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A CONVERSE TO A THEOREM OF KOMLÓS FOR CONVEX SUBSETS OF L_1

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A theorem of Komlós is a subsequence version of the strong law of large numbers. It states that if $(f_n)_n$ is a sequence of norm-bounded random variables in $L_1(\mu)$, where μ is a probability measure, then there exists a subsequence $(g_k)_k$ of $(f_n)_n$ and $f \in L_1(\mu)$ such that for all further subsequences $(h_m)_m$, the sequence of successive arithmetic means of $(h_m)_m$ converges to f almost everywhere.

In this paper we show that, conversely, if C is a convex subset of $L_1(\mu)$ satisfying the conclusion of Komlós' theorem, then C must be L_1-norm bounded.

Introduction. A version of the strong law of large numbers in probability theory states that if $(f_n)_{n=1}^{\infty}$ is a sequence of independent, scalar-valued integrable functions (random variables), on a probability measure space (Ω, Σ, μ), each having the same distribution with mean m, then

$$\frac{1}{n}\sum_{j=1}^{n} f_j \underset{n}{\rightarrow} m \quad \text{almost everywhere.}$$

In (1967) Komlós **[Ko]** showed that arbitrary sequences of integrable random variables whose absolute values have uniformly bounded expectations always have subsequences that satisfy a version of the strong law. Indeed, for all sequences $(f_n)_{n=1}^{\infty}$ in $L_1(\mu)$ with

$$\sup_n \int_{\Omega} |f_n| \, d\mu < \infty,$$

there exists a subsequence $(g_k)_{k=1}^{\infty}$ of $(f_n)_n$ and $f \in L_1(\mu)$ such that all further subsequences $(h_m)_m$ of $(g_k)_k$ satisfy

$$\frac{1}{N}\sum_{m=1}^{N} h_m \underset{N}{\rightarrow} f \quad \text{almost everywhere.}$$

This result became the archetype for what Chatterji **[C2]** in the early 1970s called "the subsequence principle in probability theory". This heuristic principle led Chatterji **[C1]**, **[C2]**, **[C3]** (see also Gaposhkin **[Ga]**) to find subsequence versions of the central limit theorem and the law of the iterated logarithm, analogous to Komlós's subsequence version of the strong law.

Chatterji **[C1]** and Gaposhkin **[Ga]** extended Komlós's theorem to all L_p spaces, for $0 < p < 2$. Aldous **[A]** and Berkes and Péter **[B-P]**, amongst others, continued the investigation of the subsequence principle using the notion of an exchangeable sequence of random variables.

A recent extension of Komlós's theorem, due to N. J. Kalton, may be found in Godefroy **[Go]**. Kalton strengthens the conclusion of Komlós's theorem so that the Cesàro means converge almost everywhere and in weak L_1.

For other recent developments concerning Komlós's theorem and further references, we refer the reader to Balder **[B1]**, **[B2]**, **[B3]** and Trautner **[T]**.

In this paper we show that every convex set C in $L_1(\mu)$ that satisfies the conclusion of Komlós's theorem, must be L_1-norm bounded. To prove this we proceed by contradiction. We create a sliding hump sequence of functions on the domain Ω, each a member of C, for which certain convex combinations have Cesàro averages with an L_0-limit that lies outside of $L_1(\mu)$.

Finally, we characterize those convex subsets of L_1 that are almost everywhere Cesàro compact in the sense of the conclusion of Komlós's theorem, using a result of Bukhvalov and Lozanovski **[B-L]**.

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1. Preliminaries and Komlós sets. $\mathbf{N}$ denotes the set of all positive integers, while "the scalars" refers to the real or complex numbers. For a Banach space X, $\mathbf{B}_X$ denotes the closed unit ball of X.

