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SOLUTIONS OF THE STATIONARY AND NONSTATIONARY NAVIER-STOKES EQUATIONS IN EXTERIOR DOMAINS

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It is shown that a nonstationary exterior Navier-Stokes flow tends to a small stationary flow in L^2 like $t^{-3/4}$ as $t \to \infty$.

0. Introduction. In this paper we are concerned with the stationary Navier-Stokes equations

$$(0.1)\qquad \begin{aligned} &(w \cdot D)w - \Delta w + D\overline{p} = f, \quad D \cdot w = 0 \quad \text{in } G, \\ &w = 0 \quad \text{on } \partial G \quad (D = \text{grad}), \end{aligned}$$

and the nonstationary Navier-Stokes equations

$$\begin{aligned} v_t + (v \cdot D)v - \Delta v + D\overline{\overline{p}} &= f \quad \text{in } G \times (0, \infty), \\ D \cdot v &= 0 \quad \text{in } G \times (0, \infty), \\ v &= 0 \quad \text{on } \partial G \times (0, \infty), \\ v|_{t=0} &= a + w \quad \text{in } G \quad (v_t = \partial v / \partial t). \end{aligned}$$

Here and in what follows G denotes a smooth exterior domain of R^3, $f = f(x)$ is a prescribed vector field, and $\overline{p}$ (resp. $\overline{\overline{p}}$) represents unknown stationary (resp. nonstationary) scalar pressure which can be determined by the stationary solution w via (0.1) (resp. nonstationary solution v via (0.2)).

As is well known, it was shown by Finn [**8**, **9**] that (0.1) admits a small solution

$$(0.3)\qquad \begin{aligned} &w \in L^\infty(G; R^3), \qquad Dw \in L^3(G; R^9), \\ &C_0 = \sup_{x \in G} |x|\,|w(x)| < \infty. \end{aligned}$$

If $C_0 < 1/2$ the Finn's solution w may be formed as a limit of a nonstationary solution v as $t \to \infty$ in local or global L^2-norms (cf. Heywood [**15**, **14**], Galdi and Rionero [**11**], Miyakawa and Sohr [**23**]) and in other norms (cf. Heywood [**16**], Masuda [**20**]). Moreover it has recently proved (cf. Borchers and Miyakawa [**4**]) that every weak solution of (0.2) tends the Finn's solution in $L^2(G; R^3)$

like $t^{-(3/p-3/2)/2}$ with $6/5 < p < 2$, provided $C_0 < 1/2$ and $a \in L^2(G; R^3) \cap L^p(G; R^3)$.

In this paper we are only interested in the case $w \in L^3(G; R^3)$, $Dw \in L^{3/2}(G; R^9)$, or $Dw \in L^r(G; R^9) \cap L^p(G; R^9)$ with $1 < r < 3/2 < p < 2$. Under certain smallness assumptions on w we show now that every weak solution of (0.2) tends to the stationary solution w in $L^2(G; R^3)$ like the sharp decay rate $t^{-3/4}$.

1. Notation and main result. In this paper we use the following spaces.

L^p = the Lebesgue spaces $L^p(G; R^3)$, with $\|\cdot\|_p$ the associated norm,

C_σ^∞ = the set of compactly supported solenoidal in $C^\infty(G; R^3)$,

$W^{k,p}$ = the Sobolev space $W^{k,p}(G; R^3)$,

J^p = the completion of C_σ^∞ in L^p,

$W_\sigma^{1,p}$ = the completion of C_σ^∞ in $W^{1,p}$,

$\widehat{W}_\sigma^{1,p}$ = the completion of C_σ^∞ under the norm $\|D\cdot\|_p$,

$\widehat{W}_\sigma^{2,p}$ = the space $\{u \in \widehat{W}_\sigma^{1,3p/(3-p)};\ D^2u \in L^p(G; R^{27})\}$ for $1 < p < 3$,

$W^{-1,2}$ = the dual of $W_\sigma^{1,2}$,

$\widehat{W}^{-1,p}$ = the dual of $\widehat{W}_\sigma^{1,p/(p-1)}$, with $\|\cdot\|_{-1,p}$ the associated norm.

Moreover for $1 < r < \infty$ and $n \geq 1$, we denote by r' the real $r/(r-1)$, by $(\cdot,\cdot)$ the inner product in $L^2(G; R^n)$, by P the bounded projection from L^r onto J^r (cf. [22]), by A the Stokes operators $-P\Delta$ with the domain $W_\sigma^{1,r} \cap W^{2,r}$, by $\overline{A}$ the Laplacian $-\Delta$ with the domain $W^{2,r}(R^3; R^3)$, and by C a positive constant which may vary from line to line, but is always independent of the quantities t, T, u, v, w, f, u_k, and a.

