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THE DUAL PAIR $(U(1), U(1))$ OVER A p-ADIC FIELD

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We find an explicit decomposition for the metaplectic representation restricted to either member of the dual reductive pair $(U(1), U(1))$ in $\widetilde{\mathrm{SL}}(2, F)$, where F is a p-adic field, with p odd.

1. Introduction and preliminaries. Let F be a p-adic field of odd residual characteristic with q being the order of the residue class field. Let $\mathcal{O}$ be the ring of integers, $\mathcal{P}$ the prime ideal, $\mathcal{U}$ the units, π a prime element, and ν the valuation on F. Let $G = \mathrm{SL}(2, F)$.

For $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, let $x(\sigma) = c$ if $c \neq 0$, and let $x(\sigma) = d$ if $c = 0$. Define a 2-cocycle on G by

$$\alpha(g_1, g_2) = (x(g_1), x(g_2))(-x(g_1)x(g_2), x(g_1 g_2)).$$

This cocycle determines a nontrivial 2-sheeted covering group $\widetilde{G}$ of G **[G1]**.

Let E be a quadratic extension of F, and $x \mapsto \bar{x}$ the Galois action. The group $U(1)$ which preserves the Hermitian form $(x, y) \mapsto x\bar{y}$ on E is isomorphic to the group N^1 of norm one elements in E. The pair of subgroups $(U(1), U(1))$ of SL(2) form a dual reductive pair **[H]**. This dual pair is one of the simplest examples over a p-adic field. Some other basic examples of dual reductive pairs are discussed in **[G2]**. In this paper we determine the decomposition of the oscillator representation of $\widetilde{G}$ upon restriction to $U(1) \subset \widetilde{G}$.

The results in this paper have recently been applied by Rogawski to the problem of calculating the multiplicities of certain automorphic representations π of $U(\mathbf{A})$ in the discrete spectrum of $L^2(U(k)\backslash U(\mathbf{A}))$, where U is a unitary group in 3 variables defined relative to a quadratic extension of number fields K/k **[R1, R2]**. I would like to thank Rogawski for several stimulating conversations and for encouraging me to publish this paper.

Let τ be a character of F. Choose a normalized measure μ so that $\mu(\mathcal{O}) = q^{\frac{\omega(\tau)}{2}}$, where $\omega(\tau)$ is the conductor of τ. Denote this measure by $d_\tau x$. Then if we define the Fourier transform on $S(F)$, the space of locally compact functions on F with compact support, by

$$\hat{f}(x) = \int f(y)\tau(2xy)\, d_\tau y,$$

we have $\hat{\hat{f}}(x) = f(-x)$. For $a \in F$, we set $\tau_a(x) = \tau(ax)$. Let

$$\kappa(\tau) = \lim_{m\to-\infty} \int_{\mathscr{P}^m} \tau(x^2)\, d_\tau x.$$

Recall **[Sh]** that $\kappa(\tau) = 1$ if $\omega(\tau)$ even, and

$$\kappa(\tau) = G(\tau) = q^{-\frac{1}{2}} \sum_{x\in\mathscr{O}/\mathscr{P}} \tau(\pi^{n-1}x^2)$$

if $n = \omega(\tau)$ is odd. For $u \in \mathscr{U}$, let $(\frac{u}{\mathscr{P}}) = 1$ if u is a square, and $(\frac{u}{\mathscr{P}}) = -1$ otherwise. Then we have $G(\tau)^2 = (\frac{-1}{\mathscr{P}})$ and $G(\tau_u) = (\frac{u}{\mathscr{P}})G(\tau)$ for $u \in \mathscr{U}$.

We now define the metaplectic representation $W = W^\tau$ of $\widetilde{G}$ associated to the quadratic form $Q(x) = x^2$ by specifying the action on generators **[G1]**. Here $\zeta = \pm 1$, and $|a|$ is the absolute value on F.

$$W\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right), \zeta\right) f(x) = \zeta\tau(bx^2)f(x),$$

$$W\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right), \zeta\right) f(x) = \zeta|a|^{\frac{1}{2}}\frac{\kappa(\tau)}{\kappa(\tau_a)} f(ax),$$

$$W\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right), \zeta\right) f(x) = \zeta\kappa(\tau)\hat{f}(x).$$

The cocycle defining $\widetilde{G}$ splits on the compact subgroup $K = \mathrm{SL}_2(\mathscr{O})$ by a function $s : K \to Z_2$. K thus lifts as a subgroup of $\widetilde{G}$ by $k \mapsto (k, s(k))$, and we may thus restrict W to obtain a representation of K on $S(F)$. Note that $U(1) \subset K$. Our goal is to find the characters of $U(1)$ which appear in the restriction of W to $U(1)$.

Let $S(\mathscr{P}^r, \mathscr{P}^s)$ be the space of functions on F which have support on $\mathscr{P}^r$ and which are constant on cosets of $\mathscr{P}^s$ in $\mathscr{P}^r$. Suppose $\omega(\tau) = n \geq 1$. Then $S(\mathscr{O}, \mathscr{P}^n)$ is invariant under W^τ restricted to K, and the group

$$K_n = \left\{k \in K | k \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right) \bmod \mathscr{P}^n\right\}$$

acts trivially on $S(\mathscr{O}, \mathscr{P}^n)$. We thus obtain a representation $W_n = W_n^\tau$ of $K/K_n \cong \mathrm{SL}_2(\mathscr{O}/\mathscr{P}^n)$ on $S(\mathscr{O}, \mathscr{P}^n)$. Note that we may consider τ as a character of $\mathscr{O}/\mathscr{P}^n$.

2. Calculation of the trace. In this section we calculate the trace of $W_n(t)$, where t denotes either an element of T or its image in $\mathrm{Sl}_2(\mathscr{O}/\mathscr{P}^n)$, and

$$T = \left\{ \begin{pmatrix} a & b \\ b\alpha & a \end{pmatrix} \middle| a^2 - b^2\alpha = 1 \right\}$$

is the torus in G corresponding to the quadratic extension $E = F(\sqrt{\alpha})$. It will suffice to let $\alpha = \tau$ or $\alpha = \varepsilon$, a primitive $(q-1)$ st root of unity in $\mathscr{O}$.

LEMMA 1. *For $t = \begin{pmatrix} a & b \\ b\alpha & a \end{pmatrix} \in T$, we have the decomposition*

$$(1) \qquad (t, s(t)) = \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & 0 \\ b\alpha a & 1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}, 1 \right) \times \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \gamma(t) \right),$$

where $\gamma(t) = (a, b)$ if $\alpha = \varepsilon, b \in \mathscr{U}$, and $a \notin \mathscr{U}$, and $\gamma(t) = 1$ otherwise. Also,

$$(2) \quad \left(\begin{pmatrix} 1 & 0 \\ b\alpha a & 1 \end{pmatrix}, 1 \right) = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & -b\alpha a \\ 0 & 1 \end{pmatrix}, 1 \right) \times \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right).$$

Proof. Both statements are clearly true if $b=0$, so we suppose $b \neq 0$. A calculation shows that the right side of (1) equals $(t, (a, b\alpha a)\gamma(t))$. We must therefore show that $s(t) = (a, b\alpha a)\gamma(t)$ for $t \neq \pm I$. Recall [G] that $s(t) = (b\alpha, a)$ if $b \neq 0$ and $b\alpha \notin \mathscr{U}$, and $s(t) = 1$ otherwise. First suppose $\alpha = \pi$. In this case $a \in \mathscr{U}$, so $(a, b\pi a) = (a, b\pi) = s(t)$. Now suppose $\alpha = \varepsilon$. If $b \notin \mathscr{U}$, then $b^2\varepsilon \in \mathscr{P}^2 \Rightarrow a^2 = 1 + b^2\varepsilon \in 1 + \mathscr{P}^2 \subset \mathscr{U} \Rightarrow a \in \mathscr{U}$. Then $\gamma(t) = 1$, so $(a, b\varepsilon a)\gamma(t) = (a, b\varepsilon) = s(t)$. If $b \in \mathscr{U}$, then $s(t) = 1$, so we must show $(a, b\varepsilon a)\gamma(t) = 1$. If $a \in \mathscr{U}$, then $\gamma(t) = 1$ so we need $(a, b\varepsilon a) = 1$, which is true since $a \in \mathscr{U}$ and $b \in \mathscr{U}$. If $a \notin \mathscr{U}$, then $\gamma(t) = (a, b)$, so we must show $(a, b\varepsilon a)(a, b) = 1 \Leftrightarrow (a, \varepsilon a) = 1$. But $a \notin \mathscr{U} \Rightarrow a^2 \in \mathscr{P}^2 \Rightarrow 1+b^2\varepsilon \in \mathscr{P}^2 \Rightarrow -b^2\varepsilon \in 1+\mathscr{P}^2$. This shows $-\varepsilon \in \mathscr{U}^2$, so $(a, \varepsilon a) = (a, (-\varepsilon)(-a)) = (a, -\varepsilon)(a, -a) = (a, -a) = 1$.