Throughout this paper Ω will be a non-empty set, Σ a σ-algebra of subsets of Ω, and μ will be a complete, positive, σ-finite, countably additive measure on Σ. $L_p(\mu)$ is the F-space or Banach space of all (equivalence classes of) measurable functions $f: \Omega \to$ the scalars for which $\|f\|_p < \infty$,

$$\|f\|_1 := \int_\Omega |f| \, d\mu,$$
$$\|f\|_\infty := \text{ess-sup}\{|f(\omega)| : \omega \in \Omega\},$$

and

$$\|f\|_0 := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(E_n)} \int_{E_n} \frac{|f|}{1+|f|}\, d\mu.$$

Here $(E_n)_{n=1}^{\infty}$ is a Σ-partition of Ω into sets with $0 < \mu(E_n) < \infty$, for each n. Such a Σ-partition exists as μ is σ-finite. If μ is finite we have the simpler definition,

$$\|f\|_0 := \int_{\Omega} \frac{|f|}{1+|f|}\, d\mu.$$

The $L_0(\mu)$-topology restricted to $L_1(\mu)$ will be called the topology of convergence locally in measure (clm); or the topology of convergence in measure (cm) when μ is finite. θ will denote the zero element in $L_1(\mu)$.

1.1. DEFINITION. A subset S of $L_0(\mu)$ will be called a Komlós set if for every sequence $(f_n)_{n=1}^{\infty}$ in S, there exists a subsequence $(g_k)_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ and $f \in S$ such that for every subsequence $(h_m)_{m=1}^{\infty}$ of $(g_k)_{k=1}^{\infty}$,

$$\frac{1}{N} \sum_{m=1}^{N} h_m \underset{N}{\to} f \quad \text{almost everywhere.}$$

Komlós showed that $\mathbf{B}_{L_1(\mu)}$ is a Komlós set.

Note that if $(f_n)_{n=1}^{\infty}$ is a sequence in $L_0(\mu)$ and $f_n \underset{n}{\to} f$ almost everywhere, then

$$\frac{1}{N} \sum_{n=1}^{N} f_n \underset{N}{\to} f \quad \text{almost everywhere.}$$

It follows that every clm-compact subset S of $L_0(\mu)$ must be a Komlós set. Consequently, even when Komlós sets are contained in $L_1(\mu)$, they need not be L_1-norm bounded (see §2 for an example). Further, it is easy to check that Komlós sets are forced to be L_0-closed. So, the concept of a Komlós subset of L_1 lies strictly between that of a clm-closed set and a clm-compact set in L_1.

2. Convex Komlós sets in L_1 are norm bounded.

2.1. THEOREM. *Let (Ω, Σ, μ) be a finite measure space. Suppose C is a subset of $L_1(\mu)$ that is convex and a Komlós set. Then C must be $\|\cdot\|_1$-bounded.*

Proof. Suppose, to get a contradiction, that C fails to be norm bounded. Then there exists a sequence $(g_n)_{n=1}^{\infty}$ in C such that $\|g_n\|_1 \underset{n}{\to} \infty$.

By assumption, C is a Komlós set. So, by passing to a subsequence if necessary, we may assume that there exists $g \in C$ such that

$$\frac{1}{N}\sum_{m=1}^{N} h_m \underset{N}{\to} g \quad \text{almost everywhere},$$

for every subsequence $(h_m)_m$ of $(g_n)_n$.

Note that $C - g$ is another convex Komlós set in $L_1(\mu)$, $\theta \in C - g$ and

$$\frac{1}{N}\sum_{m=1}^{N}(h_m - g) \underset{N}{\to} \theta \quad \text{almost everywhere},$$

for all subsequences $(h_m)_m$ of $(g_n)_n$. Clearly, by relabelling each $g_n - g$ as g_n and $C - g$ as C, we have that the following is true. C is a convex Komlós set in $L_1(\mu)$, $(g_n)_n$ is a sequence in C with $\|g_n\|_1 \underset{n}{\to} \infty$, $\theta \in C$ and for every subsequence $(h_m)_m$ of $(g_n)_n$,

$$\frac{1}{N}\sum_{m=1}^{N} h_m \underset{N}{\to} \theta \quad \text{almost everywhere.}$$

We shall now use $(g_n)_n$ to construct another sequence $(f_n)_n$ in C such that $f_n \underset{n}{\to} \theta$ almost everywhere and $\|f_n\|_1 \underset{n}{\to} \infty$. Let $u_1 := 1$ and $f_1 := g_{u_1}$. Since $\|g_n\|_1 \underset{n}{\to} \infty$, there exists $u_2 \in \mathbf{N}$ with $u_2 > u_1$ such that

$$\|g_{u_2}\|_1 > \|g_{u_1}\|_1 + 2(2^2).$$

Define f_2 by

$$f_2 := \frac{1}{2}(g_{u_1} + g_{u_2}),$$

$f_2 \in C$ because C is convex. Also,

$$\|f_2\|_1 \geq \frac{1}{2}(\|g_{u_2}\|_1 - \|g_{u_1}\|_1) > \frac{1}{2} \cdot 2(2^2) = 2^2.$$

