be primarily interested in support sets that arise from f-measures associated with regular integral summability methods. A support set of an arbitrary f-measure can be trivial. For example, if F is any closed set in $\mathbb{H}^*$, then F is the support set of some separating f-measure: We define $\mu(A) = 1$ if there is a closed set B of $\mathbb{H}$ with

$B \subset A$ and $F \subset B^*$ while $\mu(A) = 0$ if $\mu(\mathbb{H}-A) = 1$. But support sets of f-measures associated with regular integral summability methods are nontrivial. We show, for example, that they all have cardinality $2^{\mathfrak{c}}$. First, however, we need some preliminary results.

3.1. LEMMA. *Let K be a nonnegative regular integral summability method and let $A \subset \mathbb{H}$. If $\mu_K(A) = 1$, then there is a discrete collection $\langle F_n \colon n \in \omega\rangle$ of closed intervals in $\mathbb{H}$ such that if $J \subset \omega$ and $|J| = \omega$, then $A_J = \bigcup_{n\in J} A \cap F_n$ is not a null set. Moreover, if the complement of J is infinite, then A_J is not μ_K-measurable.*

Proof. Let $S_0 = T_0 = 0$ and select S_1 such that $s > S_1$ implies that $17/16 > \int_0^\infty K(s, t)\chi_A(t)\,dt > 15/16$. Now select T_1 such that $\int_0^{T_1} K(S_1, t)\chi_A(t)\,dt > 7/8$. We proceed recursively: If $S_0 < S_1 < \cdots < S_{n-1}$ and $T_0 < T_1 < \cdots < T_{n-1}$ have been selected, select $S_n > S_{n-1}$ such that $s \geq S_n$ implies that

$$\int_0^{T_{n-1}+1} K(s, t)\,dt < 1/16.$$

We may then select $T_n > T_{n-1} + 1$ such that

$$\int_{T_{n-1}+1}^{T_n} K(S_n, t)\chi_A(t)\,dt > 7/8 \quad \text{and hence}$$

$$\int_{T_n}^{\infty} K(S_n, t)\chi_A(t)\,dt < 3/16.$$

Let $F_n = [T_{n-1} + 1, T_n]$ for $n \in \mathbb{N}$ and let J be an infinite subset of ω. Since $\langle F_n \colon n \in \omega\rangle$ has been constructed to be a discrete collection of closed intervals, we only need to establish the other conclusions. To this end, observe that if $n \in J$, then

$$\int_0^\infty K(S_n, t)\chi_{A_J}(t)\,dt \geq \int_{T_{n-1}+1}^{T_n} K(S_n, t)\chi_A(t)\,dt > 7/8$$

and, if $n \notin J$, then

$$\int_0^\infty K(S_n, t)\chi_{A_J}(t)\,dt$$
$$\leq \int_0^{T_{n-1}+1} K(S_n, t)\,dt + \int_{T_n}^{\infty} K(S_n, t)\chi_A(t)\,dt < 1/4.$$

Note that if the complement of J is finite, then $A - A_J$ is bounded and hence $\mu_K(A_J) = 1$. If the complement of J is infinite, then

$$\liminf_{j\in J} \int_0^\infty K(S_j, t)\chi_{A_J}(t)\,dt \geq 7/8 \quad \text{and}$$

$$\limsup_{j\in\omega - J} \int_0^\infty K(S_j, t)\chi_{A_J}(t)\,dt \leq 1/4$$

and hence A_J is not μ_K-measurable.

3.2. PROPOSITION. *Let μ be any f-measure. If F is closed in $\mathbb{H}$, and F is not a μ-null set, then $F^* \cap S_\mu \neq \varnothing$.*

Proof. Since F is not a null set, then for all $A \in \mathcal{F}_\mu$, $F \not\subseteq \mathbb{H} - A$. Then $F \cap A \neq \varnothing$ for all $A \in \mathcal{F}_\mu$, and so there is a z-ultrafilter q on $\mathbb{H}$ which contains $\{F\} \cup \mathcal{F}_\mu$. Clearly $q \in F^* \cap S_\mu$.

3.3. THEOREM. *If K is any nonnegative regular integral summability method, then S_{μ_K} contains a copy of $\beta\mathbb{N}$ and hence $|S_{\mu_K}| = 2^{\mathfrak{c}}$.*

Proof. Let $\langle F_n : n \in \omega\rangle$ be as in 3.1 (with $A = \mathbb{H}$). Let $\mathcal{E}$ be an infinite family of pairwise disjoint infinite subsets of ω. Let $F_E = \bigcup_{n\in E} F_n$ for all $E \in \mathcal{E}$. By 3.2, $(F_E)^* \cap S_{\mu_K} \neq \varnothing$ for all $E \in \mathcal{E}$ and, since $F_E \cap F_J = \varnothing$ for $E \neq J$, $(F_E)^* \cap (F_J)^* = \varnothing$. We conclude that $|S_{\mu_K}| \geq \mathcal{E} \geq \omega$. Then S_{μ_K} is closed and infinite, and so by [**11**, 9.12], S_{μ_K} contains a copy of $\beta\mathbb{N}$.