LEMMA 2. *Suppose $t = \begin{pmatrix} a & b \\ b\alpha & a \end{pmatrix} \in T$ and $a \in \mathscr{U}$. Then for $f \in S(\mathscr{O}, \mathscr{P}^n)$,*

$$(W_n(t, s(t))f)(x) = \frac{\kappa(\tau)}{\kappa(\tau_a)} \sum_{s \in \mathscr{O}/\mathscr{P}^n} K_{b\alpha a}(ax, s)\tau\left(\frac{b}{a}s^2\right) f(s),$$

where, for $c \in \mathscr{O}$,

$$K_c(x\,,\,s) = q^{-n} \sum_{\mathscr{O}/\mathscr{P}^n} \tau(-cr^2)\tau(-2xr)\tau(2rs).$$

Proof. For any $\phi \in S(\mathscr{O}\,,\,\mathscr{P}^n)$, we have, for $c \in \mathscr{O}$,

$$\begin{aligned}
&\left(W\left(\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right), 1\right)\phi\right)(x) \\
&\quad = \left(W\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), 1\right) W\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1\right) W\left(\left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right), 1\right)\right. \\
&\qquad\qquad \left.\times\, W\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1\right)\phi\right)(x) \\
&\quad = \frac{\kappa(\tau)}{\kappa(\tau_{-1})}\left(W\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1\right) W\left(\left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right), 1\right)\right. \\
&\qquad\qquad \left.\times\, W\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1\right)\phi\right)(-x) \\
&\quad = \frac{\kappa(\tau)^2}{\kappa(\tau_{-1})}\left(W\left(\left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right), 1\right) W\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1\right)\phi\right)^{\wedge}(-x).
\end{aligned}$$

But $\phi \in S(\mathscr{O}\,,\,\mathscr{P}^n) \Rightarrow \hat{\phi} \in S(\mathscr{O}\,,\,\mathscr{P}^n)$, so for $c \in \mathscr{O}$, we have

$$W\left(\left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right), 1\right) W\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1\right)\phi \in S(\mathscr{O}\,,\,\mathscr{P}^n).$$

For any $\psi \in S(\mathscr{O}\,,\,\mathscr{P}^n)$, we have

$$\begin{aligned}
\hat{\psi}(x) &= \int \psi(y)\tau(2xy)\,d_\tau y = \int_{\mathscr{O}} \psi(y)\tau(2xy)\,d_\tau y \\
&= \sum_{r \in \mathscr{O}/\mathscr{P}^n} \int_{\mathscr{P}^n} \psi(r+y)\tau(2x(r+y))\,d_\tau y \\
&= \sum_{r \in \mathscr{O}/\mathscr{P}^n} \psi(r)\tau(2xr) \int_{\mathscr{P}^n} \tau(2xy)\,d_\tau y.
\end{aligned}$$

But $y \mapsto \tau(2xy)$ is trivial on $\mathscr{P}^n \Leftrightarrow x \in \mathscr{O}$, so $\hat{\psi}(x) = 0$ if $x \notin \mathscr{O}$, and if $x \in \mathscr{O}$, we have

$$\hat{\psi}(x) = q^{-\frac{n}{2}} \sum_{r \in \mathscr{O}/\mathscr{P}^n} \psi(r)\tau(2xr).$$

Therefore,

$$\begin{aligned}
&\left(W\left(\begin{pmatrix}1&0\\c&1\end{pmatrix}, 1\right)\phi\right)(x)\\
&= \frac{\kappa(\tau)^2}{\kappa(\tau_{-1})}q^{-\frac{n}{2}}\sum_{r\in\mathscr{O}/\mathscr{P}^n}\left(W\left(\begin{pmatrix}1&-c\\0&1\end{pmatrix}, 1\right)\right.\\
&\qquad\qquad\left.\times W\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}, 1\right)\phi\right)(r)\tau(-2xr)\\
&= \frac{\kappa(\tau)^2}{\kappa(\tau_{-1})}q^{-\frac{n}{2}}\sum_{r\in\mathscr{O}/\mathscr{P}^n}\tau(-cr^2)\left(W\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}, 1\right)\phi\right)(r)\tau(-2xr)\\
&= \frac{\kappa(\tau)^3}{\kappa(\tau_{-1})}q^{-n}\sum_{r\in\mathscr{O}/\mathscr{P}^n}\tau(-cr^2)\tau(-2xr)\sum_{s\in\mathscr{O}/\mathscr{P}^n}\phi(s)\tau(2rs).
\end{aligned}$$

But

$$\frac{\kappa(\tau)^3}{\kappa(\tau_{-1})} = 1,$$

so we get

$$\left(W\left(\begin{pmatrix}1&0\\c&1\end{pmatrix}, 1\right)\phi\right)(x) = \sum_{s\in\mathscr{O}/\mathscr{P}^n}K_c(x, s)\phi(s),$$

where for $c\in\mathscr{O}$,

$$K_c(x, s) = q^{-n}\sum_{r\in\mathscr{O}/\mathscr{P}^n}\tau(-cr^2)\tau(-2xr)\tau(2rs).$$

Now we calculate the action of $W_n(t, s(t))$ for $a\in\mathscr{U}$. Note that in this case, $\gamma(t) = 1$. For $f\in S(\mathscr{O}, \mathscr{P}^n)$, we have

$$\begin{aligned}
&(W_n(t, s(t))f)(x)\\
&= \left(W\left(\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}, 1\right)W\left(\begin{pmatrix}1&0\\b\alpha a&1\end{pmatrix}, 1\right)W\left(\begin{pmatrix}1&\frac{b}{a}\\0&1\end{pmatrix}, 1\right)f\right)(x)\\
&= \frac{\kappa(\tau)}{\kappa(\tau_a)}\left(W\left(\begin{pmatrix}1&0\\b\alpha a&1\end{pmatrix}, 1\right)W\left(\begin{pmatrix}1&\frac{b}{a}\\0&1\end{pmatrix}, 1\right)f\right)(ax)\\
&= \frac{\kappa(\tau)}{\kappa(\tau_a)}\sum_{s\in\mathscr{O}/\mathscr{P}^n}K_{b\alpha a}(ax, s)\tau\left(\frac{b}{a}s^2\right)f(s).
\end{aligned}$$

Here we used the fact that

$$a\in\mathscr{U}\Rightarrow W\left(\begin{pmatrix}1&\frac{b}{a}\\0&1\end{pmatrix}, 1\right)f\in S(\mathscr{O}, \mathscr{P}^n).$$

This completes the proof of Lemma 2.

If $a \in \mathscr{U}$, the action of $W_n(t, s(t))$ is therefore given by the kernel

$$\frac{\kappa(\tau)}{\kappa(\tau_a)} K_{b\alpha a}(ax, s)\tau\left(\frac{b}{a}s^2\right).$$

We now use this kernel to calculate the trace of $W_n(t, s(t))$ when $a \in \mathscr{U}$. The kernel is a function defined on $\mathscr{O}/\mathscr{P}^n \times \mathscr{O}/\mathscr{P}^n$, so we have

$$\begin{aligned}
\text{(3) trace } W_n(t, s(t)) &= \sum_{s\in\mathscr{O}/\mathscr{P}^n} \frac{\kappa(\tau)}{\kappa(\tau_a)} K_{b\alpha a}(as, s)\tau(\frac{b}{a}s^2) \\
&= \frac{\kappa(\tau)}{\kappa(\tau_a)} \sum_{s\in\mathscr{O}/\mathscr{P}^n} q^{-n} \sum_{r\in\mathscr{O}/\mathscr{P}^n} \tau(-b\alpha ar^2)\tau(2rs(1-a))\tau\left(\frac{b}{a}s^2\right) \\
&= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{-n} \sum_{r\in\mathscr{O}/\mathscr{P}^n} \tau(-b\alpha ar^2) \sum_{s\in\mathscr{O}/\mathscr{P}^n} \tau\left(\frac{b}{a}s^2 + 2r(1-a)s\right).
\end{aligned}$$

Suppose $\nu(b) = k$. The inner sum can be written

$$\begin{aligned}
&\sum_{u\in\mathscr{O}/\mathscr{P}^{n-k}} \sum_{v\in\mathscr{P}^{n-k}/\mathscr{P}^n} \tau\left(\frac{b}{a}(u+v)^2\right) \tau(2r(1-a)(u+v)) \\
&\quad = \sum_{u\in\mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right) \tau(2r(1-a)u) \sum_{v\in\mathscr{P}^{n-k}/\mathscr{P}^n} \tau(2r(1-a)v)
\end{aligned}$$

since $\frac{b}{a}uv \in \mathscr{P}^n$ and $\frac{b}{a}v^2 \in \mathscr{P}^n$.

