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L^p-INTEGRABILITY OF THE SECOND ORDER DERIVATIVES OF GREEN POTENTIALS IN CONVEX DOMAINS

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We give estimates in L^p, $1 < p \leq 2$, of the second order derivatives of the Green potential of $f \in L^p$, for convex domains. This is done by interpolating between estimates in L^1 and L^2 of functions in atomic H^1 and L^2, respectively. The crucial step is obtaining the atomic estimate which is done by adapting to the present situation, a technique introduced by Dahlberg and Kenig.

0. Introduction. The purpose of this paper is to prove that for a convex domain Ω in $\mathbf{R}^n$, the second order derivatives of the Green potential of f, $\nabla_2 Gf$, are in $L^p(\Omega)$ if $f \in L^p(\Omega)$ with $1 < p \leq 2$. To avoid technicalities we assume throughout the paper that $n \geq 3$. The main results of the paper are

THEOREM 1. *Suppose D is a convex domain above a Lipschitz graph in $\mathbf{R}^n$, i.e., $D = \{x_n > \varphi(x')\}$ where φ is a convex Lipschitz function with Lipschitz constant bounded by M and $x' \in \mathbf{R}^{n-1}$. Let G be the Green function for D and let the Green potential for $f \in L^p(D)$, $1 < p \leq 2$, be denoted by Gf. Then we have that*

$$\int_D |\nabla_2 Gf|^p \, dx \leq c \int_D |f|^p \, dx,$$

where ∇_2 denotes the second order derivatives and the constant c only depends on the Lipschitz constant M. □

Via a patching argument we will derive the case of a bounded convex domain from the results of Theorem 1.

THEOREM 2. *Let Ω be an open, bounded and convex domain in $\mathbf{R}^n$, $n \geq 3$. Let G be its Green function. Suppose $1 < p \leq 2$. Then $\nabla_2 Gf \in L^p(\Omega)$ and*

$$\int_\Omega |\nabla_2 Gf|^p dx \leq C \int_\Omega |f|^p dx,$$

where C can be taken to depend only on the Lipschitz character of the domain. □

We note that a Lipschitz requirement is superfluous in the case of a bounded domain since every bounded and convex domain is a Lipschitz domain. That the theorems are not true for any Lipschitz domains follows from simple examples, see e.g. [**D**]. It is readily seen from the following simple example that we have to require $p \leq 2$ in general. Let Ω be an infinite cone in $\mathbf{R}^2$ with vertex at the origin and opening angle θ. Let $\tilde{v}(x, y) = y$ in $\{(x, y) : y > 0\}$. Take $v = \tilde{v}(z^\alpha)$ where $\alpha = \pi/\theta$. Then the second order derivatives of v behave as $|z|^{\alpha-2}$. Hence, $|\nabla_2 v| \in L^p$ $(|z| < 1)$ iff $p(2-\alpha) < 2$. The work in this area can provisionally be divided into two groups, see [**MSa**] and [**MP**]. In the first group attention is focused on global smoothness conditions on the boundary; in the other group the singularities are localized, and one considers a finite number of singularities of a specific type on the boundary, e.g. such as edges, polyhedral angles, conical points, etc. It is desirable to treat not only a finite number of singularities, but to give a global smoothness condition in the spirit of the first group of works, allowing for (not necessarily localized) singularities of the type mentioned in the other group. Convexity is of course such a condition. Although the case $p = 2$ of Theorem 2 is a classical result of Kadlec [**Ka**], for other p's one has considered domains with a finite number of singularities on the boundary. We again refer to [**MP**], see also [**Ko**], especially the powerful method of [**MP**] for estimates in related L^p-spaces and Hölder classes for more general elliptic boundary value problems and domains, with a finite number of singularities on the boundary. Recent results are contained in [**JK**] for estimates in a bounded Lipschitz domain with data in Sobolev and Besov spaces. Theorem 2 is well-known to be true for a bounded smooth, i.e. $C^{1,1}$, domain, cf. [**G**]. For $p = 2$, Theorem 2 is also true for combinations of the smooth and convex case. Here 'combinations' is taken to mean bounded Lipschitz domains fulfilling a uniform outer ball condition. Thus the domain admits not only a finite number of singularities on the boundary. See [**A**].

In fact, it is a classical result that for Ω open and bounded, and $f \in L^2(\Omega)$, Gf is the unique solution in $H_0^1(\Omega)$ solving

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \gamma(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where γ is the trace operator on the boundary of Ω. Further, if Ω is of the type mentioned above, then also $\|\nabla_2 u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$. The proof for a convex domain in this case, $p = 2$, rests on the following simple application of Green's formula. We have

$$\begin{aligned}\int_\Omega |\nabla_2 u|^2 dx &= \sum_{ij} \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}\, dx \\ &= \sum_{ij} -\int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial^3 u}{\partial x_i^2 \partial x_j}\, dx + \text{boundary integral} \\ &= \sum_{ij} \int_\Omega \frac{\partial^2 u}{\partial x_j^2} \frac{\partial^2 u}{\partial x_i^2}\, dx + \text{boundary integral} \\ &= \int_\Omega |\Delta u|^2\, dx + \text{boundary integral.}\end{aligned}$$

The boundary integral, in the case of zero boundary condition, turns out to be

$$\int_{\partial\Omega} (\operatorname{tr}\mathscr{B}) \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma,$$

where $\operatorname{tr}\mathscr{B}$ is the trace of the second fundamental quadratic form on the boundary, i.e. the mean curvature. For a convex domain we have $\operatorname{tr}\mathscr{B} \leq 0$ and consequently

$$\int_\Omega |\nabla_2 u|^2\, dx \leq \int_\Omega |\Delta u|^2\, dx.$$

Of course, in the above considerations a suitable approximation of the boundary with smooth boundaries has to be used.

The natural way to try to extend the result to other p's is to interpolate. However, it is not immediately clear how to extend the estimates to $L^1(\Omega)$ since, as indicated above, the available proof technique is strongly L^2-dependent. Moreover, if $f \in L^1$ then it is in general not true that $\nabla_2 Gf \in L^1$. In fact, $\nabla_2 Gf \in L^1$ may fail even if $f \in C^\infty(\overline{\Omega})$, see **[JK]**. However, one can use the atomic space H^1_{at} as a substitute for L^1 when interpolating, to obtain the desired result for $1 < p \leq 2$. See **[CW]**.

Theorem 2 follows from Theorem 1. The proof is given in §4. To prove Theorem 1 we will adapt to the present situation, a method that originated with Dahlberg and Kenig, **[DK]**, for solutions to Laplace's equation. As indicated above we interpolate between L^2 and atomic H^1. We thus have to show that $\nabla_2 Gf$ is a bounded operator on L^2 and from H^1_{at} into L^1 respectively. The L^2 case is a more or less direct consequence of the estimate

$$\int_\Omega |\nabla_2 u|^2\,dx \le \int_\Omega |\Delta u|^2\,dx\,,$$

for a convex domain, together with some suitable local formulation. For our purposes a function $a \in L^\infty(\mathbf{R}^n)$ is an atom if

(i) $\operatorname{supp} a \subset B = B_{R_a}(P_a)$
(ii) $\|a\|_\infty \le 1/|B|$
(iii) $\int a\,dx = 0$,

and $H^1_{at}(\mathbf{R}^n) = \{f \in L^1(\mathbf{R}^n) : f = \sum \lambda_j a_j\,,\ \sum |\lambda_j| < \infty\,,\ a_j \text{ is an atom}\}$ with norm $\|\cdot\| = \sum|\lambda_j|$. It follows that it is sufficient to show that there is a constant c such that $\int_D |\nabla_2 Ga|\,dx < c$ for all atoms a. Using the scaling properties of atoms the problem can be reduced to consider the case of atoms with support in a ball with radius one. For such atoms two cases arises, depending on whether the distance of the ball to ∂D is less than or greater than some strictly positive number. It will be seen that it is sufficient to show that Ga has the right decay outside some ball of fixed radius. This is shown, utilizing a variant of the method of **[DK]**, by using a reflection principle to have a solution u, extending Ga, of a uniformly elliptic PDE operator L, with bounded measurable coefficients and ellipticity constant only depending on the Lipschitz constant M, and $Lu = 0$ outside the support of the reflected atom. Thus the representation formula of **[SW]** applies to give a better decay than the fundamental solution outside the support of the extended atom. This will take care of the case when the distance from the support of the atom to ∂D is small. When this distance is large, the radius of the smallest ball containing the support of the extended atom might be arbitrarily large. In order to have the decay outside a ball of uniform radius, we have to substract off a solution, corresponding to the reflected part of the atom, with the same decay properties.