3.4. COROLLARY. *If K is a nonnegative regular integral summability method, then $w(S_{\mu_K}) = \mathfrak{c}$.*

Proof. This follows from 3.3 and the fact that $w(\beta\mathbb{N}) = \mathfrak{c} = w(\beta\mathbb{H})$. (See [**8**, 3.6.12, 3.5.3].)

We will denote the boundary in X of a subset A of a space X by $\mathrm{bdy}_X A$, and we will make use of a function Ex that extends open sets of $\mathbb{H}$ to open sets of $\beta\mathbb{H}$ defined as follows:

$$\mathrm{Ex}(U) = \beta\mathbb{H} - \mathrm{cl}_{\beta\mathbb{H}}(\mathbb{H} - U)$$

for all U open in $\mathbb{H}$. From [**5**, 3.1, 3.2] we have the following facts about Ex.

3.5. PROPOSITION. *Let U be an open set in $\mathbb{H}$.*
(1) $\mathbb{H} \cap \mathrm{Ex}(U) = U$; *hence* $\mathrm{cl}_{\beta\mathbb{H}}\,\mathrm{Ex}(U) = \mathrm{cl}_{\beta\mathbb{H}}\, U$.
(2) $\mathrm{bdy}_{\beta\mathbb{H}}\,\mathrm{Ex}(U) = \mathrm{cl}_{\beta\mathbb{H}}\,\mathrm{bdy}_{\mathbb{H}}\, U$.

It is evident that $\{\mathrm{Ex}(U) : U$ is open in $\mathbb{H}\}$ is a base for the topology on $\beta\mathbb{H}$.

We now give some relations between properties of measures and those of the support sets they generate. First we note the following.

3.6. PROPOSITION. *If μ is any f-measure and G is open in $\beta\mathbb{H}$ with $S_\mu \subset G$, then $\mu(G \cap \mathbb{H}) = 1$.*

Proof. Let G be open in $\beta\mathbb{H}$ with $\mu(G \cap \mathbb{H}) \neq 1$. Then $\mu(\mathbb{H} - G) \neq 0$ and so, by 3.2, $(\mathbb{H} - G)^* \cap S_\mu \neq \varnothing$. Then $S_\mu \not\subset G$.

3.7. PROPOSITION. *Let μ be any separating measure.*

(1) *If B is closed in $\mathbb{H}$ with $\mu(B) = 0$, then $B^* \cap S_\mu = \varnothing$.*

(2) *If U is open in $\mathbb{H}$ with $\mu(U) = 1$, then $S_\mu \subset \mathrm{Ex}(U)$.*

(3) *If μ is strongly separating, then for all closed subsets F of $\mathbb{H}$, $S_\mu \subset F^*$ if and only if $\mu(F) = 1$.*

Proof. (1) Let $p \in S_\mu$ and let B be closed in $\mathbb{H}$ with $\mu(B) = 0$. Now $\mathbb{H} - B$ is open and $\mu(\mathbb{H} - B) = 1$. Since μ is separating, there is a closed $A \subset \mathbb{H} - B$ with $\mu(A) = 1$. Then $p \in \mathrm{cl}_{\beta\mathbb{H}} A$, and so $p \notin B^*$.

(2) Let $\mu(U) = 1$. Then $\mu(\mathbb{H} - U) = 0$ and so, by (1), $S_\mu \subset \mathrm{Ex}(U)$.

(3) Let F be closed in $\mathbb{H}$ and suppose that F does not have μ-measure 1. Let $E \subset \mathbb{H} - F$ with $\mu(E \cup F) = 1$. Now E is not μ-null, and so, by 3.2, $S_\mu \cap E^* \neq \varnothing$. We conclude that $S_\mu \not\subset F^*$. The converse is trivial.

A point $p \in \mathbb{H}^*$ is called a *remote point* (resp. *far point*) of $\mathbb{H}$ if $p \notin \mathrm{cl}_{\beta\mathbb{H}} A$ for any nowhere dense (resp. closed discrete) $A \subset \mathbb{H}$ (see [**5**, 1.4]), and p is a *large point* of $\mathbb{H}$ if $p \notin \mathrm{cl}_{\beta\mathbb{H}} A$ for any closed set $A \subset \mathbb{H}$ where $m(A) < \infty$ (see [**17**]).