Now we make preparations for stating our main result. The existence of the stationary solutions w is guaranteed by the following.

LEMMA 1.1. *Let $1 < r \leq 3/2 < p < 2$, and $f \in C_\sigma^\infty$. Then there is a small $h > 0$ such that* (0.1) *admits a unique solution within the class*

$$\{w \in \widehat{W}_\sigma^{1,r} \cap \widehat{W}_\sigma^{1,p};\ \|Dw\|_{3/2} \leq h\},$$

provided that $\|f\|_{-1,3/2} \leq h^2$. Moreover

$$\|Dw\|_r + \|Dw\|_p \leq C(\|f\|_{-1,r} + \|f\|_{-1,p}).$$

From (0.1) and (0.2) we see that $u = v - w$ and $\hat{p} = \overline{p} - \overline{\overline{p}}$ solve the problem

$$(1.1)\qquad \begin{aligned} &u_t + (u \cdot D)u - \Delta u + (u \cdot D)w + (w \cdot D)u + D\hat{p} = 0, \\ &D \cdot u = 0 \quad \text{in } G \times (0, \infty), \\ &u = 0 \quad \text{on } \partial G \times (0, \infty), \\ &u|_{t=0} = a \quad \text{in } G. \end{aligned}$$

Weak solutions are given in the following sense.

DEFINITION 1.1. Let $a \in J^2$, and $w \in \widehat{W}_\sigma^{1,3/2}$ solve (0.1). A weakly continuous function u: $[0, \infty) \to J^2$ is said to be a weak solution of (1.1) if $u(0) = a$, $u \in L^\infty(0, \infty; J^2) \cap L^2(0, \infty; \widehat{W}_\sigma^{1,2})$,

$$(1.2)\qquad \|u(t)\|_2^2 + \int_s^t \|Du(z)\|_2^2\, dz \le \|u(s)\|_2^2,$$

$$(1.3)\qquad \begin{aligned} (u(t), g(t)) + \int_s^t ((Du, Dg) + ((u \cdot D)w, g) \\ + ((w \cdot D)u, g) - (u, g_z))\, dz \\ = (u(s), g(s)) - \int_s^t ((u \cdot D)u, g)\, dz \end{aligned}$$

for all $t > s \ge 0$ and all $g \in C([0, \infty); W_\sigma^{1,2}) \cap C^1([0, \infty); J^2)$, where $g_z = \partial g/\partial z$.

The existence of weak solutions to (1.1) is guaranteed by the following.

LEMMA 1.2. *Let $a \in J^2$, and $w \in \widehat{W}_\sigma^{1,3/2}$ such that $\|Dw\|_{3/2} < 1/8$. Then (1.1) admits a weak solution.*

We are now in a position to state our main result.

THEOREM 1.1. *Let $1 < r < 3/2 < p < 2$, $a \in J^2 \cap L^1$, and let $w \in W_\sigma^{1,r} \cap W_\sigma^{1,p}$ such that w solves (0.1) and $\|Dw\|_r + \|Dw\|_p$ is sufficiently small. Then every weak solution of (1.1) possesses the sharp decay property*

$$\|u(t)\|_2 = O(t^{-3/4}).$$

Section 2 is concerned with the proof of Lemmas 1.1 and 1.2. In **[23]**, it has been obtained an existence result on weak solutions of (1.1) with w the Finn's solution such that $C_0 < 1/2$. However, the argument of [23] heavily depends on the property (0.3). In §3, with the use of the approach developed from [7], we shall show sharp decay estimates of solutions to the linearized equations of (1.1). If w only satisfies (0.3) and $C_0 < 1/2$, such estimates seem unavailable. Theorem 1.1 will be proved in §4 by making use of the estimates carried out in §3 and studying the time average $t^{-1}\int_0^t \|u(s)\|_2\,ds$. A similar technique has been used in [23, 4]. However, we have not used the spectral decomposition of the Stokes operator A in L^2 as usually used in earlier work concerning the L^2 decay problem. Moreover our proof seems much simpler.

It should be noted that the L^2 decay problem of (1.1) with $w = 0$ stems from Leray [19], and has affirmatively been solved (cf. [24, 3, 2] and the references therein). If $1 < p < 2$ and u is a weak solution of (1.1) with $w = 0$, it has been proved that $\|u(t)\|_2 = O(t^{-(3/p-3/2)/2})$ provided $u(0) \in J^2 \cap L^p$ (cf. [2]), and $\|u(t)\|_2 = O(t^{-3/4})$ provided $u(0) \in J^2 \cap L^1$ and $\|e^{-tA}a\|_2 \le Ct^{-3/4}\|a\|_1$ (cf. [3]).