Acknowledgment. During the final preparation of this paper, the author learned that Tom Wolff has obtained (unpublished) weak type $(1, 1)$ estimates for the second order derivatives of the Green potential. The method of proof uses the maximum principle and so does not apply to other boundary conditions, e.g. the Neumann problem. These problems can be treated by the methods of this paper and the results will appear elsewhere. The results of Wolff have been reproved by Steve Fromm, MIT, and used for higher order regularity properties of the solution in the case of higher regularity of the data. He also

considers, corresponding to less regularity, the case of one derivative in the x-variable and one in the y-variable, of the Green function $G(x, y)$. These mentioned results will be contained in a forthcoming thesis.

1. Preliminaries. Let in the following $\varphi: \mathbf{R}^{n-1} \to \mathbf{R}$ be a convex Lipschitz function with Lipschitz constant bounded by $M > 0$. Let $D = \{(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} : x_n > \varphi(x')\}$ denote the domain above the graph of φ in $\mathbf{R}^n$. Define $D_R = D \cap B_R(0)$ for $R > 0$ where $B_R(0)$ is the ball centered at the origin with radius R. When deriving the estimates for H^1_{at} we will lean on estimates for the L^2 case. The L^2 case includes the case of an atom. We reserve the notation a for an atom.

As is well-known, if Ω is an open set in $\mathbf{R}^n$, $n \geq 3$, then Ω has a Green function G_Ω, or just G, and if $\Omega_1 \subset \Omega_2$ then $G_{\Omega_1} \leq G_{\Omega_2}$, cf. [H]. We denote by Gf the Green potential of f, i.e.

$$Gf(x) = \int_\Omega G(x, y) f(y)\, dy, \qquad x \in \Omega.$$

The following is a result of Dahlberg, [D].

THEOREM 1.1. *Let $D \subset \mathbf{R}^n$ be a Lipschitz domain and set $p_2 = 4/3$, $p_n = 3n(n+3)^{-1}$ for $n \geq 3$. Then there is a number $\varepsilon = \varepsilon(D) > 0$ such that if $1 < p < p_n + \varepsilon$ and q is given by $1/q = 1/p - 1/n$, then*

$$\left(\int_D |\nabla Gf|^q\, dP\right)^{1/q} \leq C \left(\int_D |f|^p\, dP\right)^{1/p}$$

where C only depends on p and D. Also, there is a constant C only depending on D such that

$$|\{P \in D :\ \nabla Gf(P) > \lambda\}| \leq C \left(\lambda^{-1} \int_D |f| dP\right)^{n/n-1},$$

where $|E|$ denotes the Lebesgue measure of a set E. □

For future reference we note the following simple consequence of the above theorem.

REMARK 1.2. For $1 < p \leq 2$ we have that

$$\left(\int_\Omega |\nabla Gf|^p\, dx\right)^{1/p} \leq C \left(\int_\Omega |f|^p\, dx\right)^{1/p},$$

for Ω an open, bounded and convex domain in $\mathbf{R}^n$, $n \geq 3$. To see this, note that since $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ for the estimate of Theorem 1.1 to be true, we have that $q > p$, and therefore a simple application of Hölder's inequality to the gradient side yields the statement in the case when $1 < p \leq p_n$. Since $p = p_n$ corresponds to $q = 3$ the statement follows also for the range $p_n < p \leq 2$. □

Another straightforward consequence of Theorem 1.1 is

LEMMA 1.3. *Suppose Ω is a bounded Lipschitz domain in $\mathbf{R}^n$, $n \geq 3$ and that $f \in L^2(\Omega)$. Then Gf is the unique solution in $H_0^1(\Omega)$ of the Poisson equation*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \gamma(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where γ is the restriction operator on $\partial\Omega$. □

We recall some properties of uniformly elliptic divergence form PDE operators. Let $A(x) = (a_{ij}(x))$ be an $n \times n$-dimensional symmetric matrix valued function in an open set $\mathscr{O}$ where the entries $a_{ij}(x)$ are realvalued measurable functions. Let L denote the operator $-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j}$. Then L is uniformly elliptic with ellipticity constant λ in $\mathscr{O}$ if there is a $\lambda \geq 1$ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2,$$

for all $\xi \in \mathbf{R}^n$. Suppose f is a distribution in $\mathscr{O}$. We call u a (weak) solution of $Lu = f$ in $\mathscr{O}$ if $u \in L^2_{1,\mathrm{loc}}(\mathscr{O})$ and $\int_{\mathscr{O}} < A\nabla u, \nabla\varphi > dx = f(\varphi)$ for all $\varphi \in C_0^\infty(\mathscr{O})$. Here $L^2_{1,\mathrm{loc}}(\mathscr{O})$ denotes the space of functions in $L^2_{\mathrm{loc}}(\mathscr{O})$ with distributional derivatives of first order in $L^2_{\mathrm{loc}}(\mathscr{O})$. As in **[DK]** we introduce the following reflection procedure for a solution of a divergence form PDE operator, which will be used in the sequel. Let, as above, D be the domain above a convex Lipschitz graph with Lipschitz constant bounded by M. Let $\Phi : D \to D^-$ be reflection in the boundary $\partial\Omega$ along the x_n-axis, given by by the bi-Lipschitzian map $\Phi(x', x_n) = (x', 2\varphi(x') - x_n)$. Define $A(x) = (a_{ij}(x))$ to be an $n \times n$-dimensional symmetric matrix valued function given by

$$A(x) = \begin{cases} I(x) & \text{for } x \in D, \\ B(x) & \text{for } x \in D^-, \end{cases}$$

where $I(x)$ is the identy matrix and

$$B(x) = \begin{pmatrix} 1 & 0 & \dots & 0 & 2\frac{\partial\varphi}{\partial x_1}(x') \\ 0 & 1 & \dots & 0 & 2\frac{\partial\varphi}{\partial x_2}(x') \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 2\frac{\partial\varphi}{\partial x_{n-1}}(x') \\ 2\frac{\partial\varphi}{\partial x_1}(x') & 2\frac{\partial\varphi}{\partial x_2}(x') & \dots & 2\frac{\partial\varphi}{\partial x_{n-1}}(x') & 1+4|\nabla\varphi(x')|^2 \end{pmatrix}.$$

From now on, we let L be the operator corresponding to this particular A unless otherwise indicated. It is not difficult to see that L is a uniformly elliptic, self-adjoint, divergence form operator with bounded real valued measurable coefficients and ellipticity constant λ only depending on the Lipschitz constant M.