Consider the sum

$$\sum_{v\in\mathscr{P}^{n-k}/\mathscr{P}^n} \tau(2r(1-a)v).$$

Since $a \in \mathscr{U}$, we may have $\nu(a-1) = 0$ or $\nu(a-1) > 0$. Suppose first that $\nu(a-1) = 0$. Then $\tau_{2r(1-a)}$ is trivial on $\mathscr{P}^{n-k} \Leftrightarrow \omega(\tau_{2r(1-a)}) \leq n-k \Leftrightarrow r \in \mathscr{P}^k$. If $r \notin \mathscr{P}^k$, we have

$$\sum_{v\in\mathscr{P}^{n-k}/\mathscr{P}^n} \tau(2r(1-a)v) = 0,$$

and (3) therefore equals

$$\text{(4)} \quad \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{-n} q^k \sum_{r\in\mathscr{P}^k/\mathscr{P}^n} \tau(-b\alpha ar^2) \sum_{u\in\mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}u^2 + 2r(1-au)\right).$$

The inner sum in (4) equals

$$(5)\quad \sum_{u\in\mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}\left(u^2+\frac{2r(1-a)a}{b}u\right)\right)$$
$$=\tau\left(-\frac{r^2(1-a)^2a}{b}\right)\sum_{u\in\mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}\left(u+\frac{r(1-a)a}{b}\right)^2\right).$$

Since $\nu(b)=k$ and $v\in\mathscr{P}^k$, we have $\nu(\frac{r(1-a)a)}{b})=\nu(r)-\nu(b)\geq 0$, so $\{u+\frac{r(1-a)a}{b}\}=\mathscr{O}/\mathscr{P}^{n-k}$ and (5) equals

$$\tau\left(-\frac{r^2(1-a)^2a}{b}\right)\sum_{u\in\mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right).$$

So if $a\in\mathscr{U}$, $a-1\in\mathscr{U}$, and $\nu(b)=k$, we have

$$(6)\ \text{trace}\ W_n(t, s(t))=\frac{\kappa(\tau)}{\kappa(\tau_a)}q^{k-n}\sum_{r\in\mathscr{P}^k/\mathscr{P}^n}\tau(-b\alpha ar^2)$$
$$\times\tau\left(-\frac{r^2(1-a)^2a}{b}\right)\sum_{u\in\mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right)$$
$$=\frac{\kappa(\tau)}{\kappa(\tau_a)}q^{k-n}\sum_{r\in\mathscr{P}^k/\mathscr{P}^n}\tau(cr^2)\sum_{u\in\mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right),$$

where $c=-\frac{2a^2(a-1)}{b}$.

Now we consider the sum

$$\sum_{v\in\mathscr{P}^{n-k}/\mathscr{P}^n}\tau(2r(1-a)v)$$

in the case when $\nu(a-1)>0$. We have $a^2-1=b^2\alpha\Rightarrow\nu(a-1)+\nu(a+1)=2\nu(b)+\nu(\alpha)$. Since $a-1\in\mathscr{P}$, we have $a+1=(a-1)+2\in\mathscr{U}$, so $\nu(a-1)=2\nu(b)+\nu(\alpha)$. We therefore have $\nu(a-1)>\nu(b)=k$. This shows that $\tau_{2r(1-a)}$ is trivial on $\mathscr{P}^{n-k}$ for all $r\in\mathscr{O}/\mathscr{P}^n$, and so (3) implies

$$(7)\qquad \text{trace}\ W_n(t, s(t))=\frac{\kappa(\tau)}{\kappa(\tau_a)}q^{k-n}\sum_{r\in\mathscr{O}/\mathscr{P}^n}\tau(-b\alpha ar^2)$$
$$\times\sum_{u\in\mathscr{O}/\mathscr{P}^{n-k}}\tau\left(\frac{b}{a}u^2+2r(1-a)u\right).$$

Considering (5) again, we have $\nu(\frac{r(1-a)a}{b}) > 0$, so if $a \in \mathscr{U}$, $a - 1 \in \mathscr{P}$, and $\nu(b) = k$, we have

$$(8)\ \operatorname{trace} W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathscr{O}/\mathscr{P}^n} \tau(-b\alpha a r^2) \times \tau\left(-\frac{r^2(1-a)^2 a}{b}\right) \sum_{u \in \mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right)$$
$$= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{-n} q^k \sum_{r \in \mathscr{O}/\mathscr{P}^n} \tau(cr^2) \sum_{u \in \mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right),$$

where $c = -\frac{2a^2(1-a)}{b}$.

We summarize (6) and (8) as follows

Lemma 3. *Suppose $a \in \mathscr{U}$ and $\nu(b) = k$. Let $c = -\frac{2a^2(1-a)}{b}$. Then*

$$(9)\quad \operatorname{trace} W_n(t, s(t)) = \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} \sum_{r \in \mathscr{P}^l/\mathscr{P}^n} \tau(cr^2) \sum_{u \in \mathscr{O}/\mathscr{P}^{n-k}} \tau\left(\frac{b}{a}u^2\right),$$

where $l = k$ if $a - 1 \in \mathscr{U}$ and $l = 0$ if $a - 1 \in \mathscr{P}$.

To calculate these sums we need

Lemma 4. *If $\omega(\tau) = n$ then $\sum_{x \in \mathscr{O}/\mathscr{P}^n} \tau(x^2) = q^{\frac{n}{2}} \kappa(\tau)$.*

Proof. Suppose n is even. Then

$$\sum_{x \in \mathscr{O}/\mathscr{P}^n} \tau(x^2) = \sum_{u \in \mathscr{O}/\mathscr{P}^{\frac{n}{2}}} \sum_{v \in \mathscr{P}^{\frac{n}{2}}/\mathscr{P}^n} \tau((u+v)^2)$$
$$= \sum_{u \in \mathscr{O}/\mathscr{P}^{\frac{n}{2}}} \tau(u^2) \sum_{v \mathscr{P}^{\frac{n}{2}}/\mathscr{P}^n} \tau(2uv).$$

But $v \mapsto \tau(2uv)$ is trivial on $\mathscr{P}^{\frac{n}{2}}/\mathscr{P}^n \Leftrightarrow u = 0$, so the sum is just $q^{\frac{n}{2}}$ in this case.

If n is odd, then

$$\sum_{x \in \mathscr{O}/\mathscr{P}^n} \tau(x^2) = \sum_{u \in \mathscr{O}/\mathscr{P}^{\frac{n+1}{2}}} \tau(u^2) \sum_{v \in \mathscr{P}^{\frac{n+1}{2}}/\mathscr{P}^n} \tau(2uv).$$

In this case, $v \mapsto \tau(2uv)$ is trivial on $\mathscr{P}^{\frac{n+1}{2}} \Leftrightarrow u \in \mathscr{P}^{\frac{n-1}{2}}$, so the sum equals

$$q^{\frac{n-1}{2}} \sum_{u \in \mathscr{P}^{\frac{n-1}{2}}/\mathscr{P}^{\frac{n+1}{2}}} \tau(u^2).$$

Writing $u = \pi^{\frac{n-1}{2}}$, with $r \in \mathcal{O}/\mathcal{P}$, the sum equals

$$q^{\frac{n-1}{2}} \sum_{r \in \mathcal{O}/\mathcal{P}} \tau(\pi^{n-1} r^2) = q^{\frac{n-1}{2}} q^{\frac{1}{2}} G(\tau) = q^{\frac{n}{2}} G(\tau).$$

This completes the proof of Lemma 4.

Now we apply Lemma 4 to the sums in (9). First, $\omega(\tau_{\frac{b}{a}}) = \omega(\tau) - \nu(\frac{b}{a}) = n - k$, so

$$\sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau\left(\frac{b}{a} u^2\right) = q^{\frac{n-k}{2}} \kappa(\tau_{\frac{b}{a}}).$$

Suppose $\nu(a-1) = 0$. Then

$$\sum_{r \in \mathcal{P}^k/\mathcal{P}^n} \tau(cr^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau(c\pi^{2k} u^2).$$

Since $\nu(c) = \nu(\frac{a-1}{b}) = -\nu(b) = -k$, $\omega(\tau_{c\pi^{2k}}) = n - 2k - \nu(c) = n - k$, and we have

$$\sum_{r \in \mathcal{P}^k/\mathcal{P}^n} \tau(cr^2) = q^{\frac{n-k}{2}} \kappa(\tau_{c\pi^{2k}}) = q^{\frac{n-k}{2}} \kappa(\tau_c).$$

Now suppose $\nu(a-1) > 0$ and consider

$$\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2).$$

If $\alpha = \varepsilon$ then $\nu(a-1) = 2\nu(b) = 2k$. We write

$$\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau(cu^2) \sum_{v \in \mathcal{P}^{n-k}/\mathcal{P}^n} \tau(2cuv).$$

But $\omega(\tau_{2cu}) = n - \nu(cu) \le n - k \Leftrightarrow \nu(cu) \ge k$, which is true for all $u \in \mathcal{O}$, so

$$\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) = q^k \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k}} \tau(cu^2) = q^k q^{\frac{n-k}{2}} \kappa(\tau_c),$$

where we used Lemma 4 since $\omega(\tau_c) = n - k$.

If $\alpha = \pi$, then $\nu(a-1) = 2\nu(b) + 1 = 2k + 1$. We write

$$\sum_{r \in \mathcal{O}/\mathcal{P}^n} \tau(cr^2) = \sum_{u \in \mathcal{O}/\mathcal{P}^{n-k-1}} \sum_{v \in \mathcal{P}^{n-k-1}/\mathcal{P}^n} \tau(c(u+v)^2)$$

and argue as above to obtain

$$\sum_{r \in \mathscr{O}/\mathscr{P}^n} \tau(cr^2) = q^{k+1} q^{\frac{n-k-1}{2}} \kappa(\tau_c).$$

Suppose that $a \in \mathscr{U}$ and $\nu(b) = k \geq 0$. We have now shown that if $\nu(a-1) = 0$, then we have

$$\text{(10)} \qquad \begin{aligned} \operatorname{trace} W_n(t, s(t)) &= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} q^{\frac{n-k}{2}} \kappa(\tau_c) q^{\frac{n-k}{2}} \kappa(\tau_{\frac{b}{a}}) \\ &= \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_{\frac{b}{a}}). \end{aligned}$$

If $\nu(a-1) > 0$ and $\alpha = \varepsilon$,

$$\text{(11)} \qquad \begin{aligned} \operatorname{trace} W_n(t, s(t)) &= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} q^k q^{\frac{n-k}{2}} \kappa(\tau_c) q^{\frac{n-k}{2}} \kappa(\tau_{\frac{b}{a}}) \\ &= q^k \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_{\frac{b}{a}}). \end{aligned}$$

If $\nu(a-1) > 0$ and $\alpha = \pi$,

$$\text{(12)} \qquad \begin{aligned} \operatorname{trace} W_n(t, s(t)) &= \frac{\kappa(\tau)}{\kappa(\tau_a)} q^{k-n} q^{k+1} q^{\frac{n-k-1}{2}} \kappa(\tau_c) q^{\frac{n-k}{2}} \kappa(\tau_{\frac{b}{a}}) \\ &= q^{\frac{2k+1}{2}} \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_{\frac{b}{a}}). \end{aligned}$$

We can summarize (10), (11), and (12) as follows.