3.8. LEMMA. *If μ is collapsing, A is closed in $\mathbb{H}$ and $\mu(A) = L$, then there is a $B \subset A$ with B closed and nowhere dense and $\mu(B) = L$.*

Proof. Let $a_0 = 0$ and, for $n > 0$, let a_n be an irrational number such that $a_{n+1} \geq a_n + 1$. Let the rationals in (a_n, a_{n+1}) be contained in an open set G_n of Lebesgue measure less than 2^{-n} with $G_n \subset (a_n, a_{n+1})$. Let $B = \bigcup_{n\in\omega}(A \cap [a_n, a_{n+1}] - G_n)$. B is a closed set and clearly $B \subset A$.

Now B contains no positive rationals, and therefore B is nowhere dense. Since $m(A-B) \leq \sum_{n=0}^{\infty} m(G_n) \leq \sum_{n=0}^{\infty} 2^{-n} = 2$, and, since μ is collapsing, $\mu(A-B) = 0$. Thus $\mu(A) = \mu(B) = L$.

3.9. PROPOSITION. *If μ is a collapsing f-measure, then every point of S_μ is large and no point of S_μ is remote.*

Proof. Let $p \in S_\mu$. If A is closed and $m(A) < \infty$, then $\mu(A) = 0$. By 1.3(2), μ is separating and so $p \notin \mathrm{cl}_{\beta\mathbb{H}} A$ by 3.7(1). We conclude that p is large.

Now $\mu(\mathbb{H}) = 1$ and so there is, by 3.8, $B \in \mathscr{F}_\mu$ with B nowhere dense. Since $p \in \mathrm{cl}_{\beta\mathbb{H}} B$, p is not remote.

3.10. COROLLARY. *If μ is a collapsing f-measure, then S_μ contains no nonempty G_δ-subsets of $\mathbb{H}^*$ and hence is nowhere dense in $\mathbb{H}^*$.*

Proof. In [5, 4.2], it is shown that if G is a nonempty G_δ-subset of $\mathbb{H}^*$, then G contains $2^{\mathfrak{c}}$ remote points of $\mathbb{H}$. By 3.9, none of these is in S_μ.

In fact, however, μ does not have to be collapsing to guarantee that S_μ is nowhere dense. We see that μ can merely be fine.

3.11. PROPOSITION. *If μ is any f-measure, then the following are equivalent:*

(1) *S_μ is nowhere dense in $\mathbb{H}^*$.*

(2) *μ is fine.*

Proof. (1) $\Rightarrow$ (2) Let U be open and unbounded. Since S_μ is nowhere dense, there is $p \in (\mathbb{H}^* \cap \mathrm{Ex}(U)) - S_\mu$. Then $p \notin \mathrm{cl}_{\beta\mathbb{H}}(\mathbb{H}-U)$ and $p \notin S_\mu$, and so there is $B \in \mathscr{F}_\mu$ with $p \notin \mathrm{cl}_{\beta H} B$. Let V be a neighborhood of p in $\beta\mathbb{H}$ that misses both $(\mathbb{H}-U)$ and B. Then $\mathbb{H} \cap V \subset (\mathbb{H}-B) \cap U$ and, since $p \in \mathbb{H}^* \cap V$, $\mathbb{H} \cap V$ is unbounded. Now $\mu(\mathbb{H}-B) = 0$ and so $\mu(\mathbb{H} \cap V) = 0$. We conclude that μ is fine.

(2) $\Rightarrow$ (1) Assume (2) and suppose (1) is false. Then there is $p \in \mathrm{int}_{\mathbb{H}^*} S_\mu = G \cap \mathbb{H}^*$ for some G open in $\beta\mathbb{H}$. There is U open in $\mathbb{H}$ with $p \in \mathrm{Ex}(U) \subset G$. Now U is unbounded and so there is $V \subset U$ with V open and unbounded with $\mu(V) = 0$. Since V is unbounded, there is $q \in \mathbb{H}^* \cap \mathrm{Ex}(V) \subset \mathbb{H}^* \cap \mathrm{Ex}(U) \subset \mathbb{H}^* \cap G = \mathrm{int}_{\mathbb{H}^*} S_\mu$. Then $q \notin \mathrm{cl}_{\beta\mathbb{H}}(\mathbb{H}-V)$ since $q \in \mathrm{Ex}(V)$ but $\mu(\mathbb{H}-V) = 1$, contradicting that $q \in S_\mu$.

We remark that **[18]** contains a similar result for the support set of a nonnegative regular matrix summability method.

For a topological space X, a subset A of X is a *P-set* in X if every G_δ-set in X containing A is a neighborhood of A.

