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DIFFERENTIAL-DIFFERENCE OPERATORS AND MONODROMY REPRESENTATIONS OF HECKE ALGEBRAS

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Associated to any finite reflection group G on an Euclidean space there is a parametrized commutative algebra of differential-difference operators with as many parameters as there are conjugacy classes of reflections. The Hecke algebra of the group can be represented by monodromy action on the space of functions annihilated by each differential-difference operator in the algebra. For each irreducible representation of G the differential-difference equations lead to a linear system of first-order meromorphic differential equations corresponding to an integrable connection over the G-orbits of regular points in the complexification of the Euclidean space. The fundamental group is the generalized Artin braid group belonging to G, and its monodromy representation factors over the Hecke algebra of G. Monodromy has long been of importance in the study of special functions of several variables, for example, the hyperlogarithms of Lappo-Danilevsky are used to express the flat sections and the work of Riemann on the monodromy of the hypergeometric equation is applied to the case of dihedral groups.

Orthogonal polynomials and special functions of classical type in several variables arise from analysis on root systems. Generally there is an underlying definite integral with a number of parameters. To evaluate such an integral in closed form means to obtain a formula in terms of known special functions, especially the gamma function. These formulas are generally meromorphic and allow analytic continuation of the parameter values into regions where the integral is no longer defined. In order to understand the singularities one is led to deep problems in Coxeter and Artin groups, Hecke algebras, their representations, and differential equations. Certain polynomials in a parameter such as Poincaré series and generic degrees of representations are associated to such groups. The logarithms of their zero-sets are closely related to the aforementioned integrals.

In previous work [**Du1**, 3] mainly concerned with orthogonal polynomials associated to finite reflection groups the author constructed a commutative algebra of differential-difference operators for such groups, with as many parameters as there are conjugacy classes of reflections. These operators are a natural generalization of partial differentiation and can be used to construct group-invariant differential operators (Heckman [**He3**]). The main topic of this paper is the theory of functions annihilated by each differential-difference operator in the algebra. Nonconstant functions of this type have singularities on the reflecting hyperplanes of the group.

For each irreducible representation of the group the differential-difference equation leads to a linear system of first-order differential equations corresponding to an integrable connection on a certain holomorphic vector bundle over the space of G-orbits of regular points in the complexification of $\mathbb{R}^N$ (here G denotes a finite orthogonal reflection group acting on $\mathbb{R}^N$). Brieskorn [**Br**] showed that the fundamental group of this space is the Artin generalized braid group $\widetilde{G}$ belonging to G. A monodromy representation of $\widetilde{G}$ is realized on the space (the "horizontal sections") of solutions of the system by means of analytic continuation. The monodromy representation of the group algebra $\mathbb{C}\widetilde{G}$ factors over the Hecke algebra of G and corresponds to the original representation of G when the parameter values give a semisimple specialization of the generic algebra of G.

Monodromy representations and Hecke algebras provide important tools and motivation for the study of special functions of several variables. Riemann studied the hypergeometric differential equation and its monodromy (this is applied to dihedral groups in the present work). Heckman and Opdam [**He1**, **2**; **HO**; **O1,2,3**] introduced hypergeometric functions associated to root systems of Weyl groups and the related monodromy actions of affine Weyl groups. Heckman [**He2**] then showed how the Hecke algebra appears in the monodromy of the Heckman-Opdam differential operator.

Opdam [**O4**] constructed differential equations of Bessel type for every finite reflection group (not necessarily crystallographic) and produced representations of the Hecke algebra isomorphic to the regular representation. His paper stimulated the research leading to this work. The approach here is more classical and aims to construct the systems of equations and representations quite explicitly. There is also a greater emphasis on irreducible representations and characters here. It is notable that Opdam has found a unified approach to the Macdonald-Mehta integrals using analysis on the Hecke algebra. His proof exhibits the remarkable correspondence between the singularities of the integral formulas and zeros of the Poincaré polynomials.

Cherednik [**Ch1**, **2**] also has investigated Hecke algebras and differential equations for special functions, especially those motivated by applications in physics.

Matsuo [**Mat**] constructed trigonometric Knizhnik-Zamolodchikov equations associated with root systems of Weyl type. In this work he used differential-difference operators and an integrable connection for group-algebra-valued functions periodic for the group lattice. He also found the relationship to the (generalized) zonal spherical functions of Heckman and Opdam.

The differential-difference operators provide an elegant approach to constructing integrable systems of first-order equations associated to root systems. For example, the Knizhnik-Zamolodchikov equations are a special case for the Coxeter group of type A_N, the symmetric group.

This paper is organized as follows:

(1) The differential-difference operators, analytic vector bundles and integrable connections associated to irreducible representations;

(2) construction of fundamental solutions of the differential equation in terms of Lappo-Danilevsky's hyperlogarithms, entire dependence on the parameters, the monodromy representation, the Hecke algebra;

(3) exceptional parameter values, generic degrees of representations, relations to semisimplicity of Hecke algebra;

(4) representations of the Hecke algebra of a dihedral group, explicit solutions in terms of hypergeometric functions;

(5) further questions.

1. Differential-difference operators and integrable connections. Suppose G is a finite reflection group acting effectively on $\mathbb{R}^N$ with the (reduced) set $\{v_i : i = 1, \dots, m\}$ of positive roots, numbered so that $\{v_i : i = 1, \dots, N\}$ is the set of simple roots. Thus $\{v_1, v_2, \dots, v_N\}$ is a basis and for each j, $v_j = \sum_{i=1}^N a_{ij} v_i$ with $a_{ij} \geq 0$. Let σ_i denote the reflection along v_i (thus $x\sigma_i := x - 2(\langle x, v_i\rangle/|v_i|^2)v_i$, where the inner product is $\langle x, y\rangle := \sum_{j=1}^N x_j y_j$ and the norm is $|x| := (\sum_{j=1}^N |x_j|^2)^{1/2}$ $(x, y \in \mathbb{C}^N)$). Then G is generated by $\{\sigma_1, \sigma_2, \dots, \sigma_N\}$, the simple reflections. Assume further that $|v_i| = |v_j|$ whenever σ_i is conjugate to σ_j, and thus $v_i w = v_j$ for some $w \in G$. Denote the set of G-regular points by $\mathbb{R}^N_{\text{reg}} = \{x \in \mathbb{R}^N : \langle x, v_i\rangle \neq 0$ for $i = 1, \dots, m\}$ (that is, $x\sigma_i \neq x$ for all i), and the complexification by $\mathbb{C}^N_{\text{reg}} = \{x \in \mathbb{C}^N : \langle x, v_i\rangle \neq 0,\ i = 1, \dots, m\}$. The connected components of $\mathbb{R}^N_{\text{reg}}$ are the chambers and their intersection with the sphere $S = \{x \in \mathbb{R}^N : |x| = 1\}$ form the Coxeter complex. One chamber is distinguished (depending on the choice of simple roots), namely $\mathscr{C} := \{x \in \mathbb{R}^N : \langle x, v_i \rangle > 0, \ i = 1, \dots, m\}$. The chambers correspond uniquely to elements of G by $w \leftrightarrow \mathscr{C}w$ (references for these basic facts are Benson and Grove [**BG**, Ch. 6], Bourbaki [**Bo**], Humphreys [**Hu**]). There is a well-determined labeling of the walls (a wall is a set $(\mathscr{C}w)^- \cap v_i^\perp$ of codimension one, where $v_i^\perp = \{x \in \mathbb{R}^N : \langle x, v_i \rangle = 0\}$) of each chamber by $\{1, 2, \dots, N\}$ so that $w^{-1}\sigma_i w$ is the reflection in wall #i of the chamber $\mathscr{C}w$.

There is a geometric interpretation of any expression $w = \sigma_{i_n} \cdots \sigma_{i_2}\sigma_{i_1}$ $(1 \le i_j \le N$, all $j)$ in terms of simple reflections: a point $x_0 \in \mathscr{C}$ can be joined to $x_0 w \in \mathscr{C}w$ by a sequence of paths, each from one chamber to an adjacent one, namely $x_0 w_{j-1} \to x_0 w_j$ through wall #i_j (with $j = 1, 2, \dots, n$) where $w_j := \sigma_{i_j} \cdots \sigma_{i_2}\sigma_{i_1} = w_{j-1}(w_{j-1}^{-1}\sigma_{i_j}, w_{j-1})$, $w_0 = 1$. An example of such a path is $\gamma_j(t) := ((1-t)x_0 + t x_0 \sigma_{i_j})w_{j-1}$. The length of w, denoted $l(w)$, is the number of factors in the shortest expression for w as a product of simple reflections, and is thus the least number of walls that must be traversed to get to $\mathscr{C}w$ from $\mathscr{C}$. Later we will replace paths like γ_j by paths in $\mathbb{C}^N_{\text{reg}}$ joining adjacent chambers.

Choose complex parameters α_i, $i = 1, \dots, m$, such that $\alpha_i = \alpha_j$ whenever σ_i is conjugate to σ_j. Following Heckman and Opdam [**HO**] we call $\{\alpha_1, \dots, \alpha_m\}$ a multiplicity function. More precisely, suppose there are m_0 classes of reflections and let β_i denote the value of α_j when σ_j is in class #i $(i = 1, \dots, m_0)$; then $(\beta_1, \beta_2, \dots)$ is the multiplicity function.

The differential-difference operators are defined [**Du1**] as follows: let $h(x) := \prod_{i=1}^m \langle x, v_i \rangle^{\alpha_i}$, a multi-valued analytic function on $\mathbb{C}^N_{\text{reg}}$ with the principal branch used on $\mathscr{C}$, where $\langle x, v_i \rangle > 0$ for all i; this function is positively homogeneous of degree $\sum_{i=1}^m \alpha_i$. The h-gradient ∇_h is defined by

$$\nabla_h f(x) := \nabla f(x) + \sum_{j=1}^m \alpha_j \frac{f(x) - f(x\sigma_j)}{\langle x, v_j \rangle} v_j$$

(for any smooth function f on $\mathbb{R}^N_{\text{reg}}$). Let $e_1, e_2, \dots, e_N$ be the standard unit basis vectors ($e_1 = (1, 0, \dots)$, etc.) for $\mathbb{R}^N$, and let $T_i := \langle e_i, \nabla_h \rangle$, $i = 1, \dots, N$. It was shown in [**Du1**] that $T_i T_j = T_j T_i$ $(1 \le i, j \le N)$ and T_i is an endomorphism on polynomial functions.

Our goal is to describe the linear space (and G-module) $\mathscr{D}_\alpha$ of smooth functions f on $\mathbb{R}^N_{\text{reg}}$ such that $\nabla_h f = 0$. The decomposition of $\mathscr{D}_\alpha$ into irreducible components leads to integrable differential equations with rational function coefficients thus allowing analytic continuation. Let Ω denote the identification space $\mathbb{C}^N_{\text{reg}}/G$ and fix a base-point $x_0 \in \mathscr{C}$. We will show that the fundamental group $\pi_1(\Omega, x_0)$ is represented on each component of $\mathscr{D}_\alpha$.