2. Proof of Lemmas 1.1, 1.2. To begin with we give the estimate (cf. [2, Theorem 3.6] or [12, 18] for a similar consideration)

$$\|Du\|_p \le C\sup\{|(Du, Dv)|;\ v \in C_\sigma^\infty,\ \|Dv\|_{p'} = 1\} \quad \text{for } 1 < p < n,\ u \in \widehat{W}_\sigma^{1,p}, \tag{2.1}$$

and the Sobolev inequality (cf. [13])

$$\|u\|_{3p/(3-p)} \le 2p(3-p)^{-1}3^{-1/2}\|Du\|_p \quad \text{for } 1 < p < n,\ u \in \widehat{W}_\sigma^{1,p}. \tag{2.2}$$

Proof of Lemma 1.1. Let r and p be given in Lemma 1.1. We rewrite (0.1) in the abstract form $Aw + P(w\cdot D)w = f$, $w \in \widehat{W}_\sigma^{1,r} \cap \widehat{W}_\sigma^{1,p}$. Since the proof of [5, (3.1)] implies that A can be extended as a bounded and invertible operator from $\widehat{W}_\sigma^{2,q}$ onto J^q with $1 < q < 3/2$, we can set

$$H\colon \widehat{W}_\sigma^{1,r} \cap \widehat{W}_\sigma^{1,p} \to \widehat{W}_\sigma^{2,3p/(6-p)} \quad \text{such that } Hw = A^{-1}(f - P(w\cdot D)w).$$

Let $w \in \widehat{W}_\sigma^{1,r} \cap \widehat{W}_\sigma^{1,p}$, $r < s < p$, and $v \in C_\sigma^\infty$ with $\|Dv\|_{s'} = 1$. Integrating by parts and using the divergence condition $D\cdot w = 0$, we have

$$\begin{aligned}(DHw, Dv) &= (f, v) - ((w\cdot D)w, v)\\ &= (f, v) + ((w\cdot D)v, w)\\ &\le (f, v) + \|w\|_3\|w\|_{3s/(3-s)}\|Dv\|_{s'},\end{aligned}$$

that is, by (2.1)–(2.2),

$$\|DHw\|_s \le C(\|f\|_{-1,s} + \|Dw\|_s\|Dw\|_{3/2}).$$

Similarly, for w, $w^* \in W_\sigma^{1,r} \cap W_\sigma^{1,p}$ we have

$$\|DHw - DHw^*\|_s \le C(\|Dw\|_{3/2} + \|Dw^*\|_{3/2})\|Dw - Dw^*\|_s.$$

Consequently, the desired assertion follows immediately from the contraction mapping principle. The proof is complete.

In [**23**], Miyakawa and Sohr proved that (1.1) admits a weak solution in case w is the Finn's solution and $C_0 < 1/2$. However, as for our case, the argument of [**23**] does not work somewhere. Now we give our proof in a slightly different way. Similar to [**23**], we also study approximate solutions of (1.1) by applying a technique developed from [**6**].

Proof of Lemma 1.2. Let $k > 1$. We set $J_k = k(k+A)^{-1}$ and $I_k = k(k+\overline{A})^{-1}E$, where E denotes the extension operator such that $Eu = u$ in G and $Eu = 0$ outside G. With the use of the notation above, we have

$$(2.3)\qquad \|J_k u\|_p \le C(k)\|u\|_r, \quad \|I_k u\|_p \le C(k)\|u\|_r \quad \text{for } 1 < r < p \le \infty,\ u \in J^r,$$

$$(2.4)\qquad \|I_k u\|_r \le \|u\|_r, \quad \|J_k u\|_r \le C\|u\|_r \quad \text{for } 1 < r < \infty,\ u \in J^r,$$

where C is independent of k. (2.3) is a consequence of the Sobolev embedding theorem and L^r-estimates. The first inequality in (2.4) follows from the proof of [**1**, Lemma 10.1], and the second one from [**2**, Theorem 1.2].

Now we proceed to the evolution equation

$$(2.5)\qquad (d/dt)u_k + Au_k = F_k(u_k), \quad u_k(0) = J_k a \quad \text{in } J^2,$$

where $F_k(u) = F_k(u, u)$ with

$$F_k(u, v) = -P(J_k u \cdot D)v - P(J_k w \cdot D)u - P(I_k u \cdot D)I_k w.$$

For $u, v \in W_\sigma^{1,2}$, we have

$$\begin{aligned}(2.6)\qquad &\|F_k(u, v)\|_2 + \|P(J_k v \cdot D)u\|_2 \\ &\le \|J_k u\|_\infty\|Dv\|_2 + \|J_k w\|_\infty\|Du\|_2 \\ &\quad + \|I_k u\|_6\|DI_k w\|_3 + \|J_k v\|_\infty\|Du\|_2 \\ &\le C(k)(\|u\|_6\|Dv\|_2 + \|w\|_3\|Du\|_2 \\ &\quad + \|u\|_6\|I_k DEw\|_3 + \|v\|_6\|Du\|_2), \quad \text{by (2.3)}, \\ &\le C(k)\|Du\|_2(\|Dv\|_2 + \|Dw\|_{3/2}), \quad \text{by (2.2)}.\end{aligned}$$