3.12. THEOREM. *Let μ be separating. The following are equivalent.*

(1) *μ has the APO.*

(2) *S_μ is a P-set in $\mathbb{H}^*$.*

Proof. (1) $\Rightarrow$ (2) Let $S_\mu \subset \bigcap_{n\in\omega} G_n$ where each $G_n = P_n \cap \mathbb{H}^*$ for some open set P_n of $\beta\mathbb{H}$. Since S_μ is compact, there is an open set $V_n \subset \mathbb{H}$ with $S_\mu \subset \operatorname{Ex}(V_n) \subset P_n$. By 3.6, $\mu(V_n) = 1$ and so $\mu(\mathbb{H} - V_n) = 0$ for all $n \in \omega$ and $\mathbb{H} - V_n$ is closed. Since μ has the APO, there is a family $\langle B_n : n \in \omega\rangle$ with $B_n \Delta(\mathbb{H} - V_n) \subset [0, j_n]$ for some $j_n \in \mathbb{H}$ and $\mu(\operatorname{cl}_{\mathbb{H}} \bigcup_{n\in\omega} B_n) = 0$. Let $G = \mathbb{H} - \operatorname{cl}_{\mathbb{H}} \bigcup_{n\in\omega} B_n$. G is open in $\mathbb{H}$ and $\mu(G) = 1$. By 3.7(2), $S_\mu \subset \operatorname{Ex}(G)$. We show that $\mathbb{H}^* \cap \operatorname{Ex}(G) \subset \bigcap_{n\in\omega} G_n$.

Let $p \in \mathbb{H}^* \cap \operatorname{Ex}(G)$ and let $n \in \omega$. We will show that $p \in \operatorname{Ex}(V_n)$. Since $p \notin \operatorname{cl}_{\beta\mathbb{H}}(\mathbb{H} - G)$, there is a neighborhood V of p in $\beta\mathbb{H}$ with $V \cap (\mathbb{H} - G) = \varnothing$. Then $\mathbb{H} \cap V \subset G$. Since $p \notin \mathbb{H}$, $p \notin [0, j_n]$. Let $W = V - [0, j_n]$. W is a neighborhood of p in $\beta\mathbb{H}$. We show next that $W \cap (\mathbb{H} - V_n) = \varnothing$. Let $x \in W \cap \mathbb{H}$. Now $x \in G$ and so $x \notin B_n$. Since $x \notin [0, j_n]$, $x \notin B_n \Delta(\mathbb{H} - V_n) \supset (\mathbb{H} - V_n) - B_n$. Then $x \in (V_n \cup B_n) - B_n \subset V_n$, and so $W \cap (\mathbb{H} - V_n) = \varnothing$. We conclude that $p \in \mathbb{H}^* \cap \operatorname{Ex}(V_n) \subset \mathbb{H}^* \cap P_n = G_n$.

(2) $\Rightarrow$ (1) Let $\langle G_n : n \in \omega\rangle$ be a decreasing family of open sets in $\mathbb{H}$ with $\mu(G_n) = 1$ for all $n \in \omega$. By 3.7(1), $S_\mu \subset \bigcap_{n\in\omega} \operatorname{Ex}(G_n)$ and so there is an open set G of $\mathbb{H}$ with $S_\mu \subset \mathbb{H}^* \cap \operatorname{Ex}(G) \subset \mathbb{H}^* \cap \operatorname{cl}_{\beta\mathbb{H}} \operatorname{Ex}(G) \subset \bigcap_{n\in\omega} \operatorname{Ex}(G_n)$. By 3.6, $\mu(G) = 1$. We claim $G - G_n$ is bounded for all $n \in \omega$.

Note first that $\mathbb{H}^* \cap \operatorname{cl}_{\beta\mathbb{H}} \operatorname{Ex}(G) = (\operatorname{cl}_{\mathbb{H}} G)^*$ and hence, for all $n \in \omega$, $(\operatorname{cl}_{\mathbb{H}} G)^* \cap (\mathbb{H} - G_n)^* = \varnothing$. Now suppose $G - G_n$ is not bounded. Pick an increasing sequence $\langle x_j : j \in \omega\rangle \subset G - G_n$ with $x_j \nearrow \infty$. Let p be a limit point in $\beta\mathbb{H}$ of $\{x_j : j \in \omega\}$. Then $p \in \mathbb{H}^* \cap \operatorname{cl}_{\beta\mathbb{H}} G \cap \operatorname{cl}_{\beta\mathbb{H}}(\mathbb{H} - G_n)$, a contradiction.

See **[3]** for a result similar to 3.12 in the matrix setting.