For a function u on D we define u^- on D^- by $u^- = u\circ\Phi^{-1}$ and we put

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for } x \in D, \\ -u^-(x) & \text{for } x \in D^-. \end{cases}$$

We next indicate a proof showing that if $u \in H_0^1(D_R)$ where $D_R = D\cap B_R(0)$, then $u^- \in H_0^1(\Phi(D_R))$. Approximate u with smooth functions u_j in $H_0^1(D_R)$ and smooth the boundary with $\varphi_a(x') = \varphi * \phi_a(x') + Ma$, where $a>0$, $*$ denotes convolution and ϕ_a is a approximate identity; $\phi_a(x) = (1/a^{n-2})\phi(x/a)$ with $0\le\phi\in C_0^\infty(\mathbf{R}^{n-1})$, $\operatorname{supp}\phi\subset B_1(0)$ and $\int\phi_a dx' = 1$. Then we have that $v_{j,k}^- \to u^-$ in $L^2(\Phi(D_r))$ where $v_{j,k}^- = u_{j,a_{j,k}}^-(y)$. Here $a_{j,k}$ depends on j and k in a suitable manner and $u_{j,a}^-(y) = u_j\circ\Phi_a^{-1}(y)$. We have denoted by Φ_a the function defined as Φ but relative to $\partial D_a = \{(x', \varphi_a(x'))\}$. Now,

$$\frac{\partial v_{j,k}^-}{\partial y_i}(y) = \frac{\partial u_j}{\partial x_i}(\Phi_{a_{j,k}}^{-1}(y)) + 2\frac{\partial u_j}{\partial x_n}(\Phi_{a_{j,k}}^{-1}(y))\frac{\partial\varphi_{a_{jk}}}{\partial y_i}(y'),$$

for $1\le i\le n-1$ and similar for $i=n$. Hence $\int_{\Phi(D_R)}|\frac{\partial v_{j,k}^-}{\partial y_i}|^2dy \le C$ independent of i, j and k. Consequently, using weak convergence in $L^2(\Phi(D_R))$, we have that a subsequence of $\frac{\partial v_{j,k}^-}{\partial y_i}$ converges weakly so that $u^-\in H^1(\Phi(D_R))$. Furthermore, since $v_{j,k}^-\in H_0^1(\Phi(D_R))$ it follows that $u^-\in H_0^1(\Phi(D_R))$. Thus we have proved the statement. From this fact follows easily that if u is a function in D such that $\phi u\in H_0^1(D)$ for each $\phi\in C_0^\infty(\mathbf{R}^n)$, then $\psi u^-\in H_0^1(D^-)$ for each $\psi\in C_0^\infty(\mathbf{R}^n)$.

Proposition 1.4. *Let $f \in L^2(D)$ and suppose u is a solution to $-\Delta u = f$ as distributions in D. Suppose further that $\phi u \in H_0^1(D)$ for each $\phi \in C_0^\infty(\mathbf{R}^n)$, and $\nabla_2 u \in L^2(D)$. Then $\tilde{u}$ is a weak solution to $L\tilde{u} = \tilde{f}$ in $\mathbf{R}^n$.*

Proof. Take $\psi \in C_0^\infty(\mathbf{R}^n)$ and choose $R > 0$ such that $\operatorname{supp}\psi \subset B_R(0)$. Put

$$\frac{\partial \tilde{u}}{\partial x_i}(x) = \begin{cases} \frac{\partial u}{\partial x_i}(x) & \text{for } x \in D, \\ -\frac{\partial u^-}{\partial x_i}(x) & \text{for } x \in D^-. \end{cases}$$

Then, since ψu and $\psi u^- \in H_0^1$, it follows that

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial x_j}(\psi) &= -\int_{\mathbf{R}^n} \tilde{u}(x)\frac{\partial \psi}{\partial x_i}(x)\,dx \\ &= \int_{D_R} \frac{\partial u}{\partial x_i}\psi\,dx - \int_{\partial D_R} u\psi\nu_i\,d\sigma \\ &\quad - \int_{D_R^-} \frac{\partial u^-}{\partial x_i}\psi\,dx + \int_{\partial D_R^-} u^-\psi\nu_i^-\,d\sigma \\ &= \int_{B_R(0)} \frac{\partial \tilde{u}}{\partial x_i}\psi\,dx. \end{aligned}$$

Hence, $\tilde{u} \in H^1_{\mathrm{loc}}(\mathbf{R}^n)$. Next we show that for each $\psi \in C_0^\infty(\mathbf{R}^n)$ we have that $\int_{\mathbf{R}^n} < A(x)\nabla\tilde{u}(x),\, \nabla\psi > dx = f(\varphi)$. For $R > 0$ such that $\operatorname{supp}\psi \subset B_R(0)$ it follows that

$$\begin{aligned} &\int_{\mathbf{R}^n} \langle A(x)\nabla\tilde{u}(x),\, \nabla\psi\rangle\,dx \\ &\qquad = \int_D \nabla u \nabla\psi\,dx - \int_{D^-} \langle B(x)\nabla u^-,\, \nabla\psi\rangle\,dx, \end{aligned}$$

and

$$\int_D \nabla u\nabla\psi\,dx = -\int_D (\Delta u)\psi\,dx + \int_{\partial D_R} \left(\frac{\partial u}{\partial \nu}\right)\psi\,d\sigma,$$

since $\nabla_2 u \in L^2(D)$. Further, a change of variables, $x = \Phi^{-1}(y)$ having Jacobian $+1$, leads to

$$\begin{aligned} \int_{D^-} \langle B(x)\nabla u^-,\, \nabla\psi\rangle\,dx &= \int_{D\cap\Phi^{-1}(D_R^-)} \nabla u\nabla\hat{\psi}\,dx \\ &= -\int_{D\cap\Phi^{-1}(D_R^-)} (\Delta u)\hat{\psi}\,dx + \int_{\partial(D\cap\Phi^{-1}(D_R^-))} \left(\frac{\partial u}{\partial \nu}\right)\hat{\psi}\,d\sigma, \end{aligned}$$

where $\hat{\psi} = \psi \circ \Phi \in H^1(D)$ and the last equality is valid since $\nabla_2 u \in L^2(D)$. Finally, while $\psi = \hat{\psi}$ on $\partial\Omega$ we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} \langle A(x)\nabla\tilde{u}(x), \nabla\psi\rangle\, dx \\
&\quad = -\int_D (\Delta u)\psi\, dx + \int_D (\Delta u)\hat{\psi}\, dx \\
&\quad\quad + \int_{\partial D} \left(\frac{\partial u}{\partial \nu}\right) \psi\, d\sigma - \int_{\partial D} \left(\frac{\partial u}{\partial \nu}\right) \hat{\psi}\, d\sigma \\
&\quad = \int_D f\psi\, dx - \int_D f\hat{\psi}\, dx \\
&\quad = \int_D f\psi\, dx - \int_{D^-} f^-(\hat{\psi}\circ\Phi^{-1})\, dx = +\int \tilde{f}\psi\, dx. \qquad \square
\end{aligned}$$

2. The L^2-case. Let, as in the previous paragraph, D be the set above a convex Lipschitz graph φ with Lipschitz constant bounded by M. Let G be the Green function of D in $\mathbf{R}^n$, $n \geq 3$. Assume, for the time being, that $\varphi(0) = 0$. This condition, of course, has no real significance but makes the statements to follow more convenient.

LEMMA 2.1. *Suppose $f \in L^2(D)$. Then $Gf \in L^2(D_R)$ for each $R > 0$,*

$$\|Gf\|_{L^2(D_R)} \leq C_R \|f\|_{L^2(D)},$$

and $-\Delta Gf = f$ as distributions in D.