The group G acts on functions on $\mathbb{C}^N_{\text{reg}}$ by translation $R(w)f(x) := f(xw)$, $w \in G$, and $\nabla_h(R(w)f)(x) = (\nabla_h f)(xw)w^{-1}$ (see **[Du1]**). Thus $\mathscr{D}_\alpha$ is a G-module. Let $\widehat{G}$ denote the set of equivalence classes of unitary irreducible representations of G; for $\tau \in \widehat{G}$, we will denote the character by χ_τ, the degree by $n(\tau)$ and the representation matrix by $\tau(w)_{ij}$ ($\widehat{G}$ can be identified with the set of " G-graphs," see Kazhdan and Lusztig **[KL]**). It is known that χ_τ is always real.

Any complex function f on G has the Fourier series

$$f(w) = \sum_{\tau \in \widehat{G}} n(\tau) \sum_{i,j=1}^{n(\tau)} f_{\tau, ij}\tau(w)_{ji},$$

where

$$f_{\tau, ij} := \frac{1}{|G|} \sum_{w \in G} f(w)\overline{\tau(w)_{ji}}.$$

If instead, f is a function on $\mathbb{R}^N_{\text{reg}}$ and a member of some G-module then each

$$f_{\tau, ij}(x) := (1/|G|) \sum_{w \in G} f(xw)\overline{\tau(w)_{ji}}$$

is in the same module. Further

$$f_{\tau, ij}(xw) = \sum_{s=1}^{n(\tau)} f_{\tau, is}(x)\tau(w)_{sj} \qquad (x \in \mathbb{R}^N_{\text{reg}},\ w \in G).$$

Thus any function in a G-module can be expressed as a sum $\sum_{\tau \in \widehat{G}} \sum_{i=1}^{n(\tau)} f_{\tau, i}(x)$ where $f_{\tau, i}(xw) = \sum_{j=1}^{n(\tau)} f_{\tau, j}(x)\tau(w)_{ji}$ and each $f_{\tau, i}$ is in the original G-module. Note that it suffices to define $(f_{\tau, i}(x))_{i=1}^{n(\tau)}$ for $x \in \mathscr{C}$.

Applying this decomposition to $\mathscr{D}_\alpha$ we formulate the differential equation, for each $\tau \in \widehat{G}$:

(1.1) Find all $\mathbb{C}^{n(\tau)}$-valued functions $f = (f_1, f_2, \ldots, f_{n(\tau)})$ on $\mathscr{C}$

satisfying

$$\nabla f_i + \sum_{s=1}^{m} \frac{\alpha_s}{\langle x, v_s\rangle}\left(f_i(x) - \sum_{j=1}^{n(\tau)} f_j(x)\tau(\sigma_s)_{ji}\right) v_s = 0,$$
$$(x \in \mathscr{C}, \ i = 1, \dots, n(\tau)).$$

The linear space of solutions of (1.1) will be denoted $\mathscr{D}_{\alpha,\tau}$. We will show $\dim_{\mathbb{C}} \mathscr{D}_{\alpha,\tau} = n(\tau)$; thus the components of a basis will be a set of $n(\tau)^2$ linearly independent functions in $\mathscr{D}_\alpha$ (when extended to all of $\mathbb{R}^N_{\text{reg}}$).

A reference for the use of sheaf theory here is Deligne [**De**, Ch. 1]. Let $\mathscr{O}$ be the sheaf of germs of analytic functions on Ω $(= \mathbb{C}^N_{\text{reg}}/G)$. Let V_τ be the analytic vector bundle over Ω consisting locally of $\mathbb{C}^{n(\tau)}$-valued holomorphic functions satisfying $f(xw) = f(x)\tau(w)$, $x \in \bigcup_{w\in G}(Yw)$ where Y is open in $\mathbb{C}^N_{\text{reg}}$ and $Y \cap (Yw) = \varnothing$ for all $w \in G$. Thus V_τ is a locally constant sheaf and an $\mathscr{O}$-module, since if g is a section of $\mathscr{O}$ and domain $g \supset Y$, then $g(xw) = g(x)$ for all $x \in Y$, $w \in G$.

Let $\nabla_{\tau,\alpha}$ be the restriction of ∇_h to V_τ. Then $\nabla_{\tau,\alpha}$ is a connection for V_τ as an analytic vector bundle over $\mathscr{O}$; indeed,

$$\nabla_{\tau,\alpha} f(x) := \nabla f(x) + \sum_{s=1}^{m} \frac{\alpha_s}{\langle x, v_s\rangle}(f(x) - f(x)\tau(\sigma_s)) \otimes v_s$$

(where ∇f means $\sum_{i=1}^{N}(\partial f/\partial x_i) \otimes e_i$; this is a $\mathbb{C}^{n(\tau)} \otimes \mathbb{C}^N$-valued function). If f, g are local sections of V_τ, $\mathscr{O}$ respectively with a common domain, then $\nabla_{\tau,\alpha}(gf) = g\nabla_{\tau,\alpha}(f) + f \otimes \nabla g$.

1.1. Theorem. *The connection $\nabla_{\tau,\alpha}$ is integrable.*

Proof. In the standard coordinates of $\mathbb{C}^N$, let

$$(\nabla_{\tau,\alpha} f_i(x))_j = \frac{\partial f_i(x)}{\partial x_j} + \sum_{k=1}^{n(\tau)} A_{jik}(x) f_k(x).$$

Integrability in the sense of Frobenius means that

$$B_{jl,is} := \frac{\partial}{\partial x_l} A_{jis}(x) + \sum_{k=1}^{n(\tau)} A_{jik}(x) A_{lks}(x)$$

is symmetric in (j, l) for each $i, s = 1, \dots, n(\tau)$ (see Varadarajan [**V**, pp. 105–106]). Here

$$A_{jik}(x) = \sum_{t=1}^{m} \frac{\alpha_t}{\langle x, v_t\rangle}(\delta_{ik} - \tau(\sigma_\tau)_{ki})(v_t)_j,$$

and

$$B_{jl,is} - B_{lj,is} = \sum_{p=1}^{m}\sum_{t=1}^{m} \frac{\alpha_p\alpha_t}{\langle x, v_p\rangle\langle x, v_t\rangle}((v_p)_j(v_t)_l - (v_p)_l(v_t)_j)$$
$$\cdot [\delta_{is} - (\tau(\sigma_p)_{si} + \tau(\sigma_t)_{si}) + \tau(\sigma_p\sigma_t)_{si}]$$

(after a step of differentiation and grouping of terms). In the square bracket, the first two terms are symmetric in p and t, and so that part of the double sum vanishes. It remains to consider

$$\sum_{p=1}^{m}\sum_{t=1}^{m} \frac{\alpha_p\alpha_t}{\langle x, v_p\rangle\langle x, v_t\rangle}\tau(\sigma_p\sigma_t)_{si}B(v_p, v_t),$$

where B is the bilinear form $B(u, v) := u_jv_l - u_lv_j$ for $u, v \in \mathbb{R}^N$ (fixed j, l). The diagonal part $(p = t)$ of the sum vanishes because $B(v_p, v_p) = 0$. The other part is a sum over plane rotations in G and $\sum\{B(v_p, v_t)\alpha_p\alpha_t/(\langle x, v_p\rangle\langle x, v_t\rangle) : \sigma_p\sigma_t = w\} = 0$ for each rotation w by Proposition 1.7 in **[Du1]** (multiply the sum by $\tau(w)_{is}$ to get the desired sum). □

Since the coefficients of the connection are rational with no singularities in $\mathbb{C}^N_{\text{reg}}$, analytic continuation holds for the horizontal sections (solutions of $\nabla_{\tau,\alpha}f = 0$).

2. Fundamental solutions and the monodromy representation. We can now assert the existence of horizontal sections of $\nabla_{\tau,\alpha}$ as a consequence of the Frobenius integrability.

2.1. THEOREM. *Each local solution f of (1.1) at $x_0 \in \mathscr{C}$ can be extended uniquely to all of $\mathscr{C}$. The space $\mathscr{D}_{\alpha,\tau}$ is of dimension $n(\tau)$. Any fundamental solution $(f_1, f_2, \dots, f_{n(\tau)})$ at x_0 is fundamental throughout $\mathscr{C}$ (that is, $\{f_1(x), f_2(x), \dots, f_{n(\tau)}(x)\}$ is a basis for $\mathbb{C}^{n(\tau)}$ for any fixed $x \in \mathscr{C}$).*

Proof. The equation $\nabla_{\tau,\alpha}f = 0$ is equivalent to the first-order linear system

$$\frac{\partial f(x)_i}{\partial x_j} = -\sum_{k=1}^{n(\tau)} A_{jik}(x)f(x)_k \tag{2.1}$$

$(i = 1, \dots, n(\tau);\ j = 1, \dots, N)$ (for smooth complex functions $(f(x)_1, \dots, f(x)_{n(\tau)})$, using a trailing subscript to label components of the vector $f(x)$). The coefficients are analytic on $\mathbb{C}^N_{\text{reg}}$ and satisfy the Frobenius integrability condition (see, e.g., **[De**, 1.6], **[V**, p.

106]). Thus for any initial condition $f(x_0)_i = c_i$ $(1 \le i \le n(\tau))$, $c \in \mathbb{C}^{n(\tau)}$, there is a unique solution f defined on $\mathscr{C}$ satisfying (2.1) and $f(x_0) = c$. A fundamental solution $(f_1, f_2, \dots, f_{n(\tau)})$ is given by $f_i(x_0)_j = \delta_{ij}$ $(1 \le i, j \le n(\tau))$.

Before we exhibit the homogeneity properties of the horizontal sections we recall some facts about conjugacy classes of reflections from **[Du3]**. Recall from §1 that G has m_0 classes of reflections and β_i is the value of the multiplicity function on class $\#i$. Define $\lambda(\tau; \alpha) = \sum_{i=1}^m \alpha_i(1 - \chi_\tau(\sigma_i)/n(\tau))$.

2.2. LEMMA. *For $\tau \in \widehat{G}$ there are integers $n_i(\tau)$, $1 \le i \le m_0$, such that $\lambda(\tau; \alpha) = \sum_{i=1}^{m_0} \beta_i n_i(\tau)$ and $0 \le n_i(\tau) \le 2m_i$. Further $n_i(\tau) = 0$ for each i exactly when $\tau = 1$ and $n_i(\tau) = 2m_i$ for each i exactly when $\tau = \operatorname{sgn}$ (or* det).

2.3. PROPOSITION. *Suppose f is a local solution of* (2.1), *then $f(x) = \prod_{i=1}^m \langle x, v_i\rangle^{-\alpha_i} g(x) = g(x)/h(x)$ where g satisfies the equations*

$$\frac{\partial g(x)}{\partial x_j} = g(x) \sum_{i=1}^m \frac{\alpha_i}{\langle x, v_i\rangle} \tau(\sigma_i)(v_i)_j \tag{2.3}$$

$(x \in \mathscr{C}, 1 \le j \le N)$. Further $f(cx) = c^{-\lambda(\tau,\alpha)} f(x)$ and $g(cx) = c^\delta g(x)$ with $\delta = \sum_{i=1}^m \alpha_i - \lambda(\tau; \alpha)$ and $c > 0$, $x \in \mathscr{C}$.