Next choose $u_3 \in \mathbf{N}$ with $u_3 > u_2$ and

$$\|g_{u_3}\|_1 > \|g_{u_1}\|_1 + \|g_{u_2}\|_1 + 3(2^3);$$

and define

$$f_3 := \frac{1}{3}(g_{u_1} + g_{u_2} + g_{u_3}).$$

Then $f_3 \in C$ and $\|f_3\|_1 > 2^3$.

Continuing inductively, we produce a subsequence $(g_{u_n})_{n=1}^{\infty}$ of $(g_n)_n$ and a sequence $(f_n)_{n=1}^{\infty}$ in C such that $\|f_n\|_1 \underset{n}{\to} \infty$ and

$$f_n = \frac{1}{n}\sum_{j=1}^{n} g_{u_j}, \quad \text{for all } n \in \mathbf{N}.$$

From above, we know that $f_n \underset{n}{\to} \theta$ almost everywhere.

We will now inductively construct a strictly increasing sequence $(n_k)_{k=0}^{\infty}$ in $\mathbf{N}$, a nonincreasing sequence $(E_n)_{n=0}^{\infty}$ in Σ and a sequence $(\delta_k)_{k=0}^{\infty}$ of positive real numbers with the following properties. $E_1 = \Omega$; and for each $k \in \mathbf{N}$ statements (1) to (5) below are true.

(1) $\delta_k < \delta_{k-1}/2$.

(2) For each $E \in \Sigma$ with $\mu(E) < \delta_k$, we have that $\int_E |f_{n_k}|\, d\mu < 1$.

(3) $\|f_{n_k}\chi_{E_k}\|_1 > 2^k(2 + \mu(\Omega))$.

(4) $\|f_n \chi_{E_{k-1}\setminus E_k}\|_\infty < 1$, for all $n \geq n_k$.

(5) $\mu(E_k) < \delta_{k-1}$.

Define $E_0 := \Omega$, $\delta_0 := 2\mu(\Omega)$ and $n_0 := 1$. Next define $E_1 := \Omega$. Since $\|f_n\|_1 \underset{n}{\to} \infty$, we can choose $n_1 \in \mathbf{N}$ so large that $n_1 > n_0$,

$$\|f_{n_1}\chi_{E_1}\|_1 > 2^1(2 + \mu(\Omega)), \quad \text{and}$$
$$\|f_n \chi_{E_0\setminus E_1}\|_\infty < 1, \quad \text{for all } n \geq n_1.$$

By the absolute continuity of the measure $|f_{n_1}|\, d\mu$ with respect to μ, there exists $\delta_1 \in (0, \mu(\Omega))$ such that for every $E \in \Sigma$ with $\mu(E) < \delta_1$, we have

$$\int_E |f_{n_1}|\, d\mu < 1.$$

Of course, $\mu(E_1) < \delta_0$.

Fix $m \in \mathbf{N}$ with $m > 1$. Suppose that we have constructed a strictly increasing sequence $(n_k)_{k=0}^{m-1}$ in $\mathbf{N}$, a non-increasing sequence $(E_k)_{k=0}^{m-1}$ in Σ and a sequence $(\delta_k)_{k=0}^{m-1}$ of positive real numbers, such that statements (1) to (5) are true for each $k \in \{1, \dots, m-1\}$. We know that $f_n \underset{n}{\to} \theta$ almost everywhere on E_{m-1}. So we can find, with the aid of Egoroff's theorem, $E_m \in \Sigma$ with $E_m \subseteq E_{m-1}$, such that

$$\mu(E_m) < \delta_{m-1} \quad \text{and} \quad \|f_n \chi_{E_{m-1}\setminus E_m}\|_\infty \underset{n}{\to} 0.$$

But statement (4) is true for each $k \in \{1, \dots, m-1\}$; and hence we see that

$$\|f_n \chi_{\Omega \setminus E_{m-1}}\|_\infty < 1, \quad \text{for all } n \geq n_{m-1}.$$