On the other hand, given k and $T > 0$, we suppose that u_k solve (2.5) over $[0, T)$, and $u_k \in L^2(0, T; W_\sigma^{1,2} \cap W^{2,2}) \cap W^{1,2}(0, T; J^2)$. Then multiplying (2.5) by $2u_k$ and $2Au_k$, respectively, we have

$$(d/dt)\|u_k\|_2^2 + 2\|Du_k\|_2^2 = 2(F_k(u_k), u_k),$$
$$(d/dt)\|Du_k\|_2^2 + 2\|Au_k\|_2^2 = 2(F_k(u_k), Au_k).$$

The estimation of the right-hand side terms of the preceding identities can be achieved as follows.

$$\begin{aligned}
2(F_k(u_k), u_k) &= 2((I_k u_k \cdot D)u_k, I_k w), \\
&\qquad \text{since } D \cdot J_k u_k = D \cdot J_k w = D \cdot I_k u = 0, \\
&\le 2\|I_k u_k\|_6 \|Du_k\|_2 \|I_k w\|_3 \\
&\le (12/3^{-1/2})\|w\|_3 \|Du_k\|_2^2, \quad \text{by (2.4) and (2.2)}, \\
&\le 8\|Dw\|_{3/2} \|Du_k\|_2^2, \quad \text{by (2.2)}, \\
&\le \|Du_k\|_2^2, \quad \text{by setting } \|Dw\|_{3/2} < 1/8,
\end{aligned}$$

$$\begin{aligned}
&2(F_k(u_k), Au_k) \\
&\quad \le 2\|Au_k\|_2 (\|J_k u_k\|_\infty \|Du_k\|_2 + \|J_k w\|_\infty \|Du_k\|_2 \\
&\qquad\qquad + \|I_k u\|_\infty \|I_k DEw\|_2) \\
&\quad \le C(k)\|Au_k\|_2 \|Du_k\|_2 (\|u_k\|_2 + \|Dw\|_{3/2} + \|DEw\|_{3/2}), \\
&\qquad\qquad \text{by (2.3) and (2.2)}, \\
&\quad \le C(k)\|Au_k\|_2 \|Du_k\|_2 (\|u_k\|_2 + \|Dw\|_{3/2}) \\
&\quad \le 2\|Au_k\|_2^2 + C(k)\|Du_k\|_2^2 (\|u_k\|_2^2 + \|Dw\|_{3/2}^2).
\end{aligned}$$

Consequently, we have

$$\|u_k(t)\|_2^2 + \int_s^t \|Du_k(z)\|_2^2 \, dz \le \|u_k(s)\|_2^2, \qquad 0 \le s < t < T, \tag{2.7}$$

$$\begin{aligned}
&\|Du_k(t)\|_2^2 \\
&\quad \le \|DJ_k a\|_2^2 + C(k) \int_0^t \|Du_k(s)\|_2^2 (\|u_k(s)\|_2^2 + \|Dw\|_{3/2}^2)\, ds \\
&\quad \le \|DJ_k a\|_2^2 + C(k)\|J_k a\|_2^2 (\|J_k a\|_2^2 + \|Dw\|_{3/2}^2), \quad \text{by (2.7)}
\end{aligned} \tag{2.8}$$

Thus, following the same way as in the proof of [**23**, Proposition 3.4] by making use of (2.6)–(2.8), we conclude that (2.5) admits a unique global solution u_k satisfying (2.6), and $u_k \in L^2(0, T; W_\sigma^{1,2} \cap W^{2,2}) \cap W^{1,2}(0, T; J^2)$ for all $T > 0$.

To obtain a weak solution of (1.1), we need to study compactness of the sequence u_k. Let $v \in W_\sigma^{1,2}$. Applying (2.2) and (2.4) repeatedly, we have, from (2.5),