Proof. Logarithmic differentiation establishes the equation for g. For the homogeneity

$$\sum_{j=1}^N x_j \frac{\partial}{\partial x_j} f(x) = -f(x) \sum_{j=1}^m \alpha_i(I - \tau(\sigma_i)).$$

In **[Du3]** it was shown that $\sum \alpha_i(I - \tau(\sigma_i)) = \lambda(\tau; \alpha) I$ (since $\sum_{i=1}^m \alpha_i(1-\sigma_i)$ is in the center of $\mathbb{C}G$ and τ is an irreducible representation). □

2.4. COROLLARY. *If $n(\tau) = 1$ (τ is a linear character), then a local solution for* (2.1) *is $f(x) = \prod_{i=1}^m \langle x, v_i\rangle^{-\alpha_i(1-\tau(\sigma_i))}$ (where $1 - \tau(\sigma_i) = 0$ or* 2; *there are 2^{m_0} such characters).*

We will use the system (2.3) because of its more concise form. It is a particular case of Cherednik's r-matrices **[Ch1]**.

By the Frobenius condition we can use one-dimensional techniques and Lappo-Danilevsky's hyperlogarithms to produce infinite series for the fundamental solution of (2.3). The series will be defined on the universal covering space of $(\mathscr{C}^N_{\text{reg}}; x_0)$.

We adapt the hyperlogarithms from **[LD]**, see also Hille **[Hi**, pp. 355–360].

2.5. DEFINITION. For a piecewise smooth path γ in $\mathbb{C}^N$ (with domain $[0, 1]$) and a sequence of (nonzero) vectors $u_1, u_2, u_3, \dots$ such that

$$\delta := \inf\{|\langle \gamma(t), u_j\rangle|/|u_j| : j = 1, 2, 3, \dots ;\ 0 \le t \le 1\} > 0$$

define $L(\gamma; u_1, u_2, \dots, u_n; t)$ inductively by

$$L(\gamma; u_1; t) := \int_0^t \frac{\langle \gamma'(s), u_1\rangle}{\langle \gamma(s), u_1\rangle} ds,$$

and

$$L(\gamma; u_1, u_2, \dots, u_{n+1}; t) := \int_0^t L(\gamma; u_1, \dots, u_n; s) \frac{\langle \gamma'(s), u_{n+1}\rangle}{\langle \gamma(s), u_{n+1}\rangle} ds$$

(and $L(\gamma; t) := 1$), $n = 1, 2, 3, \dots$.

2.6. PROPOSITION. $|L(\gamma; u_1, \dots, u_n; t)| \le l(\gamma; t)^n/(\delta^n n!)$ *for* $0 \le t \le 1$, $n = 0, 1, 2, \dots$, *where the length function of* γ *is* $l(\gamma; t) := \int_0^t |\gamma'(s)|\, ds$.

Proof. By induction

$$|L(\gamma; u_1, \dots, u_n; t)| \le \frac{1}{\delta^{n-1}(n-1)!} \int_0^t l(\gamma; s)^{n-1} \frac{|u_n||\gamma'(s)|}{\delta |u_n|} ds = \frac{l(\gamma; t)^n}{\delta^n n!}. \qquad \square$$

Denote the fundamental solution to (2.3) at x_0 by $\mathscr{F}_\alpha$ so that $\mathscr{F}_\alpha(x_0) = I$ and

$$\frac{\partial}{\partial x_i} \mathscr{F}_\alpha(x) = \mathscr{F}_\alpha(x) \sum_{j=1}^m \frac{\alpha_j}{\langle x, v_j\rangle} \tau(\sigma_j)(v_j)_i$$

$(i = 1, \dots, N)$. The $n(\tau)$ rows of $\mathscr{F}_\alpha$ are a basis for the local solutions at x_0. The following is essentially a restatement of Theorem 9.6.1 in Hille **[Hi**, p. 356], adapted to paths in $\mathbb{C}^N$.

2.7. THEOREM. *Let* γ *be a piecewise smooth path in* $\mathbb{C}^N_{\text{reg}}$ *with* $\gamma(0) = x_0$,

$$\delta := \inf\{|\langle v_i, \gamma(t)\rangle|/|v_i| : 0 \le t \le 1,\ i = 1, \dots, m\}.$$

Then the analytic continuation of $\mathscr{F}_\alpha$ *along* γ *is given by*

$$\mathscr{F}_\alpha(\gamma(t)) = I + \sum_{n=1}^{\infty} \sum_{i_1, i_2, \dots, i_n=1}^{m} \alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_n} \cdot \tau(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}) L(\gamma; v_{i_1}, v_{i_2}, \dots, v_{i_n}; t).$$

The series is absolutely convergent and is dominated in norm by $\exp(\sum_{i=1}^m |\alpha_i| l(\gamma; t)/\delta)$.

Proof. Let $\mathscr{F}_\alpha$ be the fundamental solution. By the chain rule,

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_\alpha(\gamma(t)) &= \sum_{j=1}^{N} \frac{d\gamma_j(t)}{dt}\frac{\partial \mathscr{F}_\alpha}{\partial x_j} \\ &= \mathscr{F}_\alpha(\gamma(t)) \sum_{i=1}^{m} \frac{\alpha_i}{\langle \gamma(t), v_i\rangle} \langle \gamma'(t), v_i\rangle.\end{aligned}$$

Thus it remains to show that the series converges and satisfies this equation. Since τ is unitary the l^2-operator norms $\|\tau(\sigma_{i_1}\cdots\sigma_{i_n})\|$ all equal 1. The series is dominated by

$$\begin{aligned}1 + \sum_{n=1}^{\infty} \sum_{i_1, i_2, \dots, i_n=1}^{m} &|\alpha_{i_1}\cdots\alpha_{i_n}| l(\gamma; t)^n/(\delta^n n!) \\ &= \exp\left(\left(\sum_{i=1}^{m} |\alpha_i|\right) l(\gamma; t)/\delta\right).\end{aligned}$$

The series can be differentiated term by term (still absolutely convergent) to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{i_1, i_2, \dots, i_{n-1}=1}^{m} &\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_{n-1}} \tau(\sigma_{i_1}\cdots\sigma_{i_{n-1}}) \\ &\cdot L(\gamma; v_{i_1}, \dots, v_{i_{n-1}}; t) \sum_{i_n=1}^{m} \alpha_{i_n} \frac{\langle \gamma'(t), v_{i_n}\rangle}{\langle \gamma(t), v_{i_n}\rangle} \tau(\sigma_{i_n}) \\ = \mathscr{F}_\alpha(\gamma(t)) \sum_{i=1}^{m} \alpha_i &\frac{\langle \gamma'(t), v_i\rangle}{\langle \gamma(t), v_i\rangle} \tau(\sigma_i).\end{aligned}$$

□

2.8. Corollary. *For any piecewise smooth path* γ *in* $\mathbb{C}^N_{\text{reg}}$ *with* $\gamma(0) = x_0$, $\mathscr{F}_\alpha(\gamma(t))$ *and* $h(\gamma(t))^{-1}\mathscr{F}_\alpha(\gamma(t))$ *are entire functions of the multiplicity parameters* $\alpha_1, \dots, \alpha_m$.

Restricting the system (2.3) (or (2.1)) to a complex line passing through x_0 leads to a Fuchsian system. Indeed, fix $u \in \mathbb{C}^N$ with $u \neq 0$ and consider the differential equation for $\mathscr{G}(z) := \mathscr{F}_\alpha(x_0 + zu)$, namely

$$\frac{d}{dz}\mathscr{G}(z) = \mathscr{G}(z)\sum_{i=1}^{m} \alpha_i \frac{\langle v_i, u\rangle}{\langle v_i, x_0\rangle + z\langle v_i, u\rangle}\tau(\sigma_i).$$

This equation has regular singular points at $z = -\langle v_i, x_0\rangle / \langle v_i, u\rangle$ (when $\langle v_i, u\rangle \neq 0$), see Hille ([**Hi**, p. 354, 9.6.1]).

We return to the consideration of the vector bundle V_τ and the relation to analytic continuation.

First we define a translation representation of G induced by τ. For any $\mathbb{C}^{n(\tau)}$-valued function f whose domain is a G-invariant subset of $\mathbb{C}^N_{\text{reg}}$ let $R_\tau(w)f(x) := f(xw)\tau(w)^{-1}$ $(w \in G)$. In addition, if f is analytic and $R_\tau(w)f = f$ for all $w \in G$, then f is a local section of V_τ.

2.9. Lemma. $\nabla_{\tau,\alpha}(R_\tau(w)f)(x) = (\nabla_{\tau,\alpha}f(xw))(\tau(w)^{-1}\otimes w^{-1})$ $(x \in \mathbb{C}^N_{\text{reg}},\ w \in G)$.

Proof. The differentiation part is obvious (recall that $w^T = w^{-1}$, $w \in G$). The difference part is

$$\begin{aligned}\sum_{s=1}^{m} &\frac{\alpha_s}{\langle x, v_s\rangle}(f(xw)\tau(w)^{-1} - f(xw)\tau(w)^{-1}\tau(\sigma_s)) \otimes v_s \\ &= \left(\sum_{s=1}^{m} \frac{\alpha_s}{\langle xw, v_s w\rangle}(f(xw) - f(xw)\tau(w^{-1}\sigma_s w)) \otimes v_s w\right) \\ &\qquad\qquad \cdot (\tau(w)^{-1} \otimes w^{-1}).\end{aligned}$$

For each reflection σ_s, let $\sigma_{s'} = w^{-1}\sigma_s w$; then $\alpha_{s'} = \alpha_s$ and $v_s w = \varepsilon_s v_{s'}$ with $\varepsilon_s = \pm 1$. Thus the sum equals

$$\sum_{s'=1}^{m} \frac{\alpha_{s'}}{\langle xw, v_{s'}\rangle} f(xw)(I - \tau(\sigma_{s'})) \otimes v_{s'}. \qquad \square$$

Henceforth "path" means a piecewise smooth path in $\mathbb{C}^N_{\text{reg}}$ with domain $[0, 1]$.

The fundamental group of Ω is generated by certain paths joining x_0 to $x_0\sigma_i$, $1 \leq i \leq N$ (Brieskorn [**Br**]). More generally we consider paths joining x_0 to $x_0 w$ for some $w \in G$.

2.10. Definition. Let γ be a path joining x_0 to $x_0 w$ for some $w \in G$. Then the monodromy operator $M(\gamma;\alpha)$ on $\mathscr{D}_{\alpha,\tau}$ is defined by $M(\gamma;\alpha)f(x) = R_\tau(w)f^\gamma(x) = f^\gamma(xw)\tau(w)^{-1}$ $(x \in \mathscr{C},\ f \in \mathscr{D}_{\alpha,\tau})$, where f^γ denotes the analytic continuation of f along γ.