3.13. COROLLARY. *If μ is a fine and separating f-measure with the APO (e.g., if μ is the f-measure associated with the Cesàro means), then S_μ is a nowhere dense P-set in $\mathbb{H}^*$.*

4. Invariants for bounded strong integral summability. We return now, as promised at the end of §2, to invariants for bounded strong integral summability. Before we begin, however, we need to make a few observations.

If K is a nonnegative regular summability method, we define

$$I_K = \{f : \mathbb{H} \to \mathbb{R} : f \text{ is bounded and strongly } K\text{-summable}\}$$

and we will note some relations that hold between I_K, μ_K and S_{μ_K}. First we note that if χ_A is a characteristic function, then χ_A is strongly K-summable to L if and only if $\mu_K(A) = L$. Furthermore, L must be either 0 or 1.

In the remainder of this section, K and K' will always represent nonnegative regular integral summability methods.

4.1. LEMMA. *If $I_K = I_{K'}$, then $\mu_K(A) = 1$ if and only if $\mu_{K'}(A) = 1$ and hence K and K' have the same support set.*

Proof. Assume that $\mu_K(A) = 1$. It suffices to show that $\mu_{K'}(A) \neq 0$. Suppose, on the contrary, that $\mu_{K'}(A) = 0$. Now by 3.1, there is a closed set $F \subset A$ such that F is not μ_K-measurable and hence $\chi_F \notin I_K$. But $\mu_{K'}(F) = 0$, and so $\chi_F \in I_{K'}$. Then $I_K \neq I_{K'}$. The remainder of the claim follows from the definition of a support set.

The following theorem now follows immediately from 3.11, 3.12 and 1.6.

4.2. THEOREM. *Suppose that $I_K = I_{K'}$.*

(1) *If K is fine, then K' is fine.*

(2) *If K and K' are separating and K has the APO, then K' has the APO.*

(3) *If K and K' are separating and K is thinning, then K' has the APO.*

A property (P) is said to be an invariant for bounded strong integral summability if, whenever K and K' are regular methods such that $I_K = I_{K'}$, then K' has property (P) whenever K has property (P). The preceding theorem shows that being fine is a summability invariant for regular methods and having the APO is a summability invariant for separating methods.

We can also establish the following partial converse to 4.1.

4.3. THEOREM. *Let K and K' be collapsing and thinning. If $S_{\mu_K} = S_{\mu_{K'}}$, then $I_K = I_{K'}$.*

Proof. Suppose that f is strongly K-summable to L. Theorem 2.1 yields that there is an $A \subset \mathbb{H}$ such that

$$\mu_K(A) = 1 \quad \text{and} \quad \lim_{t\to\infty}(f(t) - L)\chi_A(t) = 0.$$

Since K is collapsing, 1.3(3) yields that there is a closed set $F \subset A$ such that $\mu_K(F) = 1$. Now, since F^* contains the support set of K', 3.7(3) yields that $\mu_{K'}(F) = 1$ and hence f is convergent in $\mu_{K'}$-density to L. Again by 2.1, f is strongly K'-summable to L.

We conclude this section by recording some connections between the types of convergence discussed in §2 and the support set of a measure. We recall that $C^*(\beta\mathbb{H})$ denotes the bounded continuous real-valued functions on $\beta\mathbb{H}$.

4.4. THEOREM. *Let $f \in C^*(\beta\mathbb{H})$. If μ is any f-measure, then $f \restriction \mathbb{H}$ is μ-statistically convergent to L if and only if $f \restriction S_\mu = L$.*

Proof. Without loss of generality, suppose that $L = 0$. First suppose $f \restriction \mathbb{H}$ is μ-statistically convergent to 0. Let $\varepsilon > 0$ be given and observe that $F_\varepsilon = \{x \in \mathbb{H}: |f(x)| \le \varepsilon\}$ is closed and $\mu(F_\varepsilon) = 1$, and hence $S_\mu \subset (F_\varepsilon)^*$. It follows that $f(p) \in [-\varepsilon, \varepsilon]$ for all $p \in S_\mu$ and $\varepsilon > 0$, and hence $f \restriction S_\mu = 0$.

Suppose next that $f \restriction \mathbb{H}$ is not μ-statistically convergent to 0. Then there is an $\varepsilon > 0$ such that $F = \{x \in \mathbb{H}: |f(x)| \ge \varepsilon\}$ is not a μ-null set. Now $F^* \cap S_\mu$ is nonempty and hence there is a $p \in S_\mu$ such that $F \in p$ and thus $|f(p)| \ge \varepsilon$. It follows that $f \restriction S_\mu \ne 0$.

The hypothesis of continuity on $\beta\mathbb{H}$ cannot be dropped from the previous theorem. For instance, if we set $f = \chi_{\mathbb{H}}$, then f is lower semicontinuous on $\beta\mathbb{H}$ and $f \restriction \mathbb{H}$ is μ-statistically convergent to 1, yet $f \restriction S_\mu = 0$.