$$\begin{aligned}
&((d/dt)u_k, v) \\
&\quad \le \|Du_k\|_2\|Dv\|_2 + \|J_k u_k\|_3\|Du_k\|_2\|v\|_6 + \|J_k w\|_3\|Du_k\|_2\|v\|_6 \\
&\qquad + \|I_k u_k\|_6\|DI_k w\|_{3/2}\|v\|_6 \\
&\quad \le \|Du_k\|_2\|Dv\|_2 + C\|v\|_6(\|u_k\|_3\|Du_k\|_2 + \|w\|_3\|Du_k\|_2 \\
&\qquad\qquad + \|u_k\|_6\|DEw\|_{3/2}) \\
&\quad \le C\|Dv\|_2(\|Du_k\|_2 + \|u_k\|_2^{1/2}\|Du_k\|_2^{3/2} + \|Du_k\|_2\|Dw\|_{3/2}) \\
&\quad \le C\|Dv\|_2(1 + \|a\|_2^{1/2} + \|Dw\|_{3/2})(\|Du_k\|_2 + \|Du_k\|_2^{3/2}), \\
&\qquad\qquad \text{by (2.7) and (2.4)},
\end{aligned}$$

with C independent of k. This together with (2.7) implies that the sequence u_k is bounded in

$$L^\infty(0, \infty; J^2) \cap L^2(0, \infty; \widehat{W}_\sigma^{1,2}) \cap W^{1,4/3}(0, T; W^{-1,2})$$

for all $0 < T < \infty$. From [**26**, Theorem 2.1 in Chapter III] it follows readily that there are a function u and a subsequence of u_k, denoted again u_k, satisfying

$$\begin{aligned}
&u_k \xrightarrow{w^*} u \text{ in } L^\infty(0, \infty; J^2), \\
&u_k \xrightarrow{w} u \text{ in } L^2(0, \infty; \widehat{W}_\sigma^{1,2}), \\
&u_k \to u \text{ strongly in } L^2_{\mathrm{loc}}(G \times (0, \infty)).
\end{aligned}$$

As in [**21**], we can check that the limit u is a weak solution of (1.1). The proof is complete.

3. Decay estimates. In this section, we let $t > 0$, $1 < r < 3/2 < p < 2$, and w be a solution of (0.1) such that $w \in \widehat{W}_\sigma^{1,r} \cap \widehat{W}_\sigma^{1,p}$, and set

$$\begin{aligned}
&Lu = Au + P(u \cdot D)w + P(w \cdot D)u, \\
&B^*u = -p(w \cdot D)u + P\sum_{i=1}^{n} u^i Dw^i, \\
&L^*u = Au + B^*u.
\end{aligned}$$

Thus, we see that

$$(Lu, v) = (u, L^*v) \quad \text{for } u, v \in W_\sigma^{1,2} \cap W^{2,2},$$

and the linearized equation of (1.1) can be stated in the form

$$(d/dt)v + Lv = 0, \quad v(0) = u.$$

Denote by $e^{-tL}u$ the solution of the preceding equation. It is the purpose of this section to prove the following.

PROPOSITION 3.1. *Suppose that* $\|Dw\|_r + \|Dw\|_p$ *is sufficiently small. Then there holds*

$$\|e^{-tL}Pu\|_2 \le Ct^{-3/4}\|u\|_1 \tag{3.1}$$

for $u \in L^1 \cap L^{6/5}$.

The preceding proposition is based on the following decay estimates.

$$\|e^{-tA}u\|_\infty \le Ct^{-1/4}\|u\|_6 \quad \text{for } u \in J^6, \tag{3.2}$$

$$\|e^{-tA}u\|_s \le Ct^{-(3/q-3/s)/2}\|u\|_q \quad \text{for } 1 < q \le s < \infty,\ u \in J^q, \tag{3.3}$$

$$\|De^{-tA}u\|_s \le Ct^{-(1+3/q-3/s)/2}\|u\|_q \quad \text{for } 1 < q \le s \le 3,\ u \in J^q. \tag{3.4}$$

The estimates (3.3) and (3.4) were recently obtained by Iwashita (cf. [**17**, Theorems 1.2, 1.3]). (3.2) will be proved in the Appendix by using the argument of [**17**].

With the use of (3.2)–(3.4), we can now prove the following.

LEMMA 3.1. *Let* $u \in C_\sigma^\infty$. *Then there hold*

$$\|e^{-tA}u\|_\infty \le Ct^{-3/4}\|u\|_2, \tag{3.5}$$

$$\begin{aligned} &\|e^{-tA}B^*u\|_\infty + \|De^{-tA}B^*u\|_3 \\ &\quad \le Ct^{-3/2p}(t+1)^{-(3/r-3/p)/2}(\|u\|_\infty + \|Du\|_3)(\|Dw\|_r + \|Dw\|_p). \end{aligned} \tag{3.6}$$

Proof. From (3.2), (3.3), (2.2) and the semigroup property of e^{-tA} we get (3.5) and

$$\begin{aligned} \|e^{-tA}B^*u\|_\infty &\le Ct^{-3/2b}\|B^*u\|_b \\ &\le Ct^{-3/2b}\|Dw\|_b(\|u\|_\infty + \|Du\|_3) \end{aligned}$$

for $b = r$, p. Moreover (3.4) and (2.2) yield

$$\|De^{-tA}B^*u\|_3 \le Ct^{-3/2b}\|Dw\|_b(\|u\|_\infty + \|Du\|_3) \quad \text{for } b = r, p.$$

Collecting terms, we get readily (3.6) and complete the proof.