2.11. Proposition. *Under the hypotheses of Definition* 2.11, $M(\gamma;\alpha)f \in \mathscr{D}_{\alpha,\tau}$.

Proof. First observe that f^γ is defined uniquely on all of $\mathscr{C}w$ and satisfies $\nabla_{\tau,\alpha} f^\gamma(xw) = 0$ for each $x \in \mathscr{C}$, because the coefficients of $\nabla_{\tau,\alpha}$ are rational functions with no singularities in $\mathbb{C}^N_{\text{reg}}$. By Lemma 2.9, $\nabla_{\tau,\alpha}(R_\tau f^\gamma) = 0$, and thus $R_\tau f^\gamma \in \mathscr{D}_{\alpha,\tau}$. □

Of course, if γ_1, γ_2 are paths with $\gamma_1(0) = \gamma_2(0) = x_0$ and $\gamma_1(1) = \gamma_2(1) = x_0 w$ (some $w \in W$), and γ_1 is homotopic to γ_2 (fixed end-points) in $\mathbb{C}^N_{\text{reg}}$, then $M(\gamma_1;\alpha) = M(\gamma_2;\alpha)$. Cherednik **[Ch1]** uses a similar definition for the monodromy action.

For any particular fundamental solution of (2.1) at x_0 $(f_1, f_2, \dots, f_{n(\tau)}) \in \mathscr{D}_{\alpha,\tau}$ such that $\{f_1(x_0), f_2(x_0), \dots, f_{n(\tau)}(x_0)\}$ is a basis for $\mathbb{C}^{n(\tau)}$) the monodromy matrix of γ is denoted $(M(\gamma;\alpha)_{ij})$ and is defined by $f_j^\gamma(x_0 w)\tau(w)^{-1} = \sum_{i=1}^{n(\tau)} M(\gamma;\alpha)_{ij} f_i(x_0)$ (by analytic continuation the formula remains valid if x_0 is replaced by any $x \in \mathscr{C}$).

Suppose γ_1, γ_2 are paths joining x_0 to $x_0 w_1$, $x_0 w_2$ respectively (thus both correspond to loops in $\Omega = \mathbb{C}^N_{\text{reg}}/G$). Thus $\gamma_1 \circ \gamma_2$ denotes the path joining x_0 to $x_0 w_1 w_2$ given by $\gamma_2(2t)$ for $0 \le t \le 1/2$ and $\gamma_1(2t-1)w_2$ for $1/2 \le t \le 1$.

2.12. Proposition. *If γ_1 and γ_2 are paths joining x_0 to $x_0 w_1$, $x_0 w_2$ respectively* (*some $w_1, w_2 \in G$*), *then*

$$M(\gamma_1 \circ \gamma_2;\alpha) = M(\gamma_1;\alpha)M(\gamma_2;\alpha).$$

Proof. Let $(M(\gamma_s;\alpha)_{ij})$ $(s = 1, 2)$ denote the monodromy matrices with respect to a fundamental solution $(f_1, \dots, f_{n(\tau)})$ of (2.1). Apply analytic continuation along γ_1 to the identity

$$f_j^{\gamma_2}(x_0 w_2) = \sum_{i=1}^{n(\tau)} M(\gamma_2;\alpha)_{ij} f_i(x_0)\tau(w_2)$$

to get

$$f_j^{\gamma_1 \cdot \gamma_2}(x_0 w_1 w_2) = \sum_{i=1}^{n(\tau)} M(\gamma_2;\alpha)_{ij} f_i^{\gamma_1}(x_0 w_1)\tau(w_2).$$

Now substitute

$$f_j^{\gamma_1}(x_0 w_1) = \sum_{k=1}^{n(\tau)} M(\gamma_1\,;\,\alpha)_{ki} f_k(x_0)\tau(w_1). \qquad \square$$

We consider the dependence of $M(\gamma\,;\,\alpha)$ on the multiplicity function α_j.

2.13. Proposition. *For a path γ joining x_0 to $x_0 w$ for some $w \in W$, let $M(\gamma\,;\,\alpha)_{ij}$ denote the monodromy matrix with respect to the fundamental solution $(f_1, \dots, f_{n(\tau)})$ with $f_i(x_0)_j = \delta_{ij}$ $(1 \le i, j \le n(\tau))$; then $M(\gamma\,;\,\alpha)_{ij}$ is an entire function of $(\alpha_1, \alpha_2, \dots)$ and $M(\gamma\,;\,0)_{ij} = \tau(w^{-1})_{ji}$.*

Proof. Let $(f_1, \dots, f_{n(\tau)})$ be the rows of $(h(x_0)/h)\mathscr{F}_\alpha$ where $\mathscr{F}_\alpha$ is the fundamental solution of (2.3) with $\mathscr{F}_\alpha(x_0) = I$, as in Theorem 2.7. Then by definition, $M(\gamma\,;\,\alpha)_{ij} = \sum_{k=1}^{n(\tau)} f_j^\gamma(x_0 w)_k \tau(w^{-1})_{ki}$ $(1 \le i, j \le n(\tau))$ where $f_j^\gamma(x_0 w)_k$ is the (j, k)-entry of $\mathscr{F}_\alpha(\gamma(1))$, which is entire in α. When each $\alpha_i = 0$, $\mathscr{F}_\alpha$ is constant and so $f_j^\gamma(x_0 w)_k = \delta_{jk}$. $\square$

By Brieskorn's theorem we can specify the generators for the fundamental group $\pi_1(\Omega, x_0)$ and assert the braid relations.

Recall that $\sigma_1, \sigma_2, \dots, \sigma_N$ are the simple reflections for the Coxeter group. The structure theorem (**[BG]**, **[Hu]**) says that G is generated by $\sigma_1, \dots, \sigma_N$ with the defining relations $(\sigma_i\sigma_j)^{m_{ij}} = 1$ for certain positive integers m_{ij} $(1 \le i, j \le N)$ satisfying $m_{ii} = 1$ and $m_{ij} = m_{ji}$ (the Coxeter graph consists of N nodes with node #i joined by an edge to node #j when $m_{ij} \ge 3$; the edge is labeled by m_{ij} when $m_{ij} \ge 4$).

For $i = 1, 2, \dots, N$, let γ_i be the path

$$\gamma_i(t) = x_0 + (r(t) - 1)(\langle x_0, v_i\rangle/|v_i|^2)v_i,$$

where r is a piecewise smooth path joining 1 to -1 in $\mathbb{C}\backslash\{0\}$ such that $\operatorname{Im} r(t) \ge 0$ for $0 \le t \le 1$. Thus γ_i joins x_0 to $x_0\sigma_i$.

2.14. Theorem (*Brieskorn* **[Br]**). *$\pi_1(\Omega\,;\,x_0)$ is generated by (the homotopy classes of) $\gamma_1, \gamma_2, \dots, \gamma_N$ subject only to the relations $(\gamma_i \circ \gamma_j \circ \gamma_i \cdots) = (\gamma_j \circ \gamma_i \circ \gamma_j \cdots)$, with m_{ij} factors on each side, for $1 \le i < j \le N$ (equality of paths being interpreted as homotopy).*

The group $\pi_1(\Omega\,;\,x_0)$ is the Artin group $\widetilde{G}$ belonging to G. It is called a generalized braid group because when $G = S_{N+1}$ (the symmetric group, type A_N) the orders of $\sigma_i\sigma_j$ are $m_{ij} = 2$ for $i + 1 < j$

and $m_{i,i+1} = 3$ *and* $\gamma_i \circ \gamma_{i+1} \circ \gamma_i = \gamma_{i+1} \circ \gamma_i \circ \gamma_{i+1}$ $(1 \le i < N)$ *are the braid relations.*

2.15. COROLLARY. *The monodromy operators* $M(\gamma_1; \alpha), \dots, M(\gamma_N; \alpha)$ *generate the monodromy group of the system* (2.1) *and satisfy the relations*

$$(M(\gamma_i;\alpha)M(\gamma_j;\alpha)M(\gamma_i;\alpha)\cdots) = (M(\gamma_j;\alpha)M(\gamma_i;\alpha)M(\gamma_j;\alpha)\cdots)$$

with m_{ij} *factors on each side* $(1 \le i < j \le N)$.

Proof. This is a consequence of the theory of local systems and analytic integrable connections on an analytic manifold (Deligne [**De**, 1.6], Varadarajan [**V**, §4]). □

We will show that $M(\gamma_j;\alpha)^2 = (1-q_j)M(\gamma_j;\alpha) + q_j I$, where $q_j = e^{-2\pi i\alpha_j}$ and thus we are led to a representation of the Hecke-Iwahori algebra of G. The proof relies on restricting the differential equation (2.3) to a complex line spanned by v_j. The method for constructing power-series solutions of a first-order system around a regular singular point is presented in Hille [**Hi**, pp. 345–352].

The following lemma is adapted to this special situation.

2.16. LEMMA. *Suppose* σ *is a unitary involution on* $\mathbb{C}^k$, $\alpha \in \mathbb{C}$, *and* $B(z)$ *is a* $k \times k$*-matrix valued analytic function on* $\{z \in \mathbb{C} : |z| < r_0\}$, *some* $r_0 > 0$ *and* $\sigma B(z)\sigma = -B(-z)$. *Then the matrix system* $\frac{d}{dz}\mathscr{G}(z) = \frac{\alpha}{z}\mathscr{G}(z)\sigma + \mathscr{G}(z)B(z)$ *has a fundamental solution* $\mathscr{G}(z) = A(z)C(z)$, *where* $C(z) = \sum_{n=0}^{\infty} z^n C_n$, *absolutely convergent for* $|z| < r_0$, $C_0 = I$, $\sigma C(z)\sigma = C(-z)$ *and, in an orthogonal decomposition of* $\mathbb{C}^k$ *in which* $\sigma = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$,

(i) $A(z) = \begin{bmatrix} z^{\alpha} I & 0 \\ 0 & z^{-\alpha} I \end{bmatrix}$, *when* $\alpha + 1/2 \notin \mathbb{Z}$,

(ii) $A(z) = \begin{bmatrix} z^{\alpha} I & 0 \\ z^{\alpha}(\log z) D & z^{-\alpha} I \end{bmatrix}$, *when* $\alpha = 1/2, 3/2, \dots$,

(iii) $A(z) = \begin{bmatrix} z^{\alpha} I & z^{-\alpha}(\log z) D \\ 0 & z^{-\alpha} I \end{bmatrix}$, *when* $\alpha = -1/2, -3/2, \dots$

(*some matrix* D).