Lemma 5. *If $a \in \mathscr{U}$ and $b \neq 0$, then*

$$\operatorname{trace} W_n(t, s(t)) = q^{\frac{\nu(a-1)}{2}} \frac{\kappa(\tau)}{\kappa(\tau_a)} \kappa(\tau_c) \kappa(\tau_{\frac{b}{a}}),$$

where $c = -\frac{2a^2(1-a)}{b}$.

To calculate trace $W_n(t, s(t))$ when $a \in \mathscr{P}$ we need another decomposition. Note that since $a \in \mathscr{P}$, we have $\alpha = \varepsilon$ and $b \in \mathscr{U}$.

Lemma 6.

$$(t, s(t)) = \left(\begin{pmatrix} -\frac{1}{b\varepsilon} & 0 \\ 0 & \frac{1}{b\varepsilon} \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & ab\varepsilon \\ 0 & 1 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \left(\begin{pmatrix} 1 & \frac{a}{b\varepsilon} \\ 0 & 1 \end{pmatrix}, 1 \right).$$

Proof. A calculation shows that the right side equals $(t, 1)$. Noting that $s(t) = 1$ in this case completes the proof.

Suppose $\nu(a) = m \geq 1$ and $\omega(\tau) = n$. Choose $f \in S(\mathscr{O}, \mathscr{P}^n)$. Using Lemma 6, we see that

$$\begin{aligned}
&(W_n(t, s(t))f)(x) \\
&= \left|-\frac{1}{b\varepsilon}\right|^{\frac{1}{2}} \frac{\kappa(\tau)}{\kappa(\tau_{-b\varepsilon})} \left(W\left(\left(\begin{smallmatrix} 1 & ab\varepsilon \\ 0 & 1 \end{smallmatrix}\right), 1\right)\right. \\
&\quad \left.\times W\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1\right) W\left(\left(\begin{smallmatrix} 1 & \frac{a}{b\varepsilon} \\ 0 & 1 \end{smallmatrix}\right), 1\right) f\right)\left(-\frac{1}{b\varepsilon}x\right) \\
&= \frac{\kappa(\tau)}{\kappa(\tau_{-b\varepsilon})}\tau\left(ab\varepsilon\left(-\frac{1}{b\varepsilon}\right)^2\right) \\
&\quad \times \left(W\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1\right) W\left(\left(\begin{smallmatrix} 1 & \frac{a}{b\varepsilon} \\ 0 & 1 \end{smallmatrix}\right), 1\right) f\right)\left(-\frac{1}{b\varepsilon}x\right) \\
&= \frac{\kappa(\tau)^2}{\kappa(\tau_{-b\varepsilon})}\tau\left(\frac{a}{b\varepsilon}x^2\right) q^{-\frac{n}{2}} \sum_{s \in \mathscr{O}/\mathscr{P}^n} \left(W\left(\left(\begin{smallmatrix} 1 & \frac{a}{b\varepsilon} \\ 0 & 1 \end{smallmatrix}\right), 1\right) f\right)(s)\tau\left(-\frac{2sx}{b\varepsilon}\right) \\
&= \frac{\kappa(\tau)^2}{\kappa(\tau_{-b\varepsilon})}\tau\left(\frac{a}{b\varepsilon}x^2\right) q^{-\frac{n}{2}} \sum_{s \in \mathscr{O}/\mathscr{P}^n} \tau\left(\frac{a}{b\varepsilon}s^2\right)\tau\left(-\frac{2sx}{b\varepsilon}\right) f(s) \\
&= \sum_{s \in \mathscr{O}/\mathscr{P}^n} K(x, s) f(s),
\end{aligned}$$

where

$$K(x, s) = q^{-\frac{n}{2}} \frac{\kappa(\tau)^2}{\kappa(\tau_{-b\varepsilon})}\tau\left(\frac{a}{b\varepsilon}x^2\right)\tau\left(\frac{a}{b\varepsilon}s^2\right)\tau\left(-\frac{2sx}{b\varepsilon}\right).$$

Since $\frac{a}{b\varepsilon} \in \mathscr{O}$ and $-\frac{2s}{b\varepsilon} \in \mathscr{O}$,

$$\begin{aligned}
\operatorname{trace} W_n(t, s(t)) &= \sum_{s \in \mathscr{O}/\mathscr{P}^n} K(s, s) \\
&= q^{-\frac{n}{2}} \frac{\kappa(\tau)^2}{\kappa(\tau_{-b\varepsilon})} \sum_{s \in \mathscr{O}/\mathscr{P}^n} \tau\left(\frac{a}{b\varepsilon}s^2\right)\tau\left(\frac{a}{b\varepsilon}s^2\right)\tau\left(-\frac{2s^2}{b\varepsilon}\right) \\
&= q^{-\frac{n}{2}} \frac{\kappa(\tau)^2}{\kappa(\tau_{-b\varepsilon})} \sum_{s \in \mathscr{O}/\mathscr{P}^n} \tau(cs^2),
\end{aligned}$$

where $c = \frac{2(a-1)}{b\varepsilon}$. Since $\nu(c) = \nu(a-1) - \nu(b) = 0$, we have $\omega(\tau_c) = n$. Using Lemma 4, we have

LEMMA 7. *If $a \in \mathscr{P}$, then*

$$\operatorname{trace} W_n(t, s(t)) = \frac{\kappa(\tau)^2}{\kappa(\tau_{-b\varepsilon})}\kappa(\tau_c),$$

where $c = \frac{2(a-1)}{b\varepsilon}$.

3. Further calculation of the trace. We now refine the formulas in Lemma 5 and Lemma 7. Suppose $E = F(\sqrt{\varepsilon})$. Letting $T_n = T \cap K_n$, we have a filtration $T \supset T_1 \supset \dots$, with $[T : T_1] = q+1$ and $[T_i : T_{i+1}] = q$ for $i \geq 1$. Let $n = \omega(\tau)$.

PROPOSITION 1. *Suppose E/F is unramified.*

(1) *For* $t \in T_k - T_{k+1}$, $k \geq 1$, $\operatorname{trace} W_n(t, s(t)) = (-1)^{n-k} q^k$.

(2) *For* $t \notin T_1$, $\operatorname{trace} W_n(t, s(t)) = \left(\frac{2(a-1)}{\mathscr{P}}\right)^n$.

Proof. Assume first $t \in T_1$. Then $a-1 \in \mathscr{P}$ and $b \in \mathscr{P}$. We have $\nu(a) = 0$. If $t \in T_k - T_{k+1}$, then $\nu(b) = k \geq 0$. We apply Lemma 5. We have $\nu(c) = \nu(\frac{b}{a}) = k$. Also, $\nu(a-1) = 2\nu(b) = 2k$. If n is even, we have $\kappa(\tau) = \kappa(\tau_a) = 1$. If in addition k is even, then $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$ and so trace $= q^k$. If n is even and k is odd, $\kappa(\tau_c) = G(\tau_c)$ and $\kappa(\tau_{\frac{b}{a}}) = G(\tau_{\frac{b}{a}})$, so trace $= q^k G(\tau_c) G(\tau_{\frac{b}{a}})$. Letting $b = u\pi^k$ and $a-1 = v\pi^{2k}$, we have $c = -\frac{2a^2 v}{u}$, so trace $= q^k \left(\frac{-2vu}{\mathscr{P}}\right)\left(\frac{ua}{\mathscr{P}}\right) G(\tau)^2 = q^k \left(\frac{2vu^2 a}{\mathscr{P}}\right) = q^k \left(\frac{2va}{\mathscr{P}}\right)$. But $a - 1 \in \mathscr{P} \Rightarrow a \in \mathscr{U}^2 \Rightarrow \left(\frac{a}{\mathscr{P}}\right) = 1$. Also, $a^2 = (1 + v\pi^{2k})^2 = 1 + 2v\pi^{2k} + v^2\pi^{4k}$, and $1 + b^2\varepsilon = 1 + u^2\pi^{2k}\varepsilon$. But $a^2 = 1 + b^2\varepsilon$, so $u^2\pi^{2k}\varepsilon = 2v\pi^{2k} + v^2\pi^{4k} \Rightarrow u^2\varepsilon = 2v + v^2\pi^{2k} \Rightarrow 2v = u^2\varepsilon - v^2\pi^{4k} = u^2\varepsilon\left(1 - \frac{v^2\pi^{2k}}{u^2\varepsilon}\right) \in u^2\varepsilon(1 + \mathscr{P}) \subset \varepsilon\mathscr{U}^2$, which implies $2v$ is not a square $\Rightarrow \left(\frac{2v}{\mathscr{P}}\right) = -1$, so trace $= -q^k$.