Proof. Estimating the Green function with the Newtonian kernel, the local L^2-estimate follows directly, for the case $n > 4$, from the estimate for the Riesz potential I_2 of smoothness 2; $I_\alpha(f)(x) = c\int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\, dy$. We need a better estimate. Since D is convex and the Laplacian is rotational invariant we might as well assume that $D \subset \{x_n > 0\}$. Hence, for $x \in D$ we have

$$|Gf(x)| \leq \int_D G(x, y)|f(y)|\, dy \leq \int_{\{x_n > 0\}} G_H(x, y)|f(y)|\, dy,$$

where G_H is the Green function for the halfspace and f is extended by zero outside D. Splitting the domain of integration into two parts where $\{|y| \leq 2|x|\}$ and $\{|y| > 2|x|\}$ respectively, it is easy to see that the local estimate can be achieved for the first part by comparing with the Newtonian kernel and estimating as in Youngs inequality. For the second part we note that $G_H(x, y) = c(|x-y|^{2-n} - |x^* - y|^{2-n})$

where $x^* = (x', -x_n)$ so that a simple application of the meanvalue theorem shows that $G_H(x, y) \leq 2(n-2)x_n|x_\theta - y|^{1-n}$ with $x_\theta = (x', (1-2\theta)x_n)$ and $0<\theta<1$. It is not difficult to see that there is a constant $c > 0$ independent of x and y such that $G_H(x, y) \leq cx_n|x-y|^{1-n}$. The inequality now follows from the theorem cited above, with $\alpha = 1$. As a consequence of this inequality it is sufficient to consider the case of compact support of f when proving the equality $-\Delta Gf = f$. For such functions the equality is an easy consequence of the definition of the Green function and the same equality for minus the Newton potential. □

Define $D_j = D_{2^j}$. Suppose $f \in L^2(D)$ and $\varphi(0) = 0$. Let $u_j \in H_0^1(D_j) \cap H^2(D_j)$ be the unique solution in $H_0^1(D_j)$ to

$$\begin{cases} -\Delta u_j = f & \text{in } D_j, \\ \gamma(u_j) = 0 & \text{on } \partial D_j. \end{cases}$$

Then $u_j = G_j f$ where G_j is the Green function of D_j. From the previous lemma follows

LEMMA 2.2. *For each $R_0 > 0$ there exists a j_0 such that for $j > j_0$ we have*

$$\|u_j\|_{L^2(D_{R_0})} \leq C_{R_0},$$

where C_{R_0} does not depend on j. □

The following lemma is a classical fact due to the convexity of D_j, cf. [**G**, p. 139].

LEMMA 2.3. *For each $R_0 > 0$ there is a j_0 such that for $j > j_0$ we have*

$$\|\nabla_2 u_j\|_{L^2(D_{R_0})} \leq \|\nabla_2 u_j\|_{L^2(D_j)} \leq \|f\|_{L^2(D_j)} \leq \|f\|_{L^2(D)},$$

where ∇_2 denotes second order derivatives, i.e.

$$\|\nabla_2 u_j\|_{L^2(D_j)} = \sum \left\| \frac{\partial^2 u_j}{\partial x_j \partial x_k} \right\|^2_{L^2(D_j)}. \qquad \square$$

Using a classical 'interpolation' inequality, cf. [**Ag**, p. 26], it now follows from the above lemmas that also the intermediate derivatives are estimated uniformly in j.

LEMMA 2.4. *For each* $R_0 > 0$ *there is a* j_0 *such that for* $j > j_0$ *we have*

$$\|\nabla u_j\|_{L^2(D_{R_0})} \leq C_{R_0},$$

where C_{R_0} *does not depend on* j. □

THEOREM 2.5. *Let* D *be the domain above a Lipschitz graph and let* $f \in L^2(D)$. *Then* $Gf(x) = \int_D G(x, y)f(y)\,dy$ *solves* $-\Delta Gf = f$ *in* D *and* $\psi Gf \in H_0^1(D)$ *for each* $\psi \in C_0^\infty(\mathbf{R}^n)$. *Furthermore,* $\nabla_2 Gf \in L^2(D)$ *and* $\int_D |\nabla_2 Gf|^2 \leq \int_D |f|^2\,dx$.

Proof. It is sufficient to consider the case $\varphi(0) = 0$. From Lemma 2.1 follows that Gf is a solution of the equation and from the same lemma also follows that $Gf(x) < \infty$ for a.e. $x \in D$. Since $G_j(x, y) \to G(x, y)$ on D, an application of the Lebesgue dominated convergence theorem gives that $G_jf(x) \to Gf(x)$ for a.e. $x \in D$. Furthermore $|G_jf|^2 \leq (G|f|)^2 \in L^1(D_{R_0})$, and so another application of the same theorem gives that $G_jf \to Gf$ in $L^2(D_{R_0})$ for each $R_0 > 0$, so that the distribution derivatives of G_jf converge to the derivatives of Gf. From the previous lemma we have that for each R_0 there is a subsequence of ∇G_jf converging weakly in $L^2(D_{R_0})$. Hence, by uniqueness of weak limits we have that $\nabla Gf \in L^2(D_{R_0})$. Therefore, ∇Gf is defined a.e. globally and belongs to L^2 locally. Moreover, $\psi G_jf \in H_0^1(D_{R_0})$ by choosing R_0 and j large enough. Again, the previous lemma shows that $\nabla(\psi G_jf)$ are uniformly bounded in $L^2(D_{R_0})$ so that a subsequence of ψG_jf converges to ψGf weakly in $H_0^1(D_{R_0})$. It follows that $\psi Gf \in H_0^1(D_{R_0})$. In particular, ψGf has a trace on the boundary and choosing ψ appropriately we see that $Gf = 0$ on ∂D.

From Lemma 2.3 we have that a subsequence of $\widetilde{\nabla_2 G_j}f$, defined to be zero extension of $\nabla_2 G_jf$ outside D_j, converges weakly to an element in $L^2(D)$. Consequently, $\nabla_2 Gf \in L^2(D)$. Now

$$\begin{aligned}\|\nabla_2 Gf\|^2 &= \langle \nabla_2 Gf, \nabla_2 Gf - \widetilde{\nabla_2 G_j}f\rangle + \langle \nabla_2 Gf, \widetilde{\nabla_2 G_j}f\rangle \\ &\leq \varepsilon + \|\nabla_2 Gf\|_{L^2(D)}\|\nabla_2 G_jf\|_{L^2(D_j)} \\ &\leq \varepsilon + \|\nabla_2 Gf\|_{L^2(D)}\|f\|_{L^2(D)}\end{aligned}$$

for each $\varepsilon > 0$. This implies the estimate of the lemma. □

3. The atomic estimate. In this section the following lemma is the main result.

LEMMA 3.1. *There is a constant C such that for all atoms a and all domains, D, above convex Lipschitz graphs, φ, with Lipschitz constant bounded by M, the following estimate is true;*

$$\int_D |\nabla_2 Ga|\, dx \leq C.$$

Moreover, C depends only on M. □

As mentioned in the introduction, the main fact that needs to be established is that there is a fixed radius such that for each atom a, Ga has an extension u that solves $Lu = 0$ outside a ball with this fixed radius. When the distance between the support of the atom and the boundary ∂D is large, applying the reflection procedure of §1 will give that the smallest ball containing the support of the reflected atom is large. This explains the different cases, (i)–(iii), appearing below.