Proof. Expand $B(z) = \sum_{n=0}^{\infty} z^n B_n$ (constant matrices B_n). The hypothesis $\sigma B(z)\sigma = -B(-z)$ implies $\sigma B_n \sigma = (-1)^{n+1} B_n$. In the orthonormal basis for $\mathbb{C}^k$ in which $\sigma = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ write

$$B_n = \begin{bmatrix} B_{n,11} & B_{n,12} \\ B_{n,21} & B_{n,22} \end{bmatrix}$$

(and similarly for other $k \times k$ matrices). Say a matrix D is even (respectively, odd) if $\sigma D \sigma = D$, respectively $-D$. Then D is even when $D_{n,21} = 0 = D_{n,12}$ and D is odd when $D_{n,11} = 0 = D_{n,22}$. For $n = 1, 2, 3, \dots$, let $M_n = \sum_{j=0}^{n-1} C_j B_{n-1-j}$, and let $M_0 = 0$, $C_0 = I$. The inductive hypothesis $\sigma C_j \sigma = (-1)^j C_j$ implies $\sigma M_n \sigma = (-1)^n M_n$.

Assume first that $\alpha + 1/2 \notin \mathbb{Z}$. The substitution

$$\mathscr{G}(z) = \sum_{n=0}^{\infty} \begin{bmatrix} z^{n+\alpha} I & 0 \\ 0 & z^{n-\alpha} I \end{bmatrix} C_n$$

in the rewritten system $z\frac{d}{dz}\mathscr{G}(z) - \alpha\mathscr{G}(z) - z\mathscr{G}(z)B(z) = 0$ leads to the equations

$$\begin{bmatrix} (n+\alpha)I & 0 \\ 0 & (n-\alpha)I \end{bmatrix} C_n - \alpha C_n \sigma = M_n, \qquad n = 0, 1, 2, \dots$$

(equivalently $nC_n + \alpha(\sigma C_n - C_n \sigma) = M_n$).

When n is odd, C_n is odd and $C_{n,12} = \frac{1}{n+2\alpha} M_{n,12}$, $C_{n,21} = \frac{1}{n-2\alpha} M_{n,21}$. When n is even, C_n is even and $C_n = \frac{1}{n} M_n$. This defines $\mathscr{G}$ uniquely (given $C_0 = I$).

Now assume $2\alpha = l$ a positive odd integer. Substitute

$$\mathscr{G}(z) = \sum_{n=0}^{\infty} \begin{bmatrix} I & 0 \\ (\log z)D & I \end{bmatrix} \begin{bmatrix} z^{n+\alpha} I & 0 \\ 0 & z^{n-\alpha} I \end{bmatrix} C_n$$

into the rewritten equation. The top row of the resulting equation is

$$\sum_{n=0}^{\infty} z^{n+\alpha}[nC_{n,11} - M_{n,11}, (n+2\alpha)C_{n,12} - M_{n,12}] = 0$$

and the bottom row is

$$\begin{aligned} \log z \sum_{n=0}^{\infty} & z^{n+\alpha}[nC_{n,11} - M_{n,11}, (n+2\alpha)C_{n,12} - M_{n,12}] \\ &+ \sum_{n=0}^{\infty} z^{n+\alpha} D[C_{n,11}, C_{n,12}] \\ &+ \sum_{n=0}^{\infty} z^{n-\alpha}[(n-2\alpha)C_{n,21} - M_{n,21}, nC_{n,22} - M_{n,22}] = 0. \end{aligned}$$

The first row now defines the first rows of C_n, M_n uniquely starting with $C_{0,11} = I$, $C_{0,12} = 0$. This wipes out the log z term in the bottom row. Since $n+\alpha = (n+l)-\alpha$ the second equation becomes

$$\sum_{n=0}^{\infty} z^{n+\alpha}[DC_{n-l,11} + (n-l)C_{n-l,21} - M_{n,21},$$
$$DC_{n,12} + nC_{n,22} - M_{n,22}] = 0.$$

Set $C_{0,21} = 0$, $C_{0,22} = I$; when n is odd and $n < l$, then $C_{n,21} = \frac{1}{n-l}M_{n,21}$, $C_{n,22} = 0$; when $n = l$, $C_{l,21} = 0$ (arbitrary) and $D = M_{l,21}$; the rest of the construction for C_n is obvious. Note that $C_{n-l,11} = 0$ when n is even, $C_{n-l,12} = 0$ when n is odd so that C_n has the same parity as n. A similar argument works for $2\alpha = -l$. □

2.17. THEOREM. *For $j = 1, \dots, N$ the monodromy operators $M(\gamma_j; \alpha)$ satisfy $(M(\gamma_j; \alpha) - I)(M(\gamma_j; \alpha) + e^{-2\pi i\alpha_j}I) = 0$.*

Proof. Fix j and let $\zeta(z) := x_0 + (z-1)(\langle x_0, v_j\rangle/|v_j|^2)v_j$ $(z \in \mathbb{C})$ be a complex line joining x_0 to $x_0\sigma_j$ (at $z = 1$ and -1 respectively); thus γ_j is the image under ζ of a path joining 1 to -1 in $\{z : \operatorname{Im} z \geq 0,\ z \neq 0\}$. Restrict the matrix system (2.3) to the line ζ to obtain

$$\frac{d}{dz}\mathcal{G}(\zeta(z)) = \mathcal{G}(\zeta(z))\sum_{i=1}^{m} \alpha_i \tau(\sigma_i) d_i(z)$$

where

$$d_i(z) := \frac{\langle \zeta'(z), v_i\rangle}{\langle \zeta(z), v_i\rangle} = \frac{\langle x_0, v_i\rangle - \langle x_0, v_i\sigma_j\rangle}{\langle x_0, v_i\rangle(1+z) + \langle x_0, v_i\sigma_j\rangle(1-z)}.$$

Clearly $d_i(z) = 0$ when $\langle v_i, v_j\rangle = 0$ and $d_i(z) = -d_{i'}(-z)$ when $v_i\sigma_j = v_{i'}$ (thus $\tau(\sigma_{i'}) = \tau(\sigma_j)\tau(\sigma_i)\tau(\sigma_j)$), note that $v_i\sigma_j$ is a positive root when $i \neq j$ because σ_j is simple. Also $d_j(z) = 1/z$. The pole of $d_i(z)$ with $i \neq j$ and $\langle v_i, v_j\rangle \neq 0$ is at

$$z_i = -\frac{\langle x_0, v_i\rangle + \langle x_0, v_i\sigma_j\rangle}{\langle x_0, v_i\rangle - \langle x_0, v_i\sigma_j\rangle};$$

here $z_i \in \mathbb{R}$ and $|z_i| > 1$ because $\langle x_0, v_i\rangle > 0$ and $\langle x_0, v_i\sigma_j\rangle = \langle x_0, v_{i'}\rangle > 0$.

Put $F(z) = \mathcal{G}(\zeta(z))$, and let

$$B(z) := \sum\{\alpha_i\tau(\sigma_i)d_i(z) : 1 \leq i \leq m,\ i \neq j,\ \langle v_i, v_j\rangle \neq 0\}.$$

The differential equation becomes

$$\frac{d}{dz}F(z) = \frac{\alpha_j}{z}F(z)\tau(\sigma_j) + F(z)B(z),$$

with $\tau(\sigma_j)B(z)\tau(\sigma_j) = -B(-z)$ and B being analytic in $\{z \in \mathbb{C} : |z| < r_0\}$ some $r_0 > 1$. There is an orthonormal basis for $\mathbb{C}^{n(\tau)}$ in which $\tau(\sigma_j) = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ (the size of the upper left block is $\frac{1}{2}(n(\tau) + \chi_\tau(\sigma_j))$). By the lemma, there is a fundamental solution $F(z)$ so that the analytic continuation along γ_j satisfies $F(-z) = AF(z)\tau(\sigma_j)$ for $0 < z < r_0$, where

(i) $A = \begin{bmatrix} e^{\pi i\alpha_j} I & 0 \\ 0 & -e^{-\pi i\alpha_j} I \end{bmatrix}$ when $\alpha_j + 1/2 \notin \mathbb{Z}$,

(ii) $A = e^{\pi i\alpha_j} \begin{bmatrix} I & 0 \\ D_0 & I \end{bmatrix}$ when $\alpha_j = 1/2, 3/2, 5/2, \dots$,

(iii) $A = e^{\pi i\alpha_j} \begin{bmatrix} I & D_0 \\ 0 & I \end{bmatrix}$ when $\alpha_j = -1/2, -3/2, -5/2, \dots$,

for certain matrices D_0 (indeed $D_0 = i\pi D$, the matrix D from 2.16(ii) and (iii)). To obtain a fundamental solution of (2.1) restricted to the complex line ζ multiply $F(z)$ by $\prod_{i=1}^m \langle \zeta(z), v_i\rangle^{-\alpha_i}$. As $\zeta(z)$ traverses the path γ_j from x_0 to $x_0\sigma_j$, the factors $\langle\zeta(z), v_i\rangle^{-\alpha_i}$ with $i \neq j$ satisfy $\operatorname{Re}\langle\zeta(z), v_i\rangle > 0$ and $\langle x_0\sigma_j, v_i\rangle = \langle x_0, v_i\sigma_j\rangle = \langle x_0, v_{i'}\rangle$ (and $\alpha_{i'} = \alpha_i$). Only the factor $\langle\zeta(z), v_j\rangle^{-\alpha_j} = (z\langle x_0, v_j\rangle)^{-\alpha_j}$ changes by $e^{-\pi i\alpha_j}$ along γ_j.

Hence the monodromy matrix (in a certain basis diagonalizing $\tau(\sigma_j)$) for $f(z) = (1/h(\zeta(z)))F(z)$, namely $f(-1) = M(\gamma_i;\alpha)f(1)\tau(\sigma_j)$ is

$$\begin{bmatrix} I & 0 \\ 0 & -q_j I \end{bmatrix}, \quad \begin{bmatrix} I & 0 \\ \pi i D & I \end{bmatrix}, \quad \begin{bmatrix} I & \pi i D \\ 0 & I \end{bmatrix},$$

when $\alpha_j + 1/2 \notin \mathbb{Z}$, $\alpha_j = 1/2, 3/2, \dots$, $\alpha_j = -1/2, -3/2, \dots$, respectively, where $q_j := e^{-2\pi i\alpha_j}$. In each case

$$M(\gamma_j; \alpha)^2 = (1 - q_j)M(\gamma_j; \alpha) + q_j I. \qquad \square$$

Note that the fundamental solution f constructed above generally does not satisfy $f(1) = I$.

Recall the definition of the generic complex algebra for G (e.g., see Humphreys [**Hu**, Ch. 7]) Bourbaki [**Bo**, p. 55]: for parameters a_i, b_i, $i = 1, \dots, N$ such that $a_i = a_j$, $b_i = b_j$ whenever σ_i is conjugate to σ_j, the algebra is the complex span of $\{T(w) : w \in G\}$ with the multiplication rules $T(\sigma_i)T(w) = T(\sigma_i w)$ if $l(\sigma_i w) = l(w) + 1$ and $T(\sigma_i)T(w) = a_i T(w) + b_i T(\sigma_i w)$, if $l(\sigma_i w) = l(w) - 1$ (of course $T(1) = I$). There is a presentation of this algebra: $T(\sigma_i)^2 = a_i T(\sigma_i) + b_i I$ and

$$(T(\sigma_i)T(\sigma_j)T(\sigma_i)\cdots) = (T(\sigma_j)T(\sigma_i)T(\sigma_j)\cdots)$$

m_{ij} factors on each side, $1 \le i < j \le N$. The specialization $a_i = q_i - 1$, $b_i = q_i$ is called the Hecke (or Hecke-Iwahori) algebra $H(G; q)$ (Iwahori **[I]**). Here we use $a_i = 1 - q_i$, $b_i = q_i$ but this is isomorphic to the standard Hecke algebra under the correspondence $T(\sigma_i) \mapsto -T(\sigma_i)$ (and $T(w) \mapsto (-1)^{l(w)} T(w)$).