If n is odd then $\frac{\kappa(\tau)}{\kappa(\tau_a)} = \frac{G(\tau)}{G(\tau_a)} = \left(\frac{a}{\mathscr{P}}\right)$. If k is even, then $\kappa(\tau_c) = G(\tau_c)$ and $\kappa(\tau_{\frac{b}{a}}) = G(\tau_{\frac{b}{a}})$. Arguing as in the case of n even and k odd, we have trace $= q^k \left(\frac{a}{\mathscr{P}}\right) G(\tau_c) G(\tau_{\frac{b}{a}}) = q^k \left(\frac{2v}{\mathscr{P}}\right) = -q^k$. If k is odd, then $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1 \Rightarrow$ trace $= q^k \left(\frac{a}{\mathscr{P}}\right)$. But $a - 1 \in \mathscr{P} \Rightarrow \left(\frac{a}{\mathscr{P}}\right) = 1$, so trace $= q^k$. This completes the proof of (1) of Proposition 1.

Now assume $t \notin T_1$. Then $a - 1 \in \mathscr{U}$ or $b \in \mathscr{U}$. We consider various cases: (1) $a-1 \in \mathscr{U}$, $b \in \mathscr{U}$; (2) $a-1 \in \mathscr{U}$, $b \in \mathscr{P}$; (3) $a-1 \in \mathscr{P}$, $b \in \mathscr{U}$. Case (3) cannot arise, since $a^2 - 1 = b^2\varepsilon \Rightarrow \nu(a-1) + \nu(a+1) = 2\nu(b)$. Then $\nu(a-1) > 0 \Rightarrow \nu(b) > 0$, which is a contradiction.

We first consider case (1). In this case, we have $\nu(a-1) = 0$, $\nu(b) = 0$, and we may have $a \in \mathscr{U}$ or $a \in \mathscr{P}$. Suppose first $a \in \mathscr{U}$. We use Lemma 5. If n is even, $\kappa(\tau) = \kappa(\tau_a) = 1$. Also, $\nu(\frac{b}{a}) = \nu(c) = 0$, so $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$. Since $\nu(a-1) = 0$, trace $= 1$. If n is odd, trace $= \frac{G(\tau)}{G(\tau_a)} G(\tau_c) G(\tau_{\frac{b}{a}}) = \left(\frac{a}{\mathscr{P}}\right)\left(\frac{c}{\mathscr{P}}\right)\left(\frac{ba}{\mathscr{P}}\right) G(\tau)^2 = \left(\frac{2(a-1)}{\mathscr{P}}\right)$. Now suppose $a \in \mathscr{P}$. Then we must use Lemma 7. If n is even,

$\kappa(\tau) = \kappa(\tau_{-b\varepsilon}) = \kappa(\tau_c) = 1$, so trace $= 1$. If n is odd, trace$=$ $\frac{G(\tau)}{G(\tau_{-b\varepsilon})}G(\tau_c) = \left(\frac{cb\varepsilon}{\mathscr{P}}\right) = \left(\frac{2(a-1)}{\mathscr{P}}\right)$.

We next consider case (2). Now we have $a-1 \in \mathscr{U}$ and $b \in \mathscr{P}$, so $a \in \mathscr{U}$ and we can use Lemma 5. If n is even, then $\kappa(\tau) = \kappa(\tau_a) = 1$. If in addition $\nu(b)$ is even, then $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$, so trace $= 1$. If $\nu(b)$ is odd, then trace$= G(\tau_c)G(\tau_{\frac{b}{a}})$. Writing $b = u\pi^{2k+1}$, this equals $\left(\frac{-2(a-1)u}{\mathscr{P}}\right)\left(\frac{ua}{\mathscr{P}}\right)G(\tau)^2 = \left(\frac{2a(a-1)}{\mathscr{P}}\right)$. We claim $\left(\frac{2a(a-1)}{\mathscr{P}}\right) = 1$. We have $\nu(a-1)+\nu(a+1) = 2\nu(b) \geq 2$, so $a-1 \in \mathscr{U} \Rightarrow a+1 \in \mathscr{P} \Rightarrow a = -1+d$, $d \in \mathscr{P}$. This shows $a-1 = -2+d = -2(1-\frac{1}{2}d) \in -2\mathscr{U}_1 \subset -2\mathscr{U}^2$, so $\left(\frac{a-1}{\mathscr{P}}\right) = \left(\frac{-2}{\mathscr{P}}\right)$. Also, $a = -1+d \in (-1)\mathscr{U}_1 \Rightarrow \left(\frac{a}{\mathscr{P}}\right) = \left(\frac{-1}{\mathscr{P}}\right)$. Therefore, $\left(\frac{2a(a-1)}{\mathscr{P}}\right) = \left(\frac{2}{\mathscr{P}}\right)\left(\frac{a}{\mathscr{P}}\right)\left(\frac{a-1}{\mathscr{P}}\right) = \left(\frac{2}{\mathscr{P}}\right)\left(\frac{-1}{\mathscr{P}}\right)\left(\frac{-2}{\mathscr{P}}\right) = 1$, so in this case trace$= 1$.

Now suppose n is odd. Then $\kappa(\tau) = G(\tau)$ and $\kappa(\tau_a) = G(\tau_a)$, so trace$= \left(\frac{a}{\mathscr{P}}\right)\kappa(\tau_c)\kappa(\tau_{\frac{b}{a}})$. If $\nu(b)$ is even, $b = u\pi^{2k}$, then trace$=$ $\left(\frac{a}{\mathscr{P}}\right)G(\tau_c)G(\tau_{\frac{b}{a}}) = \left(\frac{a}{\mathscr{P}}\right)\left(\frac{-2(a-1)u}{\mathscr{P}}\right)\left(\frac{ua}{\mathscr{P}}\right)G(\tau)^2 = \left(\frac{2(a-1)}{\mathscr{P}}\right)$. If $\nu(b)$ is odd, $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$, so trace$= \left(\frac{a}{\mathscr{P}}\right)$. But we saw above that $\left(\frac{2a(a-1)}{\mathscr{P}}\right) = 1$, so trace$= \left(\frac{a}{\mathscr{P}}\right) = \left(\frac{2(a-1)}{\mathscr{P}}\right)$. This finishes case (2) and thus completes the proof of Proposition 1.

Now we assume E/F is ramified, $E = F(\sqrt{\pi})$. We have a filtration $T \supset T_0 \supset T_1 \supset \dots$, where $T_n = \{\left(\begin{smallmatrix} a & b \\ b\pi & a \end{smallmatrix}\right) | a \in 1+\mathscr{P}^{2n+1}, b \in \mathscr{P}^n\}$. We have $[T : T_0] = 2$ and $[T_n : T_{n+1}] = q$ for $n \geq 1$. Recall that we have a bijection $\phi : \mathscr{O} \to T_0$, where we identify $\left(\begin{smallmatrix} a & b \\ b\pi & a \end{smallmatrix}\right) \in T_0$ with $a + b\sqrt{\pi} \in N^1$ [S]. ϕ is given by

$$\phi(x) = \frac{1+\pi x^2}{1-\pi x^2} + \sqrt{\pi}\frac{2x}{1-\pi x^2},$$

$x \in \mathscr{O}$. Representatives for $\mathscr{P}^n$ in $\mathscr{O}$ can be taken to be $\{a_0 + a_1\pi + \cdots + a_{n-1}\pi^{n-1} | a_i = 0 \text{ or } a_i = \varepsilon^j, 0 \leq j \leq q-2\}$.

PROPOSITION 2. *Suppose E/F is ramified.*

(1) *Say $t \in T_i - T_{i+1}$, $t = \phi(x)$, $x = a_i\pi^i + \cdots + a_{n-1}\pi^{n-1}$, with $a_i = \varepsilon^{j(t)}$, $0 \leq j(t) \leq q-2$. Then*

$$\operatorname{trace} W_n(t, s(t)) = q^{\frac{2i+1}{2}}(-1)^{j(t)}\left(\frac{2}{\mathscr{P}}\right)\left(\frac{-1}{\mathscr{P}}\right)^{n+i+1}G(\tau).$$

(2) *Say $t \in T - T_0$. Then* $\operatorname{trace} W_n(t, s(t)) = \left(\frac{-1}{\mathscr{P}}\right)^n$.