Before getting into the details of Lemma 3.1 we recall a standard result from PDE. Following **[LStW]** and **[GWi]**, let G_L be the Green function for L in $\mathbf{R}^n$. Then there are constants K_i only depending on the ellipticity constant λ of L such that

$$\frac{K_1}{|P-Q|^{n-2}} \leq G_L(P, Q) \leq \frac{K_2}{|P-Q|^{n-2}}.$$

We sketch a proof of the following particular, but to us useful fact.

PROPOSITION 3.2. *Suppose $f \in L^2(\mathbf{R}^n)$ and has compact support. Then the Green potential of f corresponding to L,*

$$G_L f(x) \equiv \int_{\mathbf{R}^n} G_L(x, y) f(y)\, dy\ ,$$

belongs to $H^1_{\mathrm{loc}}(\mathbf{R}^n)$ and it is a weak solution of $Lu = f$ in $\mathbf{R}^n$.

Proof. Since $G_L f$ can be estimated pointwise by the Newton potential of the absolute value of f, it is clear that $G_L f \in L^2(\mathbf{R}^n) + L_\infty(\mathbf{R}^n)$ and therefore we have that $G_L f \in L^2_{\mathrm{loc}}(\mathbf{R}^n)$. It is not difficult to see from the results of **[LStW]**, that

$$G_L^R(x, y) \leq K G_\Delta^R(x, y) \leq K|x-y|^{2-n},$$

for x, y in a compact set, and R sufficiently large. Here K does not depend on R and G_Δ^R denotes the Green function for the Laplacian in $B_R(0)$. Hence,

$$\int_{B_{R_0(0)}} |G_L^R f(x)|^2\, dx \leq C_{R_0},$$

where C_{R_0} does not depend on R and $\operatorname{supp} f \cup B_{R_0}(0) \subset B_R(0)$. It now follows that $G_L^R f \to G_L f$ in $L^2(B_{R_0})$ and consequently, the distribution derivatives of $G_L^R f$ converge to those of $G_L f$.

We claim that $\int_{B_{R_0(0)}} |\nabla G_L^R f|^2 \, dx \le C_{R_0}$ for R big enough. This follows from an easy variant of a standard Cacciopoli type inequality. To see this, choose R_0 so that $\operatorname{supp} f \subset B_{R_0}(0)$. Take $\psi \in C_0^\infty(\mathbf{R}^n)$, $0 \le \psi \le 1$, $\psi \equiv 1$ on $B_{R_0+1}(0)$ and such that $\operatorname{supp} \psi \subset B_{R_0+10}(0)$. Let R be fixed and put for convenience $u_R \equiv G_L^R f \in H_0^1(B_R(0))$. Now u solves

$$\sum_{i,j} \int_{\mathbf{R}^n} a_{i,j} D_j u_R D_i \phi \, dx = \int_{\mathbf{R}^n} f \phi \, dx,$$

for each $\phi \in C_0^\infty(B_R(0))$. Take R subject to $R > R_0 + 10$ and let $\phi = \psi u_R$. Then we have

$$\begin{aligned}
\frac{1}{\lambda} \int_{B_{R_0}(0)} |\nabla u_R|^2 \, dx &\le \frac{1}{\lambda} \int_{B_R(0)} \psi |\nabla u_R|^2 \, dx \\
&\le \sum_{i,j} \int_{B_R(0)} (a_{i,j} D_j u_R D_i u_R) \psi \, dx \\
&= \sum_{i,j} \int_{B_R(0)} a_{i,j} D_j u_R (D_i(\psi u_R) - u_R D_i \psi) \, dx \\
&= \int_{B_R(0)} f \psi u_R \, dx - \sum_{i,j} \int_{B_R(0)} a_{i,j} (D_j u_R) u_R D_i \psi \, dx.
\end{aligned}$$

Hence, using Hölder's inequality it is enough to give a uniform estimate of the L^2 norm of the derivatives $D_i \psi D_j u_R$ over the set $B_{R_0+10}(0) \backslash B_{R_0}(0)$, which is immediate from a standard Cacciopoli estimate and the uniform estimate for the potentials themselves. The claim is proved. By weak convergence of a subsequence in $L^2(B_{R_0}(0))$ we get that $\nabla G_L f \in L^2(B_{R_0}(0))$, as a consequence $\nabla G_L f \in L^2_{\mathrm{loc}}(\mathbf{R}^n)$. Using the weak convergence again, it is immediate that $G_L f$ is a weak solution in $\mathbf{R}^n$. □

Recall that for a function v in D we define

$$\tilde{v}(x) = \begin{cases} v(x) & \text{for } x \in D, \\ -v^-(x) & \text{for } x \in D^-, \end{cases}$$

where $v^-(x) = u \circ \Phi^{-1}(x)$ for $x \in D^-$. Obviously, in general $\tilde{v} \ne v$ in D^- for a function v defined in $\mathbf{R}^n$. Now, $Ga(x) = \int_D G(x, y) a(y) \, dy$ for $x \in D$. From Thm. 2.5 and Prop. 1.1 follows that $\widetilde{Ga}$ is a weak solution to $Lu = \tilde{a}$ in $\mathbf{R}^n$.

PROPOSITION 3.3. $G_L\tilde{a} = \widetilde{Ga}$.

Proof. There is an R such that supp $\tilde{a} \subset B_R(0)$ and a simple estimate gives $|\widetilde{Ga}| \le C$ in $\mathbf{R}^n \setminus B_{R+1}(0)$. A similar estimate of $G_L\tilde{a}$, using the bound of G_L, gives $|G_L\tilde{a}| \le C$ in $\mathbf{R}^n \setminus B_{R+1}(0)$. Moreover, $L(\widetilde{Ga} - G_L\tilde{a}) = 0$. It is immediate that this solution has a limit zero at infinity, that it is continuous, and therefore bounded in $\mathbf{R}^n$. Thus, Theorem 4 of [**Mo**] shows that the solution is a constant which, in view of the limit at infinity, must be zero, i.e. $G_L\tilde{a} = \widetilde{Ga}$. □

We start by showing that Lemma 3.1 can be reduced to a particularly simple situation. A straightforward verification gives that for a translation of coordinates in the integral of the lemma, the new function is expressed as the Green potential of an atom with center on the nth coordinate axis and a Green function for a domain above a convex Lipschitz graph φ with the same Lipschitz constant as the original graph and obeying $\varphi(0) = 0$. As a consequence it is sufficient to prove the lemma in this latter situation.

We now claim that we can simplify the situation even further. The following standard procedure exploits the dilation properties of atoms. Suppose D is given as the domain above the convex Lipschitz graph φ with Lipschitz constant bounded by M and suppose that $\varphi(0) = 0$. Assume a is an atom with supp $a \subset B_{R_a}(P_a)$ and put $a^+(x) = a(x)$ in D and zero otherwise. Let $\Lambda : \mathbf{R}^n \to \mathbf{R}^n$ be the C^∞-isomorphism given by $y = \Lambda(x) = \frac{1}{R_a}(x - P_a) + P_a$, $x = \Lambda^{-1}(y) = R_a(y - P_a) + P_a$ and let $\widetilde{D} = \Lambda(D)$. Now $\partial\widetilde{D} = \Lambda(\partial D)$ and $\widetilde{D}$ is convex since Λ maps lines onto lines. Further, $y' = x'/R_a$, $y_n = \frac{1}{R_a}(\varphi(x') - P_a) + P_a$ since P_a is on the x_n-axis. Let $\tilde{\varphi}(y') = \frac{1}{R_a}(\varphi(x') - P_a) + P_a$ so that $\partial\widetilde{D} = \{(y', \tilde{\varphi}(y'))\}$. Therefore $\widetilde{D}$ is a domain above a Lipschitz graph with Lipschitz constant bounded by M. Define $w(y) = R_a^{n-2}(Ga)(\Lambda^{-1}(y))$ in $\widetilde{D}$ and let $b(y) = R_a^n a(\Lambda^{-1}(y))$ for $y \in \mathbf{R}^n$. Then $-\Delta w = b$ as distributions in $\widetilde{D}$ and b is a function with the following properties,

(i) supp $b \subset \{y : |\Lambda^{-1}(y) - P_a| < R_a\} = \{y : |y - P_a| < 1\}$ so that supp $b \subset B_1(P_a)$ and P_a is on the y_n-axis,

(ii) $\|b\|_\infty = R_a^n \|a(\Lambda^{-1}(\cdot))\|_\infty \le 1$

(iii) $\int b\,dx = 0$.