2.18. THEOREM. *For any multiplicity function α, $\mathscr{D}_{\tau,\alpha}$ is a module for the Hecke algebra $H(G; q)$ of G with parameters $q_j = e^{-2\pi i \alpha_j}$. The representation is $\sum_{w\in G} c_w T(w) \mapsto \sum_{w\in G} c_w M(\gamma_w; \alpha)$ where γ_w is the path $\gamma_{j_1} \circ \gamma_{j_2} \circ \cdots \circ \gamma_{j_k}$ corresponding to the reduced expression $w = \sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_k}$ $(1 \le j_i \le N)$, $k = l(w)$ $(c_w \in \mathbb{C})$.*

Geometrically, the path γ_w can be interpreted as a sequence of paths starting at x_0 passing "around" walls $\#j_k, \#j_{k-1}, \dots, \#j_1$ finishing at $x_0 w$; and this is the minimum number of steps.

3. Special values of α and the generic degrees. The Poincaré series of G determines which values of α_i lead to semisimple specializations of the generic algebra. We recall some results from Gyoja and Uno **[GU]**, Macdonald **[M]**, and Yamane **[Y]**. For each $w \in G$ there is a vector-valued length $(l_1(w), l_2(w), \dots, l_{m_0}(w))$ (where m_0 is the number of classes of reflections) so that there is a reduced expression $w = \sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_k}$ $(1 \le ji \le N)$ with $l_1(w)$ of the factors being in class $\#1$, and so on.

Macdonald showed that $l_i(w) = n_i(w)$, the number of positive roots belonging to class $\#i$ made negative by w (see also Curtis, Iwahori, and Kilmoyer **[CIK**, Cor. 10.5, p. 113**]**). Let $q_1, q_2, \dots, q_{m_0}$ be indeterminates and let $q^{l(w)} := \Pi_{i=1}^{m_0} q_i^{l_i(w)}$. The Poincaré series (polynomial, since G is finite) is $P_G(q) := \sum_{w\in G} q^{l(w)}$. Macdonald **[M]** determined $P_G(q)$ for all the indecomposable finite Coxeter groups. Gyoja and Uno **[GU]** showed that $H(G, q)$ is semisimple for nonzero complex values of q_i exactly when $P_G(q) \ne 0$ (their proof is for $q_1 = q_2 = \cdots$ but is easily transferred to the general case). Yamane **[Y]** showed that for any particular irreducible (principal) m-dimensional representation M of (generic) $H(G; q)$ there is a polynomial in q whose zero-set consists of excluded values (for such values some of the properties are lost). Let

$$\psi_M(q) = \frac{1}{m} \sum_{w\in G} q^{l(w_0 w)} \operatorname{Tr}(M(T(w))) \operatorname{Tr}(M(T(w^{-1}))),$$

where w_0 is the longest element in G. The generic degree $d_M(q)$ of M is given by $q^{l(w_0)} P_G(q)/\psi_M(q)$ (see, e.g., Curtis **[Cu**, p. 53**]**).

Both d_M and ψ_M are polynomials in q. Yamane showed that M is an irreducible representation of the specialization of $H(G, q)$ exactly when $\psi_M(q) \neq 0$. We can show that the monodromy representation on $\mathscr{D}_{\alpha,\tau}$ corresponds to the irreducible representation τ of G, in the sense of Proposition 7.1 of **[CIK**, Proposition 2.1, p. 53] of Curtis **[Cu]**. As before, we use $(\beta_1, \beta_2, \dots)$ to denote the values of α_i on the classes of reflections.

3.1. THEOREM. *For values of the multiplicity function* $(\beta_1, \beta_2, \dots)$ *such that* $P_G(e^{-2\pi i\beta}) \neq 0$ *the representation* $T(w) \mapsto M(\gamma_w; \alpha)$ $(w \in G)$ *corresponds to* τ.

Proof. From Macdonald's **[M]** list of P_G for indecomposable Coxeter groups we see that (i) (case $m_0 = 2$, type B_N, F_4, $I_2(2k)$) the zero sets of $P_G(e^{-2\pi i\beta_1}, e^{-2\pi i\beta_2})$ are hyperplanes of the form $n_1\beta_1 + n_2\beta_2 = n_0$ (integers n_0, n_1, n_2); the complement in $\mathbb{C}^2$ is pathwise connected to $(0, 0)$ where $q = (1, 1)$. (ii) (case $m_0 = 1$, type A, D, E, H, $I_2(2k+1)$) the zero-set of $P_G(e^{-2\pi i\beta})$ is countable with no points of accumulation, and again the complement is pathwise connected to $\beta = 0$, $q = 1$. The matrix entry functions (for Proposition 2.13) are entire in (β_1, β_2) (respectively β_1) and $H(G; e^{-2\pi i\beta})$ is semisimple at each point of the path. At $\beta = 0$, $M(\gamma_w; 0) = \tau(w^{-1})^T$ (transpose) (unitarily equivalent to $\tau(w)$ since each character of G is real). The argument in **[CIK**, Prop. 7.1, p. 102], which relies on the factorization of the characteristic polynomial of a generic element $\sum_{w\in W} c_w T(w) \in H(G; q)$, proves the desired correspondence, since analytic continuation involving the coefficients of the polynomial works here. □

3.2. COROLLARY. *The representation* $T(w) \mapsto M(\gamma_w; \alpha)$ *corresponds to* τ *and is irreducible and principal when* $\psi_M(e^{-2\pi i\beta}) \neq 0$.

Proof. Suppose $\psi_M(e^{-2\pi i\beta_0}) \neq 0$ and possibly $P_G(e^{-2\pi i\beta_0}) = 0$. Join β_0 to 0 by a path on which $\psi_M(e^{-2\pi i\beta}) \neq 0$ for all β and $P_G(e^{-2\pi i\beta}) \neq 0$ for all β except possibly $\beta = \beta_0$.

At each point β on the path, $M(\gamma_w; \alpha)$ is irreducible and principal by Proposition 3.2 of Yamane **[Y]**, and corresponds to τ. Analytic continuation finishes the argument. □

We will demonstrate this corollary explicitly for the dihedral groups.

4. The dihedral groups. The even dihedral groups have two classes of reflections while the odd dihedral groups have just one. In case of one parameter there is a basis for $\mathscr{D}_{\alpha,\tau}$ in terms of hypergeometric functions. We will determine the differential equation for the two-parameter case and then specialize to one parameter. Fix an integer $k \geq 2$ and consider the dihedral group $I_2(2k)$ (of order $4k$). Let $\xi := \pi/k$ then a list of positive roots is $v_j := (\sin(j\xi/2), \cos(j\xi/2))$ with corresponding reflections

$$\sigma_j = \begin{bmatrix} \cos(j\xi) & -\sin(j\xi) \\ -\sin(j\xi) & -\cos(j\xi) \end{bmatrix}, \qquad 0 \leq j \leq 2k-1.$$

The simple roots for $I_2(2k)$ are v_0 and v_{2k-1}, while v_0 and v_{2k-2} are the simple roots for the subgroup $I_2(k)$ (containing σ_{2j}, $0 \leq j \leq k-1$). The four linear characters are covered by the general formula in Corollary 2.4. The irreducible unitary representations are τ_l, $l = 1, \dots, k-1$ for $I_2(2k)$; or $1 \leq l < k/2$ for $I_2(k)$; which are defined by

$$\tau_l(\sigma_j) = \begin{bmatrix} 0 & e^{-ijl\xi} \\ e^{ijl\xi} & 0 \end{bmatrix}.$$

The multiplicity function is taken to be $\alpha_{2j} = \alpha$, $\alpha_{2j+1} = \beta$ (for arbitrary $\alpha, \beta \in \mathbb{C}$). Because of the homogeneity of $\mathscr{D}_{\alpha,\tau}$ (degree $= -k(\alpha+\beta)$) it suffices to restrict the system (2.3) to a circle around the origin. Let $x_1(\theta) = \cos(\theta/k)$, $x_2(\theta) = \sin(\theta/k)$ with $\theta \in \mathbb{C}$, and fix the representation τ_l. Then (2.3) becomes

$$(4.1)\quad \frac{d}{d\theta}[g_1(x(\theta)), g_2(x(\theta))] = \frac{1}{k}[g_1, g_2]\left\{ \alpha \sum_{j=0}^{k-1} \tau_l(\sigma_{2j}) \cot(j\xi + \theta/k) + \beta \sum_{j=0}^{k-1} \tau_l(\sigma_{2j+1}) \cot((j+1/2)\xi + \theta/k) \right\}.$$

4.1. Proposition. *The system* (4.1) *reduces to*

$$\frac{d}{d\theta} g_1 = e^{i\theta(1-2l/k)} \left(\frac{\alpha}{\sin\theta} + \frac{i\beta}{\cos\theta} \right) g_2,$$
$$\frac{d}{d\theta} g_2 = e^{-i\theta(1-2l/k)} \left(\frac{\alpha}{\sin\theta} - \frac{i\beta}{\cos\theta} \right) g_1,$$

with either:

$\beta \neq 0$, $\quad G = I_2(2k)$, $\quad 1 \leq l \leq k-1$ *the fundamental region* $\mathscr{C}$ *corresponds to* $0 < \theta < \pi/2$;

or:
$$\beta = 0, \quad G = I_2(k), \quad 1 \le l < k/2,$$
$$\mathscr{C} \text{ corresponds to } 0 < \theta < \pi.$$

Proof. Write the terms to be summed in complex exponential form, namely,

$$\cot(j\xi + \theta/k) = -(e^{-2i\theta/k} + \omega^j)/(e^{-2i\theta/k} - \omega^j),$$

where $\omega := e^{2\pi i/k}$ (similar expressions for the β term). Since $\omega^k = 1$ each required sum can be expressed by use of

$$\frac{1}{k}\sum_{j=0}^{k-1} \frac{\omega^{jn}}{t - \omega^j} = \frac{t^{n-1}}{t^k - 1} \qquad (n = 1, 2, \dots, k, \text{ variable } t),$$

a partial fraction identity. □

Let $\delta := \frac{1}{2}(1 - \frac{2l}{k})$, a convenient parameter in subsequent work. When $\beta \ne 0$, $-1/2 < -(1/2 - 1/k) \le \delta \le (1/2 - 1/k) < 1/2$, and when $\beta = 0$, $0 < \delta \le (1/2 - 1/k) < 1/2$.