Proof. We may use Lemma 5 in all cases. Assume first $t \in T_i - T_{i+1}$. Suppose that n and i are both even. With $x = a_i\pi^i + \cdots + a_{n-1}\pi^{n-1}$, $\nu(x) = i$. If $\phi(x) = a + b\sqrt{\pi}$, then $\nu(b) = i$, $\nu(a-1) = 2i+1$, and $\nu(c) = i+1$, where $c = -\frac{2a^2(a-1)}{b}$. Then $\kappa(\tau_c) = G(\tau_c)$ and $\kappa(\tau_{\frac{b}{a}}) = 1$. Therefore trace $W_n(t, s(t)) = q^{\frac{2i+1}{2}} G(\tau_c)$. But $G(\tau_c) = \left(\frac{-2}{\mathscr{P}}\right) G(\tau_{\frac{a-1}{b}})$. Now, $\frac{a-1}{b} = \pi x$, so $G(\tau_{\frac{a-1}{b}}) = G(\tau_{\pi x}) = \left(\frac{a_i + a_{i+1}\pi + \cdots + a_{n-1}\pi^{n-i-1}}{\mathscr{P}}\right) G(\tau)$. With $a_i = \varepsilon^{j(t)}$, $a_i + a_{i+1}\pi + \cdots + a_{n-1}\pi^{n-i-1} \in \varepsilon\mathscr{U}^2$, so $G(\tau_{\frac{a-1}{b}}) = \left(\frac{\varepsilon^{j(t)}}{\mathscr{P}}\right) G(\tau) = (-1)^{j(t)} G(\tau)$. So

$$\text{trace} = q^{\frac{2i+1}{2}} \left(\frac{-2}{\mathscr{P}}\right) (-1)^{j(t)} G(\tau) = q^{\frac{2i+1}{2}} (-1)^{j(t)} \left(\frac{2}{\mathscr{P}}\right) \left(\frac{-1}{\mathscr{P}}\right)^{n+i+1} G(\tau).$$

If n is even and i is odd, then $\kappa(\tau_c) = 1$ and $\kappa(\tau_{\frac{b}{a}}) = G(\tau_{\frac{b}{a}})$, so trace $= q^{\frac{2i+1}{2}} G(\tau_c) G(\tau_{\frac{b}{a}})$. We have

$$\begin{aligned} \frac{b}{a} &= \frac{2x}{1+\pi x^2} \\ &= \frac{2a_i\pi^i}{1+\pi x^2}\left[1 + \frac{a_{i+1}}{a_i}\pi + \cdots + \frac{a_{n-1}}{a_i}\pi^{n-i-1}\right] \in \frac{2a_i\pi^i}{1+\pi x^2}\mathscr{U}^2, \end{aligned}$$

so $G(\tau_{\frac{b}{a}}) = \left(\frac{2a_i}{\mathscr{P}}\right) G(\tau) = \left(\frac{2\varepsilon^{j(t)}}{\mathscr{P}}\right) G(\tau) = \left(\frac{2}{\mathscr{P}}\right) (-1)^{j(t)} G(\tau)$. Therefore, trace $= q^{\frac{2i+1}{2}} \left(\frac{2}{\mathscr{P}}\right) (-1)^{j(t)} G(\tau)$.

If n is odd and i is even,

$$\text{trace} = q^{\frac{2i+1}{2}} \frac{G(\tau)}{G(\tau_a)} G(\tau_{\frac{b}{a}}) = q^{\frac{2i+1}{2}} \left(\frac{2}{\mathscr{P}}\right) (-1)^{j(t)} G(\tau).$$

If n is odd and i is odd,

$$\text{trace} = q^{\frac{2i+1}{2}} \frac{G(\tau)}{G(\tau_a)} G(\tau_c) = q^{\frac{2i+1}{2}} \left(\frac{2}{\mathscr{P}}\right) \left(\frac{-1}{\mathscr{P}}\right) (-1)^{j(t)} G(\tau).$$

This completes the proof of (1).

Now suppose $t \notin T_0$. For elements of T/T_0 we use $\{t\} = \{-r\}$, $r \in T_0$. We therefore write $t = \begin{pmatrix} -a & -b \\ -b\pi & -a \end{pmatrix}$, with $a \in 1 + \mathscr{P}$, $b \in \mathscr{O}$, and $c = -\frac{2a^2(a+1)}{b}$. If n is even, then $\kappa(\tau) = \kappa(\tau_a) = 1$. If in addition $\nu(b)$ is even, then $\kappa(\tau_c) = \kappa(\tau_{\frac{b}{a}}) = 1$, so trace $= 1$. If $\nu(b)$ is odd, trace $= G(\tau_{-\frac{2(a+1)}{b}}) G(\tau_{\frac{b}{a}})$. Writing $b = u\pi^{2l+1}$, this equals $\left(\frac{-1}{\mathscr{P}}\right)\left(\frac{-2(a+1)u}{\mathscr{P}}\right)\left(\frac{ua}{\mathscr{P}}\right) = \left(\frac{2a(a+1)}{\mathscr{P}}\right)$. But $\nu(a-1)+\nu(a+1) = 2\nu(b)+1$, with $\nu(a+1) = 0$ and $\nu(b) > 0$, so $a - 1 \in \mathscr{P} \Rightarrow a + 1 \in 2 + \mathscr{P} \subset 2\mathscr{U}^2 \Rightarrow$

$\left(\frac{a+1}{\mathscr{P}}\right) = \left(\frac{2}{\mathscr{P}}\right)$. Also, $a \in 1+\mathscr{P} \Rightarrow \left(\frac{a}{\mathscr{P}}\right) = 1$, so $\left(\frac{2a(a+1)}{\mathscr{P}}\right) = \left(\frac{a}{\mathscr{P}}\right) = 1$, and therefore trace $= 1$.

If n is odd, trace $= \frac{G(\tau)}{G(\tau_{-a})}\kappa(\tau_{-\frac{2(a+1)}{b}})\kappa(\tau_{\frac{b}{a}}) = \left(\frac{-1}{\mathscr{P}}\right)\kappa(\tau_{-\frac{2(a+1)}{b}})\kappa(\tau_{\frac{b}{a}})$. If $\nu(b)$ is even, write $b = u\pi^{2k}$. Then trace $= \left(\frac{-1}{\mathscr{P}}\right)\left(\frac{-2(a+1)u}{\mathscr{P}}\right)G(\tau)\left(\frac{u}{\mathscr{P}}\right)G(\tau)$ $= \left(\frac{2(a+1)}{\mathscr{P}}\right)\left(\frac{-1}{\mathscr{P}}\right)$. But we still have $a+1 \in 2\mathscr{U}^2$, so trace $= \left(\frac{-1}{\mathscr{P}}\right)$. If $\nu(b)$ is odd, $\kappa(\tau_{-\frac{2(a+1)}{b}}) = \kappa(\tau_{\frac{b}{a}}) = 1$, so trace $= \left(\frac{-1}{\mathscr{P}}\right)$. For $t \notin T_0$, therefore, trace $= \left(\frac{-1}{\mathscr{P}}\right)^n$. This completes the proof of Proposition 2.

4. Calculation of multiplicities. In this section we choose $\chi \in \widehat{T}$ with conductor $c(\chi)$ less than or equal to n, and we calculate $\langle\chi, W_n\rangle$, the multiplicity of χ in W_n, χ and W_n being considered as representations of T/T_n.

Assume first that E/F is unramified. Let us say that the conductor of the trivial character of T is zero, and we let θ_0 be the unique nontrivial character of conductor 1 such that $\theta_0^2 = 0$.

LEMMA 8. *For $t \notin T_1$, $t = \begin{pmatrix} a & b \\ b\varepsilon & a \end{pmatrix}$, we have* $\left(\frac{2(a-1)}{\mathscr{P}}\right) = -\theta_0(t)$.

Proof. We identify $t \in T$ with $\lambda = a + b\sqrt{\varepsilon} \in N^1$. Let $|x|_E$ be the valuation on E. If $|1+\lambda|_E = 1$, we can write $\lambda = \frac{1+x\sqrt{\varepsilon}}{1-x\sqrt{\varepsilon}}$, $x \in \mathscr{O}$. Then $\lambda + \lambda^{-1} + 2 = \frac{4}{1-\varepsilon x^2}$, and $2(a-1) = \lambda + \lambda^{-1} - 2 = \frac{4\varepsilon x^2}{1-\varepsilon x^2}$. It is proved in **[S-Sh]** that if $|1+\lambda|_E = 1$, then $\left(\frac{\lambda+\lambda^{-1}+2}{\mathscr{P}}\right) = \left(\frac{1-\varepsilon x^2}{\mathscr{P}}\right) = \theta_0(\lambda)$. Therefore, $\left(\frac{2(a-1)}{\mathscr{P}}\right) = \left(\frac{\lambda+\lambda^{-1}-2}{\mathscr{P}}\right) = \left(\frac{4\varepsilon x^2(1-\varepsilon x^2)}{\mathscr{P}}\right) = -\left(\frac{1-\varepsilon x^2}{\mathscr{P}}\right) = -\theta_0(t)$. If $|1+\lambda|_E > 0$, then $-\lambda \in 1+\mathscr{P}_E$ ($\mathscr{P}_E$ the prime ideal in E) and $\lambda = -s^2$, $s \in N^1$. Write $s = c + d\sqrt{\varepsilon}$. Then $\lambda = -s^2 \Rightarrow 2(a-1) = -4c^2$, so $\left(\frac{2(a-1)}{\mathscr{P}}\right) = \left(\frac{-1}{\mathscr{P}}\right)$. But we also have $\lambda = -s^2 \Rightarrow \theta_0(\lambda) = \theta_0(-s^2) = \theta_0(-1)$, and it is proved in **[S-Sh]** that $\theta_0(-1) = -\left(\frac{-1}{\mathscr{P}}\right)$. Therefore, $\left(\frac{2(a-1)}{\mathscr{P}}\right) = \left(\frac{-1}{\mathscr{P}}\right) = -\theta_0(-1) = -\theta_0(\lambda)$. This completes the proof of Lemma 8.