Proof of Proposition 3.1. Setting $v(t) = e^{-tL^*}u$ with $u \in C_\sigma^\infty$, we have obviously that $v \in C([0, \infty); L^\infty \cap W_\sigma^{1,3})$ and

$$v(t) = e^{-tA}u + \int_0^t e^{-(t-s)A}B^*v(s)\,ds.$$

This gives, by (3.4)–(3.6),

$$\begin{aligned}\|v(t)\|_\infty &+ \|Dv(t)\|_3 \\ &\le Ct^{-3/4}\|u\|_2 + C\int_0^t (t-s)^{-3/2p}(t-s+1)^{-(3/r-3/p)/2} \\ &\quad \times (\|v\|_\infty + \|Dv\|_3)\, ds(\|Dw\|_r + \|Dw\|_p).\end{aligned}$$

Setting $|||v|||_t = \sup_{0<s<t} s^{3/4}(\|v(s)\|_\infty + \|Dv(s)\|_3)$, we have

$$\begin{aligned}\|v(t)\|_\infty &+ \|Dv(t)\|_3 \\ &\le Ct^{-3/4}\|u\|_2 + C(\|Dw\|_r + \|Dw\|_p)|||v|||_t \\ &\quad \times \int_0^t (t-s)^{-3/2p}(t-s+1)^{-(3/r-3/p)/2}s^{-3/4}\, ds \\ &\le Ct^{-3/4}\|u\|_2 + Ct^{-3/4}(\|Dw\|_r + \|Dw\|_p)|||v|||_t \\ &\quad \times \int_0^t s^{-3/2p}(s+1)^{-(3/r-3/p)/2}\, ds \\ &\quad + Ct^{1/4-3/2p}(t+1)^{-(3/r-3/p)/2}(\|Dw\|_r + \|Dw\|_p)|||v|||_t \\ &\le Ct^{-3/4}(\|u\|_2 + (\|Dw\|_r + \|Dw\|_p)|||v|||_t),\end{aligned}$$

where we have used the condition $r < 3/2 < p$. Hence, if we presuppose that

$$C(\|Dw\|_r + \|Dw\|_p) < 1/2 \tag{3.7}$$

with the constant C given in the last term above, we obtain

$$\|e^{-tL^*}u\|_\infty \le Ct^{-3/4}\|u\|_2. \tag{3.8}$$

Now we take $u \in L^1 \cap L^{6/5}$ and $v \in L^2$. By (3.8) we have

$$(e^{-tL}Pu, v) = (u, e^{-tL^*}Pv) \le \|u\|_1\|e^{-tL^*}Pv\|_\infty \le Ct^{-3/4}\|u\|_1\|v\|_2$$

and therefore the validity of (3.1). The proof is complete.

4. Proof of Theorem 1.1. In this section we always suppose that the stationary solution $w \in \widehat{W}_\sigma^{1,r} \cap \widehat{W}_\sigma^{1,p}$ with $1 < r < 3/2 < p < 2$ such that (3.7) holds. Let u be a weak solution of (1.1). Then (1.2) implies

$$\|u(t)\|_2 \le t^{-1}\int_0^t \|u(s)\|_2\, ds. \tag{4.1}$$

On the other hand, taking $v \in C_\sigma^\infty$ and applying (1.3) with $g(z) = e^{-(t-z)L^*}v$, we have

$$\begin{aligned}(u(t), v) &+ \int_0^t (Lu(s), e^{-(t-s)L^*}v)\, ds - \int_0^t (u(s), L^*e^{-(t-s)L^*}v)\, ds \\ &= (a, e^{-tL^*}v) - \int_0^t ((u\cdot D)u, e^{-(t-s)L^*}v)\, ds,\end{aligned}$$

that is,

$$(u(t), v) = (e^{-tL}a, v) - \int_0^t (e^{-(t-s)L}P(u\cdot D)u(s), v)\,ds$$
$$\leq \|e^{-tL}a\|_2\|v\|_2 + \int_0^t \|e^{-(t-s)L}P(u\cdot D)u(s)\|_2\,ds\|v\|_2$$
$$\leq C\|v\|_2\left(t^{-3/4}\|a\|_1 + \int_0^t (t-s)^{-3/4}\|u(s)\|_2\|Du(s)\|_2\,ds\right),$$

where we have used (3.1). We then get

$$\|u(s)\|_2 \leq Cs^{-3/4}\|a\|_1 + C\int_0^s (s-z)^{-3/4}\|u(z)\|_2\|Du(z)\|_2\,dz.$$