4.2. COROLLARY. *Solutions* $[g_1, g_2]$ *of* (4.1) *satisfy* $g_1 = e^{i\delta\theta}u_1$, $g_2 = e^{-i\delta\theta}u_2$ *where*

$$\begin{aligned} \frac{d}{d\theta}u_1 &= -i\delta u_1 + \left(\frac{\alpha}{\sin\theta} + \frac{i\beta}{\cos\theta}\right)u_2, \\ \frac{d}{d\theta}u_2 &= i\delta u_2 + \left(\frac{\alpha}{\sin\theta} - \frac{i\beta}{\cos\theta}\right)u_1. \end{aligned} \tag{4.2}$$

The substitution $\zeta = \tan(\theta/2)$ rationalizes the system (4.2); indeed,

$$\frac{d}{d\zeta}[u_1, u_2] = [u_1, u_2]\left\{\frac{2i\delta}{1+\zeta^2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\alpha}{\zeta}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{2i\beta}{1-\zeta^2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}. \tag{4.3}$$

This is a Fuchsian system with regular singular points at 0, ± 1, $\pm i$, ∞. The hypergeometric differential equation allows three such points so this system appears to be too complicated for hypergeometric functions. But specialization to $\beta = 0$, $G = I_2(k)$ $(1 \le l < k/2$, $0 < \delta < 1/2)$ allows solutions in known functions. The connection coefficients among the different solutions of the hypergeometric equation will enable us to get the monodromy matrices explicitly.

Set $\beta = 0$ and substitute $z = \sin^2(\theta/2) = \zeta^2/(1 + \zeta^2)$,

$$u_1 = \zeta^\alpha v_1 + i\zeta^{\alpha+1}v_2, \quad u_2 = \zeta^\alpha v_1 - i\zeta^{\alpha+1}v_2.$$

Eliminating v_2 from the first-order system

$$\frac{dv_1}{dz} = \frac{\delta}{1-z}v_2, \quad \frac{dv_2}{dz} = -\frac{\delta}{z}v_1 - \frac{2\alpha+1}{2z(1-z)}v_2,$$

we obtain

$$z(1-z)\left(\frac{d}{dz}\right)^2 v_1 + (\alpha+1/2-z)\frac{d}{dz}v_1 + \delta^2 v_1 = 0. \tag{4.4}$$

This allows us to write down solutions with specified singularities at the regular singular points $z = 0, 1$ ($\theta = 0, \pi$ respectively).

4.3. THEOREM. *The following four solutions for* (4.2) *are associated to the singularities* z^0, $(1-z)^0$, $z^{1/2-\alpha}$, $(1-z)^{\alpha+1/2}$ *respectively. When* $\alpha + 1/2 \notin \mathbb{Z}$ *and* $\alpha + 1/2 \pm \delta \notin \mathbb{Z}$ *any two of the solutions form a basis for the solution space* (*the values of* α *correspond to* $q \neq -1$, $e^{\pm 2\pi i l/k}$) :

$$\begin{aligned}
[u_1^{\mathrm{I}}, u_2^{\mathrm{I}}] = {} & \frac{z^{\alpha/2}(1-z)^{-\alpha/2}}{\Gamma(\alpha+1/2)} {}_2F_1\left(\begin{matrix}\delta, -\delta\\ \alpha+1/2\end{matrix}; z\right)[1, 1] \\
& + \frac{i\delta}{\Gamma(\alpha+3/2)} z^{(\alpha+1)/2}(1-z)^{(1-\alpha)/2} \\
& \cdot {}_2F_1\left(\begin{matrix}1+\delta, 1-\delta\\ \alpha+3/2\end{matrix}; z\right)[-1, 1], \\
[u_1^{\mathrm{II}}, u_2^{\mathrm{II}}] = {} & \frac{z^{\alpha/2}(1-z)^{-\alpha/2}}{\Gamma(1/2-\alpha)} {}_2F_1\left(\begin{matrix}\delta, -\delta\\ 1/2-\alpha\end{matrix}; 1-z\right)[1, 1] \\
& + \frac{i\delta}{\Gamma(3/2-\alpha)} z^{(\alpha+1)/2}(1-z)^{(1-\alpha)/2} \\
& \cdot {}_2F_1\left(\begin{matrix}1-\delta, 1+\delta\\ 3/2-\alpha\end{matrix}; 1-z\right)[1, -1], \\
[u_1^{\mathrm{III}}, u_2^{\mathrm{III}}] = {} & \frac{z^{(1-\alpha)/2}(1-z)^{(\alpha+1)/2}}{\Gamma(3/2-\alpha)} {}_2F_1\left(\begin{matrix}1-\delta, 1+\delta\\ 3/2-\alpha\end{matrix}; z\right)[1, 1] \\
& + \frac{i}{\delta\Gamma(1/2-\alpha)} z^{-\alpha/2}(1-z)^{\alpha/2} {}_2F_1\left(\begin{matrix}\delta, -\delta\\ 1/2-\alpha\end{matrix}; z\right)[1, -1], \\
[u_1^{\mathrm{IV}}, u_2^{\mathrm{IV}}] = {} & \frac{z^{(1-\alpha)/2}(1-z)^{(\alpha+1)/2}}{\Gamma(\alpha+3/2)} {}_2F_1\left(\begin{matrix}1-\delta, 1+\delta\\ \alpha+3/2\end{matrix}; 1-z\right)[1, 1] \\
& + \frac{i}{\delta\Gamma(\alpha+1/2)} z^{-\alpha/2}(1-z)^{\alpha/2} \\
& \cdot {}_2F_1\left(\begin{matrix}\delta, -\delta\\ \alpha+1/2\end{matrix}; 1-z\right)[-1, 1].
\end{aligned}$$

Proof. The form

$$\frac{1}{\Gamma(c)}{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{\Gamma(c+n)}\frac{z^n}{n!}$$

denotes the series which is an entire function of c. Four of Kummer's 24 solutions (see [AS], p. 563) have the argument z and another four have argument $1-z$. By the identity

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right)=(1-z)^{c-a-b}{}_2F_1\left(\begin{matrix}c-a, c-b\\ c\end{matrix}; z\right),$$

half of them are redundant. The above list contains the four solutions coming from the equation (4.4) and use of $v_2=\frac{1}{\delta}(1-z)\frac{d}{dz}v_1$ (and the formulas

$$\frac{d}{dz}{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right)=\frac{ab}{c}{}_2F_1\left(\begin{matrix}a+1, b+1\\ c+1\end{matrix}; z\right),$$

and

$$\frac{d}{dz}\left(z^{1-c}{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right)\right)$$
$$=(1-c)z^{-c}{}_2F_1\left(\begin{matrix}a-c+1, b-c+1\\ 1-c\end{matrix}; z\right).$$

Next, $[u_1, u_2]=\zeta^{\alpha}v_1[1, 1]+i\zeta^{\alpha+1}v_2[1, -1]$ where $\zeta=z^{1/2}(1-z)^{1/2}$ (positive values for $0<z<1$). The restrictions on α come from the conditions that the 24 solutions all be defined. □

The paths $\gamma_1(t)=\theta_0e^{\pi it}$ and $\gamma_2(t)=\pi+(\theta_0-\pi)e^{\pi it}$ $(0\le t\le 1)$, and $x_0=(\cos(\theta_0/k), \sin(\theta_0/k))$, $0<\theta_0<\pi$) generate $\pi_1(\Omega; x_0)$. Along γ_1, z^c becomes $e^{2\pi ic}z^c$; along γ_2, $(1-z)^c$ becomes $e^{2\pi ic}(1-z)^c$ (any $c\in\mathbb{C}$). Using a superscript to denote analytic continuation, we have $(0<\theta<\pi)$:

$$[u_1^{\mathrm{I}}, u_2^{\mathrm{I}}]^{\gamma_1}(-\theta)=e^{\pi i\alpha}[u_2^{\mathrm{I}}, u_1^{\mathrm{I}}](\theta);$$
$$[u_1^{\mathrm{II}}, u_2^{\mathrm{II}}]^{\gamma_2}(2\pi-\theta)=e^{-\pi i\alpha}[u_2^{\mathrm{II}}, u_1^{\mathrm{II}}](\theta);$$
$$[u_1^{\mathrm{III}}, u_2^{\mathrm{III}}]^{\gamma_1}(-\theta)=-e^{-\pi i\alpha}[u_2^{\mathrm{III}}, u_1^{\mathrm{III}}](\theta);$$
$$[u_1^{\mathrm{IV}}, u_2^{\mathrm{IV}}]^{\gamma_2}(2\pi-\theta)=-e^{\pi i\alpha}[u_2^{\mathrm{IV}}, u_1^{\mathrm{IV}}](\theta).$$

Now let $g_1^s = e^{i\delta\theta}u_1^s$, $g_2^s = e^{-i\delta\theta}u_2^s$ with $s = \mathrm{I}, \mathrm{II}, \mathrm{III}$, or IV. Recall that the simple reflections are σ_0 and $\sigma_{2(k-1)}$; and

$$\tau_l(\sigma_0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_l(\sigma_{2(k-1)}) = \begin{bmatrix} 0 & -e^{-2\pi i\delta} \\ -e^{2\pi i\delta} & 0 \end{bmatrix}$$

(since $e^{-2(k-1)li\xi} = e^{2li\pi/k} = e^{i\pi(1-2\delta)} = -e^{-2\pi i\delta}$).

4.4. **Proposition.** *The monodromy actions are*

$$M(\gamma_1; \alpha)g^{\mathrm{I}} = e^{\pi i\alpha}g^{\mathrm{I}}, \qquad M(\gamma_2; \alpha)g^{\mathrm{II}} = -e^{-\pi i\alpha}g^{\mathrm{II}},$$
$$M(\gamma_1; \alpha)g^{\mathrm{III}} = e^{-\pi i\alpha}g^{\mathrm{III}}, \qquad M(\gamma_2; \alpha)g^{\mathrm{IV}} = e^{\pi i\alpha}g^{\mathrm{IV}}.$$

Proof. The factor $e^{i\delta\theta}$ becomes $e^{-i\delta\theta}$ along γ_1, and $[g_2, g_1] = [g_1, g_2]\tau(\sigma_0)$. This proves the $M(\gamma_1; \alpha)$ action. Along γ_2 we get

$$\begin{aligned}[g_1^{\mathrm{II}}, g_2^{\mathrm{II}}]^{\gamma_2}(2\pi - \theta) &= e^{-\pi i\alpha}[e^{i\delta(2\pi-\theta)}u_2^{\mathrm{II}}(\theta), e^{-i\delta(2\pi-\theta)}u_1^{\mathrm{II}}(\theta)] \\ &= -e^{-\pi i\alpha}[g_1^{\mathrm{II}}, g_2^{\mathrm{II}}](\theta)\tau(\sigma_{2(k-1)}).\end{aligned}$$

A similar calculation applies to g^{IV}. □

In order to obtain a fundamental solution which is entire in α we choose g^{I}, g^{II}. From the known connection coefficients ([**AS**, p. 559]) and the identity $\Gamma(c)\Gamma(1-c) = \pi/\sin(\pi c)$ we obtain the following:

4.5. **Proposition.**

$$g^{\mathrm{III}} = \frac{\Gamma(\delta)\Gamma(-\delta)}{\Gamma(1/2-\alpha+\delta)\Gamma(1/2-\alpha-\delta)}g^{\mathrm{I}} + \frac{\cos\pi\alpha}{\pi}\Gamma(\delta)\Gamma(-\delta)g^{\mathrm{II}},$$
$$g^{\mathrm{IV}} = -\Gamma(\delta)\Gamma(-\delta)\frac{\cos\pi\alpha}{\pi}g^{\mathrm{I}} - \frac{\Gamma(\delta)\Gamma(-\delta)}{\Gamma(\alpha+1/2+\delta)\Gamma(\alpha+1/2-\delta)}g^{\mathrm{II}}.$$

The coefficients are entire functions of α, *and* $0 < \delta < 1/2$.