The next result is an abstract version of 2.1.

4.5. THEOREM. *Let $f \in C^*(\mathbb{H})$. If μ is a separating f-measure with the APO, then f is μ-statistically convergent to L if and only if f is convergent in μ-density to L.*

Proof. Let $\hat{f}$ extend f to $\beta\mathbb{H}$. If f is μ-statistically convergent to L then, by 4.4, $\hat{f} \restriction S_\mu = L$. Since μ has the APO, S_μ is a P-set and hence there is an open set $V \subset \mathbb{H}$ with $S_\mu \subset \mathrm{Ex}(V) \subset \hat{f}^{\leftarrow}\{L\}$. Now $\mu(V) = 1$ by 3.5 and $\lim_{t\to\infty}(f(t)-L)\chi_V(t) = 0$, i.e. f is convergent in μ-density to L.

It is straightforward to verify that f is μ-statistically convergent to L if f is convergent in μ-density to L.

5. Topological properties of support sets. A topological space X is *zero-dimensional* if X has a base for its neighborhoods consisting of clopen (=closed and open) sets.

We show next that if μ is collapsing, then its support set is zero-dimensional. First we need some preliminary results. Although the first lemma is probably known, we do not have a reference for it, and so we include a proof for completeness.

5.1. LEMMA. *Every point of* $\mathbb{H}^*$ *has a local base consisting of sets of the form* $\mathrm{Ex}(G)$ *where* G *is the union of a discrete collection of open intervals.*

Proof. Let U be open in $\beta\mathbb{H}$ with $p \in U$. Then there are open sets V and W in $\mathbb{H}$ with $p \in \mathrm{Ex}(V) \subset \mathrm{cl}_{\beta\mathbb{H}} \mathrm{Ex}(V) \subset \mathrm{Ex}(W) \subset U$. Let $W = \bigcup_{n\in\omega}(a_n, b_n)$ where the intervals are pairwise disjoint. Let $V_n = V \cap (a_n, b_n)$. Let $c_n = \inf V_n$ and $d_n = \sup V_n$. Let $\mathscr{E} = \{(c_n, d_n) : n \in \omega\}$ and $G = \bigcup \mathscr{E}$. Then, since $\mathrm{cl}_{\mathbb{H}} V \cap \mathrm{cl}_{\mathbb{H}}\{a_n, b_n : n \in \omega\} = \varnothing$, G is a union of a discrete collection of open intervals and $p \in \mathrm{Ex}(G) \subset U$.

5.2. PROPOSITION. *If* μ *is separating and if every point of* S_μ *is a far point, then* S_μ *is zero-dimensional.*

Proof. Let $p \in S_\mu$ and let $\mathscr{D} = \{A \subset \mathbb{H} : A$ is the union of a discrete collection of closed intervals and $p \in \mathrm{Ex}(\mathrm{int}_{\mathbb{H}} A)\}$. Let $\mathscr{B} = \{\mathrm{int}_{S_\mu} A^* : A \in \mathscr{D}\}$. By 5.1, $\mathscr{B}$ is a local base in S_μ at p. Now let $B = \mathrm{int}_{S_\mu} A^* \in \mathscr{B}$. To see that B is clopen in S_μ, we note that if $q \in \mathrm{bdy}_{S_\mu} B$, then $q \in \mathrm{cl}_{\beta\mathbb{H}} A \cap \mathrm{cl}_{\beta\mathbb{H}}(\mathbb{H} - \mathrm{int}_{\mathbb{H}} A) = \mathrm{bdy}_{\beta\mathbb{H}} \mathrm{Ex}(\mathrm{int}_{\mathbb{H}} A) = \mathrm{cl}_{\beta\mathbb{H}} \mathrm{bdy}_{\mathbb{H}} \mathrm{int}_{\mathbb{H}} A$ by 3.5(2). Then $q \in S_\mu \cap (\mathrm{bdy}_{\mathbb{H}} A)^*$ but $\mathrm{bdy}_{\mathbb{H}} A$ is closed discrete, contradicting that q is a far point.

The next corollary follows from 5.2 and 3.9.

5.3. COROLLARY. *If* μ *is a collapsing* f*-measure, then* S_μ *is zero-dimensional.*

5.4. Corollary. *if K is a nonnegative separating regular integral summability method, then S_{μ_K} is zero-dimensional.*

Example 1.8(4) shows that 5.4 is not true for an arbitrary separating f-measure, and thus the hypothesis that every point of S_μ is a far point cannot be dropped in 5.2.