Since $\|f_n\|_1 \underset{n}{\to} \infty$, it follows that

$$\sup_{n \in \mathbf{N}} \|f_n \chi_{E_m}\|_1 = \infty.$$

Choose $n_m \in \mathbf{N}$ with $n_m > n_{m-1}$, such that

$$\|f_{n_m} \chi_{E_m}\|_1 > 2^m(2 + \mu(\Omega)), \quad \text{and}$$
$$\|f_n \chi_{E_{m-1} \setminus E_m}\|_\infty < 1, \quad \text{for all } n \geq n_m.$$

Now, the measure $|f_{n_m}|\,d\mu$ is absolutely continuous w.r.t. μ. Therefore there exists $\delta_m > 0$ satisfying $\delta_m < \delta_{m-1}/2$; and such that for every $E \in \Sigma$ with $\mu(E) < \delta_m$, we have that

$$\int_E |f_{n_m}|\,d\mu < 1.$$

Our inductive construction is complete.

For convenience, let us relabel each f_{n_k} as f_k. We note that statements (2), (3) and (4) above still hold true, with n_k replaced everywhere by k. We will refer to (2), (3) and (4), modified in this way, as (2)*, (3)* and (4)* respectively.

For each $k \in \mathbf{N}$, define

$$\psi_k := \sum_{j=1}^{k} \frac{1}{2^j} f_j.$$

Since $\theta \in C$, each $\psi_k \in \mathrm{co}(C) = C$. Also define, for every $m \in \mathbf{N}$,

$$\varphi_m := \left(\frac{1}{2^m}|f_m| - \sum_{j=1}^{m-1} \frac{1}{2^j}|f_j| - 1 \right) \chi_{E_m \setminus E_{m+1}}.$$

$(\psi_k)_{k=1}^\infty$ is a sequence in C, which is a Komlós set in $L_1(\mu)$. So there exists a subsequence $(\psi_{k_l})_{l=1}^\infty$ of $(\psi_k)_{k=1}^\infty$ and $q \in C$ such that

$$(\Diamond) \qquad q_N := \frac{1}{N} \sum_{l=1}^{N} \psi_{k_l} \underset{N}{\to} q \quad \text{almost everywhere.}$$

Moreover, note that $q \in C \subseteq L_1(\mu)$; so that

$$(\spadesuit) \qquad \|q\|_1 < \infty.$$

Let $k_0 := 0$. It is simple to verify that for all $N \in \mathbf{N}$,

$$q_N = \sum_{j=1}^{N} \frac{N-j+1}{N} \sum_{t=k_{j-1}+1}^{k_j} \frac{1}{2^t} f_t.$$

In the calculations below, when we have a pointwise inequality between two measurable functions, we mean that the inequality holds almost everywhere.

Fix $m \in \mathbf{N}$ and consider $E_m \backslash E_{m+1}$. Note that there is a unique $i \in \mathbf{N}$ such that $k_{i-1} < m \leq k_i$. Next fix $N \in \mathbf{N}$ with $N \geq i$. By property (4)* above, $|f_j| < 1$ on $E_m \backslash E_{m+1}$, for all $j \geq m+1$. Temporarily, let $c_m := \chi_{E_m \backslash E_{m+1}}$. Then,

$$\begin{aligned}
|q_N c_m| &= \left| \left(\sum_{1 \leq j \leq N,\, j \neq i} \frac{N-j+1}{N} \sum_{t=k_{j-1}+1}^{k_j} \frac{1}{2^t} f_t \right.\right. \\
&\qquad\qquad \left.\left. + \frac{N-i+1}{N} \sum_{t=k_{i-1}+1}^{k_i} \frac{1}{2^t} f_t \right) c_m \right| \\
&\geq \left(\frac{N-i+1}{N} \frac{1}{2^m} |f_m| - \sum_{1 \leq t < m} \frac{1}{2^t} |f_t| - \sum_{m < t \leq k_N} \frac{1}{2^t} |f_t| \right) c_m \\
&\geq \frac{-(i-1)}{N} \frac{1}{2^m} |f_m| c_m + \varphi_m + c_m - \left(\sum_{m < t \leq k_N} \frac{1}{2^t} \right) c_m \\
&\geq \varphi_m - \frac{i-1}{N} \frac{1}{2^m} |f_m| c_m.
\end{aligned}$$

Thus, we have shown the following.