Integrating the above inequality from 0 to t, we have

$$\int_0^t \|u(s)\|_2\,ds \leq Ct^{1/4}\|a\|_1 + C\int_0^t dz\int_z^t (s-z)^{-3/4}\|u(z)\|_2\|Du(z)\|_2\,ds$$
$$\leq Ct^{1/4}\|a\|_1 + Ct^{1/4}\int_0^t \|u(s)\|_2\|Du(s)\|_2\,ds$$
$$\leq Ct^{1/4}\|a\|_1 + Ct^{1/4}\|a\|_2\left(\int_0^t \|u(s)\|_2^2\,ds\right)^{1/2}, \quad \text{by (1.2).}$$

Combining this with (4.1), we have

$$\|u(t)\|_2 \leq Ct^{-3/4}\|a\|_1 + Ct^{-3/4}\|a\|_2\left(\int_0^t \|u(s)\|_2^2\,ds\right)^{1/2},$$

that is,

$$\|u(t)\|_2 \leq C_1 t^{-3/4}\left(1 + \left(\int_0^t \|u(s)\|_2^2\,ds\right)^{1/2}\right), \tag{4.2}$$

where and in what follows $C_1 = C_1(\|a\|_1, \|a\|_2)$ may vary from line to line.

Now we apply (4.2) and (1.2) to complete our proof via a boot strap iteration argument.

Note that

$$\|u(t)\|_2 \leq C_1, \quad \text{by (1.2)}, \tag{4.3}$$

and

$$\|u(t)\|_2 \leq C_1 t^{-3/4}(1 + t^{1/2}), \quad \text{by (4.2) and (4.3).} \tag{4.4}$$

Combining (4.4) with (4.3), we have

$$\|u(t)\|_2 \leq C_1 t^{-1/4}. \tag{4.5}$$

Moreover, taking (4.2) and (4.5) into account, we have

$$\|u(t)\|_2 \le C_1 t^{-3/4}(1 + t^{1/4}).$$

This together with (4.3) implies

$$(4.6) \qquad \|u(t)\|_2 \le C_1 (t+1)^{-1/2}.$$

Similarly, (4.2) and (4.6) yield

$$\|u(t)\|_2 \le C_1 t^{-3/4}(1 + \ln(t+1)),$$

and so, by (4.3),

$$(4.7) \qquad \|u(t)\|_2 \le C_1 (t+1)^{-2/3}.$$

Finally, by (4.2) and (4.7), we arrive at the desired estimate

$$\|u(t)\|_2 \le C_1 t^{-3/4}$$

and complete the proof.

REMARK 4.1. It should be noted that the validity of the assumption of Lemma 1.2 follows from the inequality $\|Dw\|_{3/2} \le \|Dw\|_r + \|Dw\|_p$ and (2.7).

Appendix: Proof of (3.2). Let Q be a domain of R^3. By $\|\cdot\|_{k,p,Q}$ and $\|\cdot\|_{p,Q}$ we denote respectively the norms of the Sobolev space $W^{k,p}(Q;R^3)$ and the Lebesgue space $L^p(Q;R^3)$. Of course, $\|\cdot\|_{k,p} = \|\cdot\|_{k,p,G}$ and $\|\cdot\|_p = \|\cdot\|_{p,G}$. $\overline{P}$ is the bounded projection from $L^p(R^3;R^3)$ onto $J^p(R^3;R^3)$, where $J^p(R^3;R^3)$ denotes the completion of the set of compactly supported solenoidal in $C^\infty(R^3;R^3)$. Let h be a constant such that $|x| < h-1$ for $x \in \partial G$, and let $g \in C^\infty(R^3;R)$ be a fixed function such that $g = 1$ for $|x| > h$ and $g = 0$ for $|x| < h-1$. Moreover we set $G_h = \{x \in G;\, |x| < h\}$.

In arriving at (3.2), we need the following lemmas.

LEMMA A.1. *Let* $1 < p \le q < \infty$, $t > 0$, $v \in L^p(R^3;R^3) \cap L^q(R^3;R^3)$, $n > 1$, *and* $u \in J^6$. *Then we have*

$$(\text{A.1}) \quad \|e^{-t\overline{A}}v\|_{\infty,R^3} \le Ct^{-3/2q}(t+1)^{-(3/p-3/q)/2}(\|v\|_{p,R^3} + \|v\|_{q,R^3}),$$

$$(\text{A.3}) \qquad \|e^{-tA}u\|_{2n,6} \le C(t^{-n}+1)\|u\|_6.$$

(A.1) *is deduced immediately by an elementary calculation.* (A.2) *is a consequence of* L^p*-estimates (cf.* **[25]**) *and the Sobolev embedding theorem. One can also refer to* **[17]** *for details.*