5.5. Proposition. *If K is a collapsing nonnegative regular integral summability method, then S_{μ_K} has no isolated points.*

Proof. Let $p \in \mathrm{Ex}(V) \cap S_{\mu_K}$. By 5.1, we may write $V = \bigcup_{n\in\omega}(a_n, b_n)$ where $\{(a_n, b_n): n \in \omega\}$ is a discrete collection. Now by 3.7(1), V is not a null set, and so there a partition of ω into disjoint sets, J and I, such that neither $A = \bigcup_{n\in J}(a_n, b_n)$ nor $B = \bigcup_{n\in I}(a_n, b_n)$ is a null set. Since $\mathrm{cl}_{\beta\mathbb{H}} A \cap \mathrm{cl}_{\beta\mathbb{H}} B = \varnothing$, we may assume that $p \notin \mathrm{cl}_{\beta\mathbb{H}} A$. Now $S_{\mu_K} \cap \mathrm{cl}_{\beta\mathbb{H}} A \neq \varnothing$, but $S_{\mu_K} \cap (\mathrm{bdy}_{\mathbb{H}} A)^* = \varnothing$, and so $\{p\} \neq \mathrm{Ex}(V) \cap S_{\mu_K}$.

A space X is an *F-space* if cozero-sets are C^*-embedded in X (see [**11**, 14.25]). It is well known that $\mathbb{H}^*$ is an F-space (see [**11**, 14.27]) and clearly C^*-embedded subsets of F-spaces are F-spaces. Thus for any f-measure μ, S_μ is an F-space. A space X is a *P'-space* if nonempty G_δ-subsets of X have nonempty interiors. $\mathbb{H}^*$ is known to be a P'-space ([**9**, 3.1]). A space X is called a *Parovičenko space* if X is a compact zero-dimensional F-space of weight $\mathfrak{c}$ without isolated points that is also a P'-space. Parovičenko spaces are of interest, among other reasons, because the continuum hypothesis is equivalent to the statement that every Parovičenko space is homeomorphic to $\mathbb{N}^*$ ([**6**]). The following question, then, arises: If μ is a collapsing f-measure, is S_μ a P'-space? We will show that this need not be the case, but first we need a lemma.

5.6. Lemma. *Let K be a collapsing regular nonnegative integral summability method. Suppose there exists a sequence $\langle G_n : n \in \mathbb{N}\rangle$ of open sets in $\mathbb{H}$ with the following properties:*

(1) *$\{G_n : n \in \mathbb{N}\}$ is discrete in $\mathbb{H}$,*

(2) *for each $n \in \mathbb{N}$, $\mathrm{cl}_{\mathbb{H}} G_n$ is not a μ-null set, and*

(3) *for all sequences $\langle s_k : k \in \omega\rangle \subset \mathbb{H}$,*

$$\sup_{k\in\omega} \int_0^\infty K(s_k, t)\chi_{G_n}(t)\,dt \leq \frac{1}{2^n}.$$

Then S_{μ_K} is not a P'-space.

Proof. By 1.2, we can find, for each $n \in \mathbb{N}$, a set $F_n = \bigcup_{j\in\mathbb{N}}[c_{nj}, d_{nj}]$ where $\{[c_{nj}, d_{nj}]: j \in \mathbb{N}\}$ is a discrete family of closed intervals and $m(G_n - F_n) < \infty$. Then $\mu_K(G_n - F_n) = 0$ and so F_n is not a μ_K-null set. For all $n, j \in \mathbb{N}$, pick ε_{nj} with $c_{nj} < c_{nj} + \varepsilon_{nj} < d_{nj} - \varepsilon_{nj} < d_{nj}$ and such that $\sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}}\varepsilon_{nj}$ is finite. Let

$$H_n = \bigcup_{j\in\mathbb{N}}[c_{nj} + \varepsilon_{nj}, d_{nj} - \varepsilon_{nj}].$$

Then H_n is not a μ_K-null set. We can define a continuous function $f: \mathbb{H} \to (0, 1]$ with the following properties:

1. $f = 1/n$ on H_n,
2. $f = 1$ on $\mathbb{H} - \bigcup_{n\in\mathbb{N}} F_n$, and
3. f is piecewise linear on $\bigcup_{n,j\in\mathbb{N}}(c_{nj}, c_{nj}+\varepsilon_{nj}) \cup (d_{nj}-\varepsilon_{nj}, d_{nj})$.