($\clubsuit$) For all $m \in \mathbf{N}$, there exists $i \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ with $N \geq i$,

$$|q_N \chi_{E_m \backslash E_{m+1}}| \geq \varphi_m - \frac{i-1}{N} \frac{1}{2^m} |f_m| \chi_{E_m \backslash E_{m+1}}.$$

Again fix $m \in \mathbf{N}$. We see that

$$\begin{aligned}\int_\Omega \varphi_m \, d\mu &= \frac{1}{2^m} \int_{E_m \setminus E_{m+1}} |f_m| \, d\mu \\ &\quad - \sum_{j=1}^{m-1} \frac{1}{2^j} \int_{E_m \setminus E_{m+1}} |f_j| \, d\mu - \mu(E_m \setminus E_{m+1}) \\ &= \frac{1}{2^m} \|f_m \chi_{E_m}\|_1 - \frac{1}{2^m} \int_{E_{m+1}} |f_m| \, d\mu \\ &\quad - \sum_{j=1}^{m-1} \frac{1}{2^j} \int_{E_m \setminus E_{m+1}} |f_j| \, d\mu - \mu(E_m \setminus E_{m+1}).\end{aligned}$$

$\mu(E_{m+1}) < \delta_m$, from (5); and so by $(2)^*$,

$$\int_{E_{m+1}} |f_m| \, d\mu < 1.$$

Also, by (5) and (1) we have that for all $j \in \{1, \ldots, m-1\}$,

$$\mu(E_m \setminus E_{m+1}) \leq \mu(E_m) < \delta_{m-1} \leq \delta_j;$$

and consequently from $(2)^*$,

$$\int_{E_m \setminus E_{m+1}} |f_j| \, d\mu < 1.$$

Using $(3)^*$ above,

$$\begin{aligned}\int_\Omega \varphi_m \, d\mu &> \frac{1}{2^m} \|f_m \chi_{E_m}\|_1 - \frac{1}{2^m} - \sum_{j=1}^{m-1} \frac{1}{2^j} - \mu(\Omega) \\ &> \frac{1}{2^m} 2^m (2 + \mu(\Omega)) - 1 - \mu(\Omega) = 1.\end{aligned}$$

In summary,

$$(\heartsuit) \qquad \int_\Omega \varphi_m \, d\mu > 1, \quad \text{for all } m \in \mathbf{N}.$$

We now estimate $\|q\|_1$ from below. Fix $m \in \mathbf{N}$. By ($\clubsuit$), there exists $i \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ with $N \geq i$,

$$|q_N(\omega)| \geq \varphi_m(\omega) - \frac{i-1}{N} \frac{1}{2^m} |f_m(\omega)|, \quad \text{for almost all } \omega \in E_m \setminus E_{m+1}.$$

From ($\diamondsuit$), we therefore have that

$$|q(\omega)| \geq \varphi_m(\omega), \quad \text{for almost all } \omega \in E_m \setminus E_{m+1}.$$

$E_1 = \Omega$, and $\mu(E_m) \underset{m}{\to} 0$, by (1) and (5). Thus, $(E_m \backslash E_{m+1})_{m=1}^{\infty}$ is a Σ-partition of Ω. Consequently, using (♠) and (♡), we are led to the following contradiction.

$$\begin{aligned}\infty > \|q\|_1 &= \sum_{m=1}^{\infty} \int_{E_m \backslash E_{m+1}} |q(\omega)| \, d\mu(\omega) \geq \sum_{m=1}^{\infty} \int_{E_m \backslash E_{m+1}} \varphi_m(\omega) \, d\mu(\omega) \\ &= \sum_{m=1}^{\infty} \int_{\Omega} \varphi_m \, d\mu \geq \sum_{m=1}^{\infty} (1)^m = \infty.\end{aligned}$$

The previous theorem extends to the case where μ is a σ-finite measure. The proof below is simpler than our original one. It was suggested by Anton Schep.