PROPOSITION 3. *Suppose E/F is unramified and $c(\chi) = i$.*

(1) *If n is even and i is even, then $\langle\chi, W_n\rangle = 1$.*
(2) *If n is even and i is odd, then $\langle\chi, W_n\rangle = 0$.*
(3) *Say n is odd and i is even. Then $\langle\chi, W_n\rangle = 0$ if $\chi \neq 1$, and $\langle 1, W_n\rangle = 1$.*
(4) *Say n is odd and i is odd. Then $\langle\chi, W_n\rangle = 1$ if $\chi \neq \theta_0$, and $\langle\theta_0, W_n\rangle = 0$.*

Proof. Suppose $n = \omega(\tau)$ is even and $c(\chi) = i > 1$. Then

$$\langle \chi , W_n \rangle = \frac{1}{(q+1)q^{n-1}} \left[q^n + \sum_{t \notin T_1} \overline{\chi}(t) + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} \overline{\chi}(t)(-1)^m q^m \right].$$

But $\sum_{t \notin T_1} \overline{\chi}(t) = \sum_{t \in T} \overline{\chi}(t) - \sum_{t \in T_1} \overline{\chi}(t) = 0$, so

$$\begin{aligned}
\langle \chi , W_n \rangle &= \frac{1}{(q+1)q^{n-1}} \\
&\quad \times \left[q^n + \sum_{m=1}^{i-2} \left[(-1)^m q^m \sum_{t \in T_m} \overline{\chi}(t) - (-1)^m q^m \sum_{t \in T_{m+1}} \overline{\chi}(t) \right] \right. \\
&\quad + \left[(-1)^{i-1} q^{i-1} \sum_{t \in T_{i-1}} \overline{\chi}(t) - (-1)^{i-1} q^{i-1} \sum_{t \in T_i} 1 \right] \\
&\quad \left. + \sum_{m=i}^{n-1} \left[(-1)^m q^m q^{n-m} - (-1)^m q^m q^{n-m-1} \right] \right] \\
&= \frac{1}{(q+1)q^{n-1}} \left[q^n - (-1)^{i-1} q^{i-1} q^{n-i} \right. \\
&\quad \left. + \sum_{m=i}^{n-1} \left[(-1)^m q^n - (-1)^m q^{n-1} \right] \right] \\
&= \frac{1}{(q+1)q^{n-1}} \left[q^n - (-1)^{i-1} q^{n-1} + (q^n - q^{n-1}) \sum_{m=i}^{n-1} (-1)^m \right].
\end{aligned}$$

If i is even, this equals one, and if i is odd, it equals zero.

If n is even and $c(\chi) = 1$, then

$$\begin{aligned}
\langle \chi , W_n \rangle &= \frac{1}{(q+1)q^{n-1}} \left[q^n + \sum_{t \notin T_1} \overline{\chi}(t) + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^m q^m \right] \\
&= \frac{1}{(q+1)q^{n-1}} \left[q^n - q^{n-1} - (q^n - q^{n-1}) \right] = 0.
\end{aligned}$$

Also, if n is even, then

$$\langle 1 , W_n \rangle = \frac{1}{(q+1)q^{n-1}} \left[q^n + \sum_{t \notin T_1} 1 + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^m q^m \right] = 1.$$

This proves (1) and (2) of Proposition 3.

Now suppose n is odd. If $c(\chi) = i > 1$ then

$$\langle \chi , W_n \rangle = \frac{1}{(q+1)q^{n-1}} \times \left[q^n - \sum_{t \notin T_1} \overline{\chi}(t)\theta_0(t) + \sum_{m=1}^{i-1} \sum_{t \in T_m - T_{m+1}} \overline{\chi}(t)(-1)^{m+1} q^m + \sum_{m=i}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^{m+1} q^m \right].$$

But $\sum_{t \notin T_1} \overline{\chi}(t)\theta_0(t) = 0$ and

$$\sum_{m=1}^{i-2} \sum_{t \in T_m - T_{m+1}} \overline{\chi}(t)(-1)^{m+1} q^m = 0,$$

so

$$\langle \chi , W_n \rangle = \frac{1}{(q+1)q^{n-1}} \left[q^n + (-1)^i q^{i-1} q^{n-i} + (q^n - q^{n-1}) \sum_{m=i}^{n-1} (-1)^{m+1} \right].$$

If i is even, this equals zero and if i is odd, it equals one.

If $c(\chi) = 1$ or $\chi = 1$, then

$$\begin{aligned}
\langle \chi , W_n \rangle &= \frac{1}{(q+1)q^{n-1}} \left[q^n - \sum_{t \notin T_1} \overline{\chi}(t)\theta_0(t) + \sum_{m=1}^{n-1} \sum_{t \in T_m - T_{m+1}} (-1)^{m+1} q^m \right] \\
&= \frac{1}{(q+1)q^{n-1}} \left[q^n - \sum_{t \in T} \overline{\chi}(t)\theta_0(t) + \sum_{t \in T_1} \overline{\chi}(t)\theta_0(t) \right] \\
&= \frac{q^n}{(q+1)q^{n-1}} - \langle \chi , \theta_0 \rangle + \frac{q^{n-1}}{(q+1)q^{n-1}} \\
&= 1 - \langle \chi , \theta_0 \rangle .
\end{aligned}$$

This completes the proof of Proposition 3.

Now we assume E/F is ramified. Let θ_0 be the unique nontrivial character of T/T_0.

PROPOSITION 4. *Let E/F be ramified. Then*

(1) $\langle 1 , W_n \rangle = 1$ *if n is even or* $\left(\frac{-1}{\mathscr{P}}\right) = 1$, *and equals* 0 *otherwise.*

(2) $\langle \theta_0 , W_n \rangle = 1 - \langle 1 , W_n \rangle$.

Proof. We have

$$\langle 1, W_n\rangle = \frac{1}{2q^n}\Bigg[q^n + \sum_{t\notin T_0}\left(\frac{-1}{\mathcal{P}}\right)^n + \sum_{i=0}^{n-1}\sum_{t\in T_i - T_{i+1}} q^{\frac{2i+1}{2}}(-1)^j\left(\frac{2}{\mathcal{P}}\right)\left(\frac{-1}{\mathcal{P}}\right)^{n+i+1} G(\tau)\Bigg],$$

where j was defined in Proposition 2. Consider $\sum_{t\in T_i - T_{i+1}}(-1)^j$. Since $a_i = \varepsilon^j$, and $h \neq i \Rightarrow a_h$ can assume the values $0, 1, \varepsilon, \dots, \varepsilon^{q-2}$, this sum is zero, so $\langle 1, W_n\rangle = \frac{1}{2q^n}[q^n + (\frac{-1}{\mathcal{P}})^n q^n]$, which gives the result.

Similarly, $\langle \theta_0, W_n\rangle = \frac{1}{2q^n}[q^n + (\frac{-1}{\mathcal{P}})^n \sum_{t\notin T_0}\theta_0(t)]$. But $\sum_{t\notin T_0}\theta_0(t) = \sum_{t\in T}\theta_0(t) - \sum_{t\in T_0}\theta_0(t) = -q^n$, so $\langle \theta_0, W_n\rangle = \frac{1}{2}[1 - (\frac{-1}{\mathcal{P}})^n]$. This completes the proof of Proposition 4.

PROPOSITION 5. *Assume $c(\chi) = m > 0$. Then $\langle \chi, W_n\rangle$ equals 0 or 1, and exactly half of the characters χ of conductor m satisfy $\langle \chi, W_n\rangle = 1$.*

Proof. We have

$$\langle \chi, W_n\rangle = \frac{1}{2q^n}\Bigg[q^n + \sum_{t\notin T_0}\overline{\chi}(t)\left(\frac{-1}{\mathcal{P}}\right)^n + \sum_{i=0}^{n-1}\sum_{T_i - T_{i+1}} \overline{\chi}(t) q^{\frac{2i+1}{2}}\left(\frac{-1}{\mathcal{P}}\right)^{n+i+1}(-1)^{j(t)}G(\tau)\Bigg],$$

where $j(t)$ is as in Proposition 2. Since χ is nontrivial on T_0, $\sum_{t\notin T_0}\overline{\chi}(t) = 0$, so

$$\begin{aligned}\langle \chi, W_n\rangle = \frac{1}{2q^n}\Bigg[q^n + \left(\frac{2}{\mathcal{P}}\right)\left(\frac{-1}{\mathcal{P}}\right)^{n+1} G(\tau) &\times \Bigg[\sum_{i=0}^{m-2}\left(\frac{-1}{\mathcal{P}}\right)^i q^{\frac{2i+1}{2}} \sum_{t\in T_i - T_{i+1}} \overline{\chi}(t)(-1)^{j(t)} \\ &+ \left(\frac{-1}{\mathcal{P}}\right)^{m-1} q^{\frac{2m-1}{2}} \sum_{t\in T_{m-1} - T_m} \overline{\chi}(t)(-1)^{j(t)} \\ &+ \sum_{i=m}^{n-1}\left(\frac{-1}{\mathcal{P}}\right)^i q^{\frac{2i+1}{2}} \sum_{t\in T_i - T_{i+1}} (-1)^{j(t)}\Bigg]\Bigg].\end{aligned}$$