Hence, b is an atom in $\mathbf{R}^n$ and solves $-\Delta w = b$ in $\widetilde{D}$. As in the case of a translation it is, as above, a straightforward verification to show that w is expressed as a Green potential. A more smooth way to show this is to argue as in Proposition 3.3. Let $\widetilde{G}$ be the Green function of $\widetilde{D}$. Then $\Delta(\widetilde{G}b - w) = 0$ in $\widetilde{D}$ and it follows from Theorem 2.5 and Proposition 1.4 that the extension by an odd reflection in the boundary, of this solution solves the homogeneous case of the differential operator L in $\mathbf{R}^n$. Now this solution is bounded and has a limit zero at infinity. Thus it must be identically zero, i.e. $w = \widetilde{G}b$. Moreover, using a translation as above we can assume that $\tilde{\varphi}(0) = 0$, and the statement is demonstrated.

In short; we have seen that without loss of generality we can restrict ourselves to consider, in Lemma 3.1, the case $\operatorname{supp} a \subset B_1(P_a)$ with P_a on the x_n-axis and $\varphi(0) = 0$.

For this situation we now split into three different cases.

PROPOSITION 3.4. *Assume $\varphi(0) = 0$. There is a $R_0 > 1$ and a $R_1 > 1$ only depending on the bound of the Lipschitz constant M such that the following is true. For each atom a, with $\operatorname{supp} a \subset B_1(P_a)$ where $P_a = (0, \dots, 0, p_a)$ is on the x_n-axis, we have that if*

(i) $p_a \leq -1$ *then* $\operatorname{supp} a \subset D^-$,

(ii) $p_a > -1$ *and* $P_a \in B_{R_1}(0)$ *then* $\operatorname{supp} \tilde{a} \subset (B_2(P_a) \cap D) \cup \Phi(B_2(P_a \cap D)) \subset B_{R_0}$,

(iii) $p_a > -1$ *and* $P_a \notin B_{R_1}(0)$ *then* $B_1(P_a) \subset B_2(P_a) \subset D$ *and* $\Phi(B_1(P_a)) \subset B_{1+2M}(P_a^-) \subset B_{2(1+M)}(P_a^-) \subset D^-$, *where* $P_a^- = -P_a = (0, \dots, 0, -p_a)$.

The proof of this result follows without difficulty from the cone property enjoyed by a Lipschitz domain.

3a. Estimates at infinity.

PROPOSITION 3.5. *Let a be an atom such that $\operatorname{supp} a \subset B_1(P_a)$ with P_a on the x_n-axis and $\varphi(0) = 0$. Let a^- be extended by zero outside D^-. Let* (i)–(iii) *be the cases of Proposition* 3.4. *There is a constant C only depending on the Lipschitz constant M and not on the atom a, such that in case*

(i) $Ga = 0$ *in* D,

(ii) $|\widetilde{Ga}(p)| \leq C$ *for* $|P| \geq R_0$,

(iii) $|(\widetilde{Ga} + G_L a^-)(p)| \leq C$ *for* $|P - P_a| \geq 2$ *and* $|G_L a^-(p)| \leq C$ *for* $|P - P_a^-| \geq 2(1 + M)$.

Proof. Assertion (i) is obvious. A simple estimate shows in case (ii) that $|Ga(P)| \leq 1/(n-2)n^2\omega_n$ for $|P - P_a| \geq 2$. Therefore all points $P \in D$ for which $|Ga(P)| > 1/(n-2)n^2\omega_n$ are contained in $B_2(P_a)$. Thus all points for which $|\widetilde{Ga}(P)| > 1/(n-2)n^2\omega_n$ are contained in B_{R_0}, which is the statement in (ii). Consider now (iii). The function $G_L a^-$ solves $Lv = a^-$ in $\mathbf{R}^n$ and supp $a^- \subset \Phi(B_1(P_a)) \subset B_{(1+2M)}(\Phi(P_a))$ where $\Phi(P_a) = -P_a = P_a^-$. For P such that $|P - P_a^-| \geq 2(1+M)$ we have from the estimate of G_L, that $|G_L a^-(P)| < C$ where C only depends on the ellipticity constant λ of L. Using Proposition 3.3, it follows that $(\widetilde{Ga} + G_L a^-)(P) = \int_{\mathbf{R}^n} G_L(P, Q)(\tilde{a} + a^-)\,dQ = \int_{\mathbf{R}^n} G_L(P, Q)\,a\,dQ$. Again, exploiting the estimate of G_L gives the desired estimate. Remembering that λ only depends on the bound of the Lipschitz constant M, the lemma follows. □

We now refine these crude estimates at infinity using the representation theorem of **[SW]**. We will have use for the following well-known result. For a proof, see Moser **[Mo]**.

THEOREM 3.6. *Suppose $R \geq 1$ and that L is a uniformly elliptic divergence form operator with realvalued, bounded and measurable coefficients in $B_R(0)$ with ellipticity constant λ. Then, if $u \in H_0^1(B_R(0))$ is a solution of $Lu = 0$ in $B_R(0)$ we have that $u \in C_{\mathrm{loc}}^{0,\alpha}(B_R(0))$. If $\|u\|_{L^\infty(B_R(0))} < \infty$ we also have*

$$|u(P) - u(0)| \leq c\|u\|_{L^\infty(B_R(0))}|P|^\alpha,$$

for $P \in B_R(0)$. Here c and $\alpha > 0$ depend only on λ and not on u or R. The exponent α is bounded away from zero in terms of λ, i.e. $1/\alpha$ is bounded in terms of λ. □

THEOREM 3.7. *Suppose L is a uniformly elliptic divergence form operator with realvalued, bounded and measurable coefficients in $\mathbf{R}^n$ and ellipticity constant λ. Let $P^\Delta \in \mathbf{R}^n$ and let g^Δ be the fundamental solution of L in $\mathbf{R}^n$ with pole at P^Δ, i.e. $g^\Delta(P) = G_L(P, P^\Delta)$, so that*

$$c_1|P - P^\Delta|^{2-n} \leq g^\Delta(P) \leq c_2|P - P^\Delta|^{2-n},$$

for constants c_i only depending on λ. Suppose u solves $Lu = 0$ weakly in $|P - P^\Delta| > R$ and that u is bounded there. Further, suppose that u is continuous on $|P - P^\Delta| \geq R$. Then there exist constants u_∞, α, $c > 0$ and $\nu > 0$ such that c and $1/\nu$ are bounded by a constant only depending on λ, and

$$u(P) = U_\infty + \alpha g^\Delta(P) + w(P),$$

for $|P-P^\Delta| > R$ *where* w *is a bounded solution of* L *in* $|P-P^\Delta| > R$ *with bound given by*

$$|w(P)| \le cR^{n-2}\|u\|_{L^\infty(|P-P^\Delta|>R)} \cdot |P-P^\Delta|^{2-n-\nu}.$$