2.2. THEOREM. *Let (Ω, Σ, μ) be a σ-finite measure space. Let C be a convex Komlós set in $L_1(\mu)$. Then C must be norm bounded.*

Proof. Fix $g \in L_1(\mu)$ such that

$$g(\omega) > 0, \quad \text{for all } \omega \in \Omega.$$

Such a g exists because μ is σ-finite. Define the finite measure ν by $d\nu := g d\mu$, and define the linear isometry T from $L_1(\mu)$ onto $L_1(\nu)$ by

$$Tf := fg^{-1}, \quad \text{for all } f \in L_1(\mu).$$

Since μ and ν have the same sets of measure zero, it is easy to see that a subset C of $L_1(\mu)$ is a Komlós set if and only if $T(C)$ is a Komlós set in $L_1(\nu)$. By Theorem 2.1, $T(C)$ is $L_1(\nu)$-norm bounded; and consequently C is $L_1(\mu)$-norm bounded. □

Note that every clm-compact subset of L_1 is automatically a Komlós set. So the example

$$C := \{n^2 \chi_{[0, 1/n]} : n \in \mathbf{N}\} \cup \{0\}$$

is a Komlós set in $L_1[0, 1]$ that fails to be L_1-norm bounded.

We also remark that a corollary to Theorem 2.1 is that every clm-compact, convex subset of $L_1(\mu)$ must be L_1-norm bounded. This is a result of Khamsi and Turpin **[K-T]**, that can be generalized to the setting of a large class of tvs topologies τ on a Banach space X (see, for example, Khamsi **[Kh]**).

3. A second dual characterization of Komlós convex sets in L_1. In this section the symbol $\cong$ will denote isometric isomorphism between

Banach spaces. Let j be the natural embedding of L_1 into L_1^{**}. It is a fact that

$$L_1^{**} = j(L_1) \oplus_1 S,$$

for some subspace S of L_1^{**}. Indeed, $L_1^* \cong L_\infty(\mu)$ and so $L_1^{**} \cong L_\infty^*$, which is isometrically isomorphic to the space of all bounded, finitely additive measures on Σ that vanish on μ-null sets. Hence, by the Yoshida-Hewitt decomposition theorem **[Y-H]** and the Radon-Nikodým theorem,

$$L_\infty^* \cong L_1 \oplus_1 pfa(\mu),$$

where $pfa(\mu)$ denotes the space of all bounded, purely finitely additive measures on Σ that vanish on μ-null sets. We identify $pfa(\mu)$ with a subspace S of L_1^{**}, and we denote by P the natural projection of L_1^{**} onto $j(L_1)$.

Recall the following result, which we will use to establish Theorem 3.1 below.

THEOREM (*Bukhvalov and Lozanovski* **[B-L]** *Theorem* 1). *Let C be a convex subset of $L_1(\mu)$ and let W be the weak*-closure of $j(C)$ in L_1^{**}.*

(a) *If C is clm-closed then $P(W) = j(C)$.*

(b) *If C is L_1-norm bounded and $P(W) = j(C)$ then C is clm-closed.*

3.1. THEOREM. *Let C be a convex subset of $L_1(\mu)$ and W be the weak*-closure of $j(C)$ in L_1^{**}. Then the following statements are equivalent.*

(a) *C is a Komlós set.*

(b) *C is L_1-norm bounded and clm-closed.*

(c) *C is L_1-norm bounded and $P(W) = j(C)$.*

Proof. (a) $\Rightarrow$ (b). By Theorem 2.2, C is L_1-norm bounded. Moreover, Komlós sets are clm-closed, as we observed above.

(b) $\Rightarrow$ (a). Fix $(f_n)_{n=1}^\infty$ in C. By Komlós's theorem **[Ko]**, there exists a subsequence $(g_k)_{k=1}^\infty$ of $(f_n)_{n=1}^\infty$ and $f \in L_1(\mu)$, such that for all subsequences $(h_m)_{m=1}^\infty$ of $(g_k)_{k=1}^\infty$ we have

$$q_N := \frac{1}{N}\sum_{m=1}^{N} h_m \underset{N}{\rightarrow} f \quad \text{almost everywhere.}$$

C is convex, and hence each $q_N \in C$. But C is clm-closed and consequently, $f \in C$.

(b) $\Leftrightarrow$ (c). This follows from **[B-L]** Theorem 1. □

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