As before, $\sum_{t\in T_i-T_{i+1}}(-1)^{j(t)}=0$ for $m\le i\le n-1$. Now consider $\sum_{t\in T_i-T_{i+1}}\overline{\chi}(t)(-1)^{j(t)}$ for $0\le i\le m-2$. Write this sum as

$$\sum_{S_1}\sum_{S_2}\overline{\chi}(\phi(a_i\pi^i+\cdots+a_{n-1}\pi^{n-1}))(-1)^{j(t)},$$

where $S_1=\{a_i, a_{i+1}, \dots, a_{m-2}|a_i\neq 0\}$, $S_2=\{a_{m-1}, \dots, a_{n-1}\}$, and ϕ is the map on $\mathscr{O}$ to T_0 which was recalled above. If $x\in\mathscr{P}^n$, then $\phi(x)\in T_n$. If $x, y\in\mathscr{O}$,

$$\frac{\phi(x)\phi(y)}{\phi(x+y)}=\frac{a-b\sqrt{\pi}}{a+b\sqrt{\pi}}=c+d\sqrt{\pi},$$

where $a=1-\pi(x^2+xy+y^2)$, $b=\pi xy(x+y)$, $c=\frac{a^2+b^2\pi}{a^2-b^2\pi}$, and $d=-\frac{2ab}{a^2-b^2\pi}$. Let $x=a_i\pi^i+\cdots+a_{m-2}\pi^{m-2}$ and $y=a_{m-1}\pi^{m-1}+\cdots+a_{n-1}\pi^{n-1}$. Then $\nu(x)=i$ and y either equals 0 or satisfies $\nu(y)\ge m-1$. We need only consider the case $y\neq 0$. Then $\nu(x+y)\ge i$, so $\nu(c)\ge 2m+1$ and $\nu(d)\ge m$. Therefore, $c+d\sqrt{\pi}\in T_m$. Since $\chi\equiv 1$ on T_m, we have $\chi(\phi(x))\chi(\phi(y))=\chi(\phi(x+y))$. This shows that

$$\sum_{t\in T_i-T_{i+1}}\overline{\chi}(t)(-1)^{j(t)}=\sum_{S_1}\overline{\chi}(\phi(x))(-1)^{j(t)}\sum_{S_2}\overline{\chi}(\phi(y)).$$

But

$$\sum_{S_2}\overline{\chi}(\phi(y))=\sum_{t\in T_{m-1}}\overline{\chi}(t)=0$$

since $\chi\not\equiv 1$ on T_{m-1}. Therefore,

$$\sum_{t\in T_i-T_{i+1}}\overline{\chi}(t)(-1)^{j(t)}=0$$

for $0\le i\le m-2$.

Next, consider

$$\sum_{t\in T_m-T_{m+1}}\overline{\chi}(t)(-1)^{j(t)}.$$

Here, $t=\phi(a_{m-1}\pi^{m-1}+\cdots+a_{n-1}\pi^{n-1})$, with $a_{m-1}=\varepsilon^{j(t)}$, $0\le j(t)\le q-2$. Let $x=a_{m-1}\pi^{m-1}$, $y=a_m\pi^m+\cdots+a_{n-1}\pi^{n-1}$. As before,

$$\frac{\phi(x)\phi(y)}{\phi(x+y)}\in T_m,$$

which makes

$$(13)\quad \sum_{t\in T_m - T_{m+1}} \overline{\chi}(t)(-1)^{j(t)} = \sum_{S_2} \overline{\chi}(\phi(x))\overline{\chi}(\phi(y))(-1)^{j(t)}$$
$$= q^{n-m} \sum_{a_{m-1}\neq 0} \overline{\chi}(\phi(a_{m-1}\pi^{m-1}))(-1)^{j(t)},$$

since $\phi(y) \in T_m$ and $\chi \equiv 1$ on T_m.

We have a map

$$\mathscr{P}^{m-1}/\mathscr{P}^m \xrightarrow{\phi} T_{m-1}/T_m \xrightarrow{\overline{\chi}} \mathbb{C}.$$

For $x, y \in \mathscr{P}^{m-1}$,

$$\frac{\phi(x)\phi(y)}{\phi(x+y)} \in T_m,$$

so $\overline{\chi}\phi$ is an additive homomorphism on $\mathscr{P}^{m-1}/\mathscr{P}^m$ to $\mathbb{C}$. Letting $\psi = \overline{\chi}\phi$, (13) becomes

$$q^{n-m}\sum_{j=0}^{q-2} \psi(\varepsilon^j \pi^{m-1})(-1)^j = q^{n-m} \sum_{x\in\mathscr{O}/\mathscr{P}} \psi(\pi^{m-1}x^2) = q^{n-m} q^{\frac{1}{2}} G(\psi).$$

(Note that $\psi_{\pi^{m-1}}$ is a character of $\mathscr{O}/\mathscr{P}$.) We can now write

$$\langle \chi, W_n\rangle = \frac{1}{2q^n}\left[q^n + \left(\frac{2}{\mathscr{P}}\right)\left(\frac{-1}{\mathscr{P}}\right)^{n+m} q^n G(\tau)G(\psi)\right],$$

which equals 0 or 1. Notice that $\psi_{\pi^{m-1}} = \tau_{\pi^{n-1}\varepsilon^i u}$ for some $0 \leq i \leq q-2$, $u \in 1+\mathscr{P}$. Then $G(\tau)G(\psi) = \left(\frac{-\varepsilon^i}{\mathscr{P}}\right) = \left(\frac{-1}{\mathscr{P}}\right)(-1)^i$, which takes on each value ± 1 for half the $q-1$ possible values of i. This completes the proof of Proposition 5.

If E/F is ramified, suppose that we replace τ by τ_u, $u \in \mathscr{U}$. Then the characters of a given conductor appearing in W_n^τ will be the same as those appearing in $W_n^{\tau_u}$ if $\left(\frac{u}{\mathscr{P}}\right) = 1$. If $\left(\frac{u}{\mathscr{P}}\right) = -1$, then the two sets of characters of a given conductor $m > 0$ appearing respectively in W_n^τ and $W_n^{\tau_\alpha}$ are disjoint. By varying τ, we thus obtain all characters of conductor $m > 0$ in the restriction to T of some W^τ.

5. Decomposition of $W^\tau|_T$. In this section we use the results of the preceding section to determine the decomposition of $W^\tau|_T$.

LEMMA 9. *For $2k > -n$, let $H_k = S(\mathscr{P}^{-k}, \mathscr{P}^{n+k})$. Then H_k is an invariant subspace for W^τ which is equivalent to $W^{\tau_\alpha}_{n+2k}$, where $\alpha = \pi^{-2k}$.*

Proof. Recall that if $\beta \in F$ and $\alpha = \beta^2$, then $W^\tau = R^{-1}W^{\tau_\alpha}R$, where $(Rf)(x) = |\beta|^{\frac{1}{2}}f(\beta x)$. Let $\beta = \pi^{-k}$. Then $\omega(\tau_\alpha) = n + 2k$. Suppose $g \in K$. Then $f \in H_k \Rightarrow Rf \in S(\mathscr{O}, \mathscr{P}^{n+2k}) \Rightarrow W^{\tau_\alpha}(g)Rf \in S(\mathscr{O}, \mathscr{P}^{n+2k}) \Rightarrow R^{-1}W^{\tau_\alpha}(g)Rf \in H_k$. Thus H_k is invariant under W^τ. Also, $W^\tau(g)f = f$ if $f \in H_k$ and $g \in K_{n+2k}$. We thus have a representation of K/K_{n+2k} on H_k which is a subrepresentation of W^τ and which is equivlent to $W^{\tau_\alpha}_{n+2k}$. This completes the proof of Lemma 8.

Suppose $W^\tau(t)f = \chi(t)f$ for all $t \in T$. If $f \in S(\mathscr{P}^r, \mathscr{P}^s)$, choose k so that $-k \leq r$ and $n + k \geq s$. Then $S(\mathscr{P}^r, \mathscr{P}^s) \subset S(\mathscr{P}^{-k}, \mathscr{P}^{n+k}) = H_k$. Then the action of W^τ on H_k is equivalent to $W^{\tau_\alpha}_{n+2k}$, $\alpha = \pi^{-2k}$, by Lemma 9. This implies χ appears in $W^{\tau_\alpha}_{n+2k}$. We apply Proposition 3 to each of the representations $W^{\tau_\alpha}_{n+2k}$, $k \geq 0$, to obtain

PROPOSITION 6. *Suppose E/F is unramified, $\omega(\tau) = n$, and $c(\chi) = i$.*

(1) *If n is even and i is even, then $\langle \chi, W^\tau|_T\rangle = 1$.*
(2) *If n is even and i is odd, then $\langle \chi, W^\tau|_T\rangle = 0$.*
(3) *If n is odd and i is even, then $\langle \chi, W^\tau|_T\rangle = 0$ if $\chi \neq 1$, and $\langle 1, W^\tau|_T\rangle = 1$.*
(4) *If n is odd and i is odd, then $\langle \chi, W^\tau|_T\rangle = 1$ if $\chi \neq \theta_0$, and $\langle \theta_0, W^\tau|_T\rangle = 0$.*

We argue in a similar fashion if E/F is ramified. Applying Propositions 4 and 5, we obtain

PROPOSITION 7. *Suppose E/F is ramified and $\omega(\tau) = n$.*

(1) *$\langle 1, W^\tau|_T\rangle = 1$ if n is even or $\left(\frac{-1}{\mathscr{P}}\right) = 1$, and equals 0 otherwise.*

(2) *$\langle \theta_0, W^\tau|_T\rangle = 1 - \langle 1, W^\tau|_T\rangle$.*

(3) *If $c(\chi) = m > 0$, then*

$$\langle \chi, W^\tau|_T\rangle = 1 \Leftrightarrow G(\tau)G(\psi) = \left(\frac{2}{\mathscr{P}}\right)\left(\frac{-1}{\mathscr{P}}\right)^{n+m},$$

where $\psi = \overline{\chi}\phi$. Otherwise, $\langle \chi, W^\tau|_T\rangle = 0$.

(4) *Exactly half the characters* χ *of a given conductor satisfy* $\langle \chi, W^{\tau}|_T \rangle = 1$.

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