Furthermore, $\alpha = K[u]/K[g^\Delta]$ *where* $K[v] = \int \langle A\nabla v, \nabla\psi\rangle\, dx$ *for a solution* v *of* L *in* $|P-P^\Delta| > R$ *and any* $\psi \in C^\infty(\mathbf{R}^n)$ *such that* $\psi \equiv 0$ *in* $|P-P^\Delta| < R+1$ *and* $\psi \equiv 1$ *in a neighbourhood of infinity.*

Proof. It is sufficient to consider the case $P^\Delta = 0$. We follow **[SW]** more or less verbatim only giving a more explicit estimate of constants in our special case, where the function is bounded outside some bounded set. Denoting by G the function $g^\Delta(P) = G_L(P, 0)$ multiplied by a suitable constant only depending on λ, we have

$$|x|^{2-n} \le G(x) \le c|x|^{2-n},$$

for some constant c only depending on λ. The function given by the well-defined quotient $u(y/|y|^2)/G(y/|y|^2)$ solves a uniformly elliptic equation in $0 < |y| < R^{-1}$ where the differential operator L' is of the same type as L and with an ellipticity constant λ' only depending on λ. It follows from the results in **[SW]** that $w(y) = (u(y/|y|^2) - u_\infty)/G(y/|y|^2)$ is the unique solution of L' in $|y| < R^{-1}$ continuously attaining the boundary values given by the function $(u(y/|y|^2)-u_\infty)/G(y/|y|^2)$, which is continuous on the boundary $|y| = R^{-1}$. As a consequence

$$\begin{aligned}\|w\|_{L^\infty(|y|<R^{-1})} &\le \max_{|y|=R^{-1}} \left|\frac{u(y/|y|^2) - u_\infty}{G(y/|y|^2)}\right| \\ &= \max_{|x|=R}\left|\frac{u(x) - u_\infty}{G(x)}\right| \le 2R^{n-2}\|u\|_{L^\infty(|x|>R)},\end{aligned}$$

since $\lim_{|x|\to\infty} u(x) = u_\infty$. Furthermore, for $|x| > R$ we have that

$$u(x) = u_\infty + w(0)G(x) + \left(w\left(\frac{x}{|x|^2}\right) - w(0)\right)G(x).$$

This is the sought for expansion of the theorem since $(w(y) - w(0))$ solves $L'g = 0$ in $|y| < R^{-1}$, and by the previous theorem

$$\left|\left(w\left(\frac{x}{|x|^2}\right) - w(0)\right)G(x)\right| \le cR^{n-2}\|u\|_{L^\infty(|x|>R)} \cdot |x|^{-\nu} \cdot |x|^{2-n},$$

for $|x| > R$. Here c depends only on λ. It follows from the results in **[SW]**, Lemma 1 and Lemma 2, that $\alpha = K[u]/K[G]$. This finishes the proof of Theorem 3.7. □

Lemma 3.8. *Assume* $\varphi(0) = 0$. *Let* a *be an atom with* supp $a \subset B_1(P_a)$ *and* P_a *on the* x_n*-axis. Let* (ii) *and* (iii) *below be the cases of Proposition* 3.4. *Then we have for atoms of case*

(ii) $\int_{D\setminus B_{2R_0}(0)} |\nabla_2 Ga|\, dx \leq C$,

(iii) $\int_{D\setminus B_2(P_a)} |\nabla_2 Ga|\, dx \leq C$.

Here C *depends only on* M *and not on the atom* a.

Proof. Let R_0 and R_1 be given as in Proposition 3.4. We first claim that for atoms of case (ii) we have for the solution $\widetilde{Ga}$ of $L\widetilde{Ga} = \tilde{a}$ that $\widetilde{Ga}(P) = w(P)$ for $|P| > R_0$ where w is a solution of $Lw = 0$ in $|P| > R_0$ with $|w(P)| \leq cR_0^{n-2}|P|^{2-n-\nu}$ and c and $1/\nu > 0$ are bounded in terms of λ and do not depend on the atom a. For atoms of case (iii) we claim the following. For the solutions $\widetilde{Ga} + G_L a^-$ and $G_L a^-$ of $Lg = a$ and $Lg = a^-$ respectively, we have that

$$\begin{cases} (\widetilde{Ga} + G_L a^-)(P) = w_1(P), \\ G_L a^-(P) = w_2(P), \end{cases}$$

where the w_i's are solutions of $Lg = 0$ in $|P - P_a| > 2$ and $|P - P_a^-| > 2(1+M)$ respectively. The w_i's fulfill, $|w_1(P)| \leq c|P - P_a|^{2-n-\nu}$ for $|P - P_a| > 2$ and $|w_2(P)| \leq c|P - P_a^-|^{2-n-\nu}$ for $|P - P_a^-| > 2(1+M)$, where c and $1/\nu > 0$ are bounded in terms of λ and do not depend on the atom a. In view of Proposition 3.5, Theorem 3.7 and the facts that the supports of a and a^- are contained in a compact subset of $B_{R_0}(0)$ or $|P - P_a| > 2$, $|P - P_a^-| > 2(1+M)$, so that the solutions are continuous on the boundary, we need only prove the following; $K[\widetilde{Ga}] = 0$ in case (ii) and $K[\widetilde{Ga} + G_L a^-] = K[G_L a^-] = 0$ in case (iii). In case (ii) we have for $\psi \in C^\infty(\mathbf{R}^n)$ such that $\psi \equiv 0$ in $|P| < R_0 + 1$ and $\psi \equiv 1$ in a neighborhood of infinity, that $1 - \psi \in C_0^\infty(\mathbf{R}^n)$. Thus

$$\begin{aligned} -K[\widetilde{Ga}] &= \int \langle A\nabla\widetilde{Ga}, \nabla(1-\psi)\rangle\, dx = \int \tilde{a}(1-\psi)\, dx \\ &= \int \tilde{a}\, dx = \int_D a\, dx - \int_{D^-} a^-\, dx = 0, \end{aligned}$$

since $\int_{D^-} a^-\, dy = \int_D a^-(\Phi(x))\, dx = \int_D a(x)\, dx$. From Proposition 3.3 we get $K[\widetilde{Ga} + G_L a^-] = K[G_L\tilde{a} + G_L a^-] = K[G_L a]$. Therefore, taking $\psi \in C^\infty(\mathbf{R}^n)$ such that $\psi \equiv 0$ in $|P - P_a| < 3$ and $\psi \equiv 1$ in a neighborhood of infinity, we get

$$\begin{aligned} -K[\widetilde{Ga} + G_L a^-] &= \int \langle A\nabla G_L a, \nabla(1-\psi)\rangle\, dx \\ &= \int a(1-\psi)\, dx = \int a\, dx = 0, \end{aligned}$$

from the vanishing mean property of atoms. In the same way we have $K[G_L a^-] = 0$. The claims are proved.