Let $\hat{f}$ extend f to $\beta\mathbb{H}$. Since H_n is not μ_K-null, there is $p_n \in S_{\mu_K} \cap \mathrm{cl}_{\beta\mathbb{H}} H_n$ and $p \in S_{\mu_K}$ such that p is a limit point of $\{p_n: n \in \mathbb{N}\}$ in $\beta\mathbb{H}$. Clearly $p \in S_{\mu_K} \cap Z(\hat{f})$. We claim that $\mathrm{int}_{S_{\mu_K}} Z(\hat{f}) = \varnothing$. Suppose not. Then there is, by 5.1, an open set $V \subset \mathbb{H}$ with $m(\mathrm{bdy}_{\mathbb{H}}(V)) = 0$ and $q \in \mathrm{Ex}(V) \cap S_{\mu_K} \subset Z(\hat{f})$. Since $q \notin \mathrm{cl}_{\mathbb{H}}(\mathbb{H} - \bigcup_{n\in\mathbb{N}} F_n)$, we may assume that $V \subset \bigcup_{n\in\mathbb{N}} \mathrm{int}_{\mathbb{H}} F_n$. We note that $\mu_K(V \cap \mathrm{int}_{\mathbb{H}} F_n) = 0$ for all $n \in \mathbb{N}$.

We will now show that $\mu_K(V) = 0$. Let $\varepsilon > 0$ and let $\langle s_k: k \in \omega\rangle$ be any increasing sequence from $\mathbb{H}$. Let $N \in \mathbb{N}$ be such that $\sum_{n>N} 1/2^n < \varepsilon$. Let $F = \bigcup_{n>N} F_n$ and $G = \bigcup_{n>N} G_n$.

Now $\mu_K(\bigcup_{n\leq N} V \cap F_n) = 0$, and so

$$\begin{aligned}
\limsup_{k\to\infty} \int_0^\infty K(s_k, t)\chi_V(t)\,dt &= \limsup_{k\to\infty} \int_0^\infty K(s_k, t)\chi_{V\cap F}(t)\,dt \\
&\leq \limsup_{k\to\infty} \int_0^\infty K(s_k, t)\chi_G(t)\,dt \\
&\leq \limsup_{k\to\infty} \sum_{n>N} \int_0^\infty K(s_k, t)\chi_{G_n}(t)\,dt \\
&\leq \sum_{n>N} \frac{1}{2^n} < \varepsilon.
\end{aligned}$$

Since $m(\mathrm{bdy}_{\mathbb{H}} V) = 0$, $\mu_K(\mathrm{cl}_{\mathbb{H}} V) = 0$, and so $\mathrm{Ex}(V) \cap S_{\mu_K} = \varnothing$, a contradiction.

5.7. PROPOSITION. *Let K be the Cesàro means. There is a sequence $\langle G_n: n \in \mathbb{N}\rangle$ of open sets satisfying all the properties of* 5.6.

Proof. We first select, for all $n \in \mathbb{N}$, open intervals (a_n, b_n) satisfying:

$$\frac{1}{b_n}(b_n - a_n) > 1 - \frac{1}{2^{2n}} \quad \text{and} \quad \frac{1}{a_n}\sum_{k=1}^{n-1}(b_k - a_k) < \frac{1}{2^{2n}}.$$

This can be done by a construction similar to that of 3.1.

Now for each $n \in \mathbb{N}$, we select, by recursion on the set $\{\langle 1, n\rangle, \langle 2, n-1\rangle, \ldots, \langle n, 1\rangle\}$, $s(k, j)$ (for $k+j = n+1$) with the following properties:

(1) $s(1, n) \in (a_n, b_n)$ and $s(k+1, j-1) \in (s(k, j), b_n)$; and
(2)

$$\frac{1}{s(k+1, j-1)}(s(k+1, j-1) - s(k, j)) \in \left(\frac{1}{2^{k+1}} - \frac{1}{2^n}, \frac{1}{2^{k+1}} - \frac{1}{2^{2n}}\right).$$

The intermediate value theorem guarantees that this recursion process is possible.

Set $T_{1,n} = (a_n, s(1, n))$ and $T_{k,j} = (s(k-1, j+1), s(k, j))$. Let $H_n = \bigcup_{l=1}^{\infty} T_{n,l}$ for each $n \in \mathbb{N}$.

The collection $\langle H_n \colon n \in \mathbb{N}\rangle$ is not quite discrete, but we can easily find, as in the proof of 5.6, $G_n \subset H_n$ with $\mu_K(H_n - G_n) = 0$ such that $\langle G_n \colon n \in \mathbb{N}\rangle$ is a discrete collection of open intervals with endpoints satisfying 1 and 2.

One may verify that $\langle G_n \colon n \in \mathbb{N}\rangle$ satisfies all the properties of 5.6.

5.7. COROLLARY. *If K is the Cesàro means, then S_{μ_K} is a compact zero-dimensional F-space of weight $\mathfrak{c}$ with no isolated points that is not a P'-space.*

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