The solutions of (4.1) *when multiplied by* $1/h(x)$ *become elements of* $\mathscr{D}_{\alpha,\tau}$. *In this case*

$$\begin{aligned}h &= \Pi_{j=0}^{k-1}\sin((\theta + j\pi)/k)^{\alpha} \\ &= (2^{1-k}\sin\theta)^{\alpha} = 2^{(2-k)\alpha}z^{\alpha/2}(1-z)^{\alpha/2} \qquad (0 < \theta < \pi).\end{aligned}$$

4.6. THEOREM. *For any* $\alpha \in \mathbb{C}$, $G = I_2(k)$, *a fundamental solution for* (2.3) *is*

$$\begin{aligned} f^{\mathrm{I}} = (\cos\theta/2)^{-2\alpha} \Bigg(& \frac{1}{\Gamma(\alpha+1/2)} {}_2F_1\left(\begin{matrix}\delta, -\delta \\ \alpha+1/2\end{matrix}; \sin^2\theta/2\right)[e^{i\delta\theta}, e^{-i\delta\theta}] \\ & - \frac{i}{2}\frac{\delta}{\Gamma(\alpha+3/2)}(\sin\theta) \\ & \cdot {}_2F_1\left(\begin{matrix}1+\delta, 1-\delta \\ \alpha+3/2\end{matrix}; \sin^2\theta/2\right)[e^{i\delta\theta}, -e^{-i\delta\theta}]\Bigg), \end{aligned}$$

$$\begin{aligned} f^{\mathrm{II}} = (\cos\theta/2)^{-2\alpha} \Bigg(& \frac{1}{\Gamma(1/2-\alpha)} {}_2F_1\left(\begin{matrix}\delta, -\delta \\ 1/2-\alpha\end{matrix}; \cos^2\theta/2\right)[e^{i\delta\theta}, e^{-i\delta\theta}] \\ & - \frac{i}{2}\frac{\delta}{\Gamma(3/2-\alpha)}(\sin\theta) \\ & \cdot {}_2F_1\left(\begin{matrix}1+\delta, 1-\delta \\ 3/2-\alpha\end{matrix}; \cos^2\theta/2\right)[e^{i\delta\theta}, -e^{-i\delta\theta}]\Bigg). \end{aligned}$$

In this basis

$$M(\gamma_1; \alpha) = \begin{bmatrix} 1 & -\dfrac{2\pi e^{-\pi i\alpha}}{\Gamma(1/2+\delta-\alpha)\Gamma(1/2-\delta-\alpha)} \\ 0 & -e^{-2\pi i\alpha} \end{bmatrix}$$

and

$$M(\gamma_2; \alpha) = \begin{bmatrix} 1 & 0 \\ \dfrac{2\pi e^{-\pi i\alpha}}{\Gamma(1/2+\delta+\alpha)\Gamma(1/2-\delta+\alpha)} & -e^{-2\pi i\alpha} \end{bmatrix}.$$

Proof. When $\alpha+1/2 \notin \mathbb{Z}$, the distinct eigenvectors of $M(\gamma_1; \alpha)$ are $(\sin\theta)^{-\alpha} g^{\mathrm{I}}$ and $(\sin\theta)^{-\alpha} g^{\mathrm{III}}$ (with eigenvalues 1, $e^{-2\pi i\alpha}$). The expansion coefficients in Proposition 4.5 lead to the matrix for $M(\gamma_1; \alpha)$ in the $(f^{\mathrm{I}}, f^{\mathrm{II}})$-basis. There is a cancellation $(1+e^{-2\pi i\alpha})/\cos\pi\alpha$ in the calculation. Since this fundamental solution is entire in α, the matrix is valid for all α. A similar proof applies to $M(\gamma_2; \alpha)$. □

4.7. COROLLARY. *The algebra generated by* $M(\gamma_1; \alpha)$ *and* $M(\gamma_2; \alpha)$ *is semisimple when* $q \neq e^{\pm 2\pi i l/k}$, *where* $q = e^{-2\pi i\alpha}$.

Proof. When the condition on q is satisfied, I, $M(\gamma_1; \alpha)$, $M(\gamma_2; \alpha)$, and $M(\gamma_1; \alpha)M(\gamma_2; \alpha)$ are linearly independent (by an elementary argument using the fact $\sin\pi\delta \neq 0$). If $\alpha \pm \delta = 1/2, 3/2, 5/2, \dots$, then $M(\gamma_1; \alpha)$ is diagonal and $\mathbb{C}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ is an invariant subspace; if $\alpha \pm \delta = -1/2, -3/2, -5/2$, then $M(\gamma_2; \alpha)$ is diagonal and $\mathbb{C}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ is invariant. □

This is an illustration of Yamane's result; see Corollary 3.2.

By calculating

$$M(\gamma_1\,;\alpha)M(\gamma_2\,;\alpha) = \begin{bmatrix} -e^{-4\pi i\alpha} - 2e^{-2\pi i\alpha}\cos(2\pi\delta) & \dfrac{2\pi e^{-3\pi i\alpha}}{\Gamma(1/2+\delta-\alpha)\Gamma(1/2-\delta-\alpha)} \\ \dfrac{-2\pi e^{-3\pi i\alpha}}{\Gamma(1/2+\alpha+\delta)\Gamma(1/2+\alpha-\delta)} & e^{-4\pi i\alpha} \end{bmatrix}$$

we can find its eigenvalues $-qe^{\pm 2\pi i\delta}$, that is, $qe^{\pm 2\pi il/k}$. Accordingly these representations of $H(I_2(k)\,;q)$ can be identified with those constructed by **[CIK**, pp. 103–104]. However, the present matrices involve not only $q^{1/2}$ (i.e., $e^{-3\pi i\alpha}$) but also values of the gamma functions $\Gamma(1/2\pm\alpha\pm\delta)$. These could be avoided by an appropriate change of scale $f^{\mathrm{II}} \to cf^{\mathrm{II}}$ but only for $q \neq e^{\pm 2\pi il/k}$, thus losing the entire dependence on α.

Up to now x_0 has been arbitrary point in $\mathscr{C}$. A likely candidate for a distinguished point would be an extreme value of $h(x)$ on the unit sphere (for $\alpha_i > 0$, studied as "peak points" in **[Du2]**). As support for this idea, consider the peak point for $I_2(k)$ $(\beta = 0)$, namely $x_1 = \cos(\pi/2k)$, $x_2 = \sin(\pi/2k)$, $\theta = \pi/2$, $z = 1/2$. Indeed, by use of Kummer's sum for ${}_2F_1\left({a,\,1-a \atop c}\,;\;1/2\right)$ and contiguity relations (**[AS**, pp. 556–558]) we can evaluate the fundamental solution $(f^{\mathrm{I}}, f^{\mathrm{II}})$ at $\theta = \pi/2$. Define the entire function $s(c, a) := (\Gamma(\frac{a+c}{2})\Gamma(\frac{1+c-a}{2}))^{-1}$. Then

$$\begin{aligned} f^{\mathrm{I}}(\pi/2) &= \sqrt{\pi/2}(s(\alpha+1/2,\delta)[\eta(1-i),\eta^{-1}(1+i)] \\ &\quad + s(\alpha+1/2,-\delta)[\eta(1+i),\eta^{-1}(1-i)]), \\ f^{\mathrm{II}}(\pi/2) &= 2^{2\alpha}\sqrt{\pi/2}(s(-\alpha+1/2,\delta)[\eta(1+i),\eta^{-1}(1-i)] \\ &\quad + s(-\alpha+1/2,-\delta)[\eta(1-i),\eta^{-1}(1+i)]), \end{aligned}$$

where $\eta := e^{i\pi\delta/2}$. The determinant of $\left[{f^{\mathrm{I}}(\pi/2) \atop f^{\mathrm{II}}(\pi/2)}\right]$ equals $-\frac{i}{\pi}2^{2\alpha+1}\sin\pi\delta$ (using $s(c, a)s(1-c, a) = (1/2\pi^2)(\sin(\pi a)+\sin(\pi c))$.

5. Further questions. When two multiplicity functions α and α' differ by integers ($\alpha_i - \alpha_i' \in \mathbb{Z}$, all i) and the specialization $H(G, q)$ is semisimple, then $M(\gamma_w\,;\alpha)$ and $M(\gamma_w\,;\alpha')$ are equivalent representations. Thus there exists a nonsingular intertwining matrix $A(\alpha, \alpha')$ such that $A(\alpha, \alpha')M(\gamma_w\,;\alpha) = M(\gamma_w\,;\alpha')A(\alpha, \alpha')$ $(w \in G)$, where $M(\gamma_w\,;\alpha)$, $M(\gamma_w\,;\alpha')$ are expressed with respect to fundamental solutions whose values at x_0 are independent of α. More specific information about $A(\alpha, \alpha')$ would be very desirable. This may be related to recurrence relations for some associated (as yet unspecified) definite integrals.

Can one find explicit bases for $\mathscr{D}_{\alpha,\tau}$ when $G = I_2(2k)$ and there are two parameters in terms of classical special functions? The differential equation (4.3) has six regular singular points but it may be possible to reduce this number by a transformation.

Are there useful distinguished points in $\mathscr{C}$ in the sense of allowing explicit evaluations of fundamental solutions? The obvious candidate is the peak point x_0 where

$$\sum_{j=1}^{m} \frac{\alpha_j}{\langle x_0, v_j\rangle} v_j = \left(\sum_{j=1}^{m} \alpha_j\right) \frac{x_0}{|x_0|^2}$$

(see **[Du2]**).

When $\alpha_i \geq 0$ for all i and $|h(x)|^2$ is an interesting weight function for orthogonal polynomials on the sphere it may be that there are interesting definite integrals associated to $\mathscr{D}_{\alpha,\tau}$, for example $|f(x)|^2|h(x)|^2$ ($f \in \mathscr{D}_{\alpha,\tau}$) with respect to surface measure on $\mathscr{C} \cap \{x \in \mathbb{R}^N : |x| = 1\}$.

Kazhdan and Lusztig (see **[KL]**, also **[Hu**, Ch. 7]) constructed a different basis for $H(G; q)$ for the purpose of a more detailed representation theory. Do the elements of their basis have an informative geometric meaning under the monodromy representation?

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