LEMMA A.2 ([**17**, *Lemmas* 5.3, 5.4] *and* (A.2)). *Let* $t > 0$, $v \in J^6$, *and* P^* *be a certain pressure such that* $p^* = Ae^{-(t+1)A}v + \Delta e^{-(t+1)A}v$. *Then*

$$\|e^{-(t+1)A}v\|_{2,6,G_h} + \|Ae^{-(t+1)A}v\|_{2,6,G_h} + \|p^*(t)\|_{3,6,G_h} \le Ct^{-1/4}\|v\|_6.$$

LEMMA A.3 ([**17**, (5.18)] *and* (A.2)). *Let* $v \in J^6$, *and* $t > 0$. *Then there is a function* v^* *such that*

$$\begin{gathered} D \cdot v^* = D \cdot (ge^{-(t+1)A}v), \\ \operatorname{supp} v^*(t) \subset \{x \in R^3 ;\, h-1 < |x| < h\}, \\ \|v^*(t)\|_{2,6} + \|(\partial/\partial t)v^*(t)\|_6 \le C(t+1)^{-1/4}\|v\|_6. \end{gathered}$$

LEMMA A.4. *Let* $t > 0$, v *and* v^* *be given in Lemma* A.3. *Then we have*

$$\|ge^{-(t+1)A}v - v^*(t)\|_\infty \le C(t+1)^{-1/4}\|v\|_6.$$

Proof. Set $u(t) = ge^{-(t+1)A}v - v^*(t)$, $u_0 = u(0)$, and

$$\begin{aligned} F(t) = p^*(t)Dg - 2(Dg \cdot D)e^{-(t+1)A}v - (\Delta g)e^{-(t+1)A}v \\ + \Delta v^*(t) - (\partial/\partial t)v^*(t), \end{aligned}$$

where p^* is given in Lemma A.2. By Lemmas A.2, A.3 we have that the support of $F(t)$ is contained in $\{x \in R^3 ;\, h-1 < |x| < h\}$, and

$$\begin{gathered} (t+1)^{1/4}\|F(t)\|_6 + \|u_0\|_{1,6} \le C\|v\|_6, \\ u_t - \Delta u + D(gp^*) = F, \quad D \cdot u = 0 \text{ in } R^3 \times (0,\infty). \end{gathered} \tag{A.3}$$

We thus rewrite u in the integral form

$$u(t) = e^{-t\overline{A}}u_0 + \int_0^t e^{-(t-s)\overline{A}}\overline{P}F(s)\,ds. \tag{A.4}$$

From (A.1), (A.3), and Sobolev's embedding theorem it follows that

$$\|e^{-t\overline{A}}u_0\|_{\infty,R^3} \le C(t+1)^{-1/4}(\|u_0\|_\infty + \|u_0\|_6) \le Ct^{-1/4}\|v\|_6,$$

and

$$\begin{aligned}
&\left\| \int_0^t e^{-(t-s)\overline{A}} \overline{P} F(s)\, ds \right\|_{\infty, R^3} \\
&\leq C \int_0^t (t-s)^{-1/2}(t-s+1)^{-3/4}(\|F(s)\|_{3, G_h} + \|F(s)\|_{6/5, G_h})\, ds \\
&\leq C \int_0^t (t-s)^{-1/2}(t-s+1)^{-3/4}\|F(s)\|_6\, ds \\
&\leq C\|v\|_6 \int_0^t (t-s)^{-1/2}(t-s+1)^{-3/4}(s+1)^{-1/4}\, ds \\
&\leq C(t+1)^{-1/4}\|v\|_6.
\end{aligned}$$

Taking (A.4) into account, we have the desired estimate and complete the proof.

Proof of (3.2). Let $v \in J^6$. By Lemmas A.1, A.2, A.3, Sobolev inequality, and Gagliardo-Nirenberg inequality (cf. **[10]**), we have

$$\begin{aligned}
\|e^{-(t+1)A}v\|_\infty &\leq \|ge^{-(t+1)A}v\|_\infty + \|e^{-(t+1)A}v\|_{\infty, G_h} \\
&\leq \|ge^{-(t+1)A}v - v^*(t)\|_\infty + C\|v^*(t)\|_{1,6} \\
&\quad + C\|e^{-(t+1)A}v\|_{1,6,G_h} \\
&\leq C(t+1)^{-1/4}\|v\|_6 \quad \text{for } t > 0, \\
\|e^{-tA}v\|_\infty &\leq C\|e^{-tA}v\|_6^{3/4}\|e^{-tA}v\|_{2,6}^{1/4} \\
&\leq C(t^{-1}+1)^{1/4}\|v\|_6 \leq Ct^{-1/4}\|v\|_6
\end{aligned}$$

for $1 > t > 0$. The proof is complete.

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