Now assume a is an atom of case (iii). Let $\mathscr{U}_m = \{x \in D : 2^m < |x - P_a| < 2^{m+1}\}$ for $m = 0, 1, 2 \ldots.$ Take $\phi \in C_0^\infty(\mathbf{R}^n)$ such that $\phi \equiv 1$ in $B_4(0) \setminus \overline{B_2(0)}$ and $\operatorname{supp} \phi \subset B_8(0) \setminus \overline{B_1}(0)$. Put

$$\phi_m(x) = \phi((x - P_a)/2^m).$$

Then $\phi_m \equiv 1$ on $\mathscr{U}_{m+1}$ and $\operatorname{supp} \phi_m \subset B_{2^{m+3}}(P_a) \setminus \overline{B_{2^m}(P_a)}$. Now $\Delta(\phi_m Ga) = \phi_m \Delta Ga + 2\langle \nabla\phi_m, \nabla Ga\rangle + (Ga)\Delta\phi_m \in L^2(D)$ and $\phi_m Ga = 0$ on ∂D. Hence,

$$\int_{\mathscr{U}_{m+1}} |\nabla_2(Ga)|^2\, dx \le \int_D |\nabla_2(\phi_m Ga)|^2\, dx \le \int_D |\Delta(\phi_m Ga)|^2\, dx.$$

Since $\operatorname{supp} \phi_m \subset \complement\overline{B_1(P_a)}$ and $\Delta Ga = 0$ in this set we have $\Delta(\phi_m Ga) = 2\nabla\phi_m \nabla Ga + Ga\Delta\phi_m$ there. The following variant of a well-known result, see **[Mo]**, reduces estimates of ∇Ga to an estimate of Ga.

LEMMA 3.9. *For any* $\eta \in C_0^\infty(\mathbf{R}^n)$ *with* $\operatorname{supp} \eta \subset \complement\overline{B_1(P_a)}$ *we have*

$$\int_D \eta^2 |\nabla Ga|^2\, dx \le 4\lambda^4 \int_D |\nabla\eta|^2 Ga^2\, dx. \qquad \square$$

The proof follows easily by noting that $\eta Ga \in H_0^1(D)$ according to Theorem 2.5 and consequently $\eta^2 Ga \in H_0^1(D \cap \{|x - P_a| > 1\})$. Returning to the proof of Lemma 3.8, the last lemma implies that

$$\begin{aligned} \int_{\mathscr{U}_{m+1}} |\nabla_2(Ga)|^2\, dx &\le 2\left[4\int_D |\nabla\phi_m|^2 |\nabla Ga|^2\, dx + \int_D |Ga|^2 |\Delta\phi_m|^2\, dx\right] \\ &\le 2\left[4\sum_i 4\lambda^4 \int_D |Ga|^2 \left|\nabla\left(\frac{\partial\phi_m}{\partial x_i}\right)\right|^2 dx \right. \\ &\qquad \left. + \int_D |Ga|^2 |\Delta\phi_m|^2\, dx\right] \\ &\le \frac{c}{(2^m)^4} \int_{D\cap\{2^m < |x - P_a| < 2^{m+3}\}} |Ga|^2\, dx, \end{aligned}$$

where c only depends on λ. From the above claim it follows that for $x \in D$ we have $\widetilde{Ga}(x) = Ga(x)$, and for $x \in D \cap \{2^m < |x - P_a| < 2^{m+3}\}$ we have , with c only depending on λ, that

$$\begin{aligned}|Ga(x)| &\le |(\widetilde{Ga} + G_L a^-)(x)| + |G_L a^-(x)| \\ &\le c|x - P_a|^{2-n-\nu} + c|x - P_a^-|^{2-n-\nu},\end{aligned}$$

if $|x - P_a^-| > 2(1 + M)$, which is the case since $B_{2(1+M)}(P_a^-) \subset D^-$. Moreover, there is a constant c only depending on the Lipschitz constant M, such that for $x \in D$ we have $|x - P_a| \le |x - P_a^-|$. Therefore, for $x \in D \cap \{2^m < |x - P_a| < 2^{m+3}\}$ we have that $|Ga(x)| \le c|x - P_a|^{2-n-\nu}$ with c only depending on M. This implies

$$\begin{aligned}\int_{\mathscr{U}_{m+1}} |\nabla_2(Ga)|^2\, dx &\le \frac{c}{(2^m)^4} \int_{D \cap \{2^m < |x - P_a| < 2^{m+3}\}} |x - P_a|^{2(2-n-\nu)}\, dx \\ &\le \frac{c}{(2^m)^4} (2^m)^n (2^m)^{2(2-n-\nu)} \\ &\le c(2^m)^{-(n+2\nu)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{\mathscr{U}_{m+1}} |\nabla_2(Ga)|\, dx &\le \sqrt{|\mathscr{U}_{m+1}| \int_{\mathscr{U}_{m+1}} |\nabla_2 Ga|^2\, dx} \\ &\le (c(2^m R_0)^n 2^{-m(n+2\nu)})^{1/2} \\ &\le c 2^{-m\nu},\end{aligned}$$

where c depends only on λ and R_0 and not on the atom a. Finally, while $2^\nu > 1$ we have

$$\int_{D \setminus B_2(P_a)} |\nabla_2 Ga|\, dx \le c \sum_{m=0}^{\infty} \left(\frac{1}{2^\nu}\right)^m \le C,$$

with a constant C only depending on M and not on the atom a.

Next we consider an atom of case (ii). Let $\mathscr{U}_m = \{p \in D : 2^m R_0 < |P| < 2^{m+1} R_0\}$ for $m = 0, 1, 2 \dots$. Take $\phi \in C_0^\infty(\mathbf{R}^n)$ such that $\phi \equiv 1$ in $B_{4R_0}(0) \setminus \overline{B_{2R_0}(0)}$ and $\operatorname{supp} \phi \subset B_{8R_0}(0) \setminus \overline{B_{R_0}(0)}$. Put $\phi_m(x) = \phi(x/2^m)$. As before we get

$$\int_{\mathscr{U}_{m+1}} |\nabla_2(Ga)|^2\, dx \le \int_D |\Delta(\phi_m Ga)|^2\, dx.$$

Since $\operatorname{supp} \phi_m \subset \{|x| > R_0\}$ and $\Delta Ga = 0$ there, we have $\Delta(\phi_m Ga) = 2\nabla\phi_m \nabla Ga + (Ga)\Delta\phi_m$ in this set. Using a lemma similar to Lemma

3.9, we obtain

$$\int_{\mathscr{U}_{m+1}} |\nabla_2(Ga)|^2\, dx \le \frac{c}{(2^m)^4} \int_{D\cap\{2^m R_0<|x|<2^{m+3}R_0\}} |Ga|^2\, dx\,,$$

where c only depends on λ. Since, by the claim, we have on $D\cap\{|x| > R_0\}$ that $|Ga(x)| = |\widetilde{Ga}(x)| = |w(x)| \le cR_0^{n-2}|x|^{2-n-\nu}$ we get in the same way as above

$$\int_{\mathscr{U}_{m+1}} |\nabla_2(Ga)|^2\, dx \le c2^{-m(n+2\nu)}\,,$$

and consequently

$$\int_{D\setminus B_{2R_0}(0)} |\nabla_2 Ga|\, dx \le c\sum_{m=0}^{\infty}\left(\frac{1}{2^\nu}\right)^m \le C\,,$$

with a constant C only depending on M and not on the atom a. This finishes the the proof of Lemma 3.8. □

3b. Proof of Lemma 3.1.

Proof of Lemma 3.1. As we have seen it is enough to prove Lemma 3.1 for the case $\operatorname{supp} a \subset B_1(P_a)$ with P_a on the x_n-axis and $\varphi(0) = 0$. Any atom of this type is either an atom of case (i), (ii) or (iii). If $\operatorname{supp} a \cap D = \varnothing$ then $Ga \equiv 0$ and the assertion of the lemma is certainly true. Let now B be either $B_{R_0}(0)$ or $B_2(P_a)$ depending on whether a is an atom of case (ii) or case (iii). Utilizing the estimates of Lemma 3.8 we get

$$\begin{aligned}
\int_D |\nabla_2 Ga|\, dx &= \int_{D\cap B} |\nabla_2 Ga|\, dx + \int_{D\setminus B} |\nabla_2 Ga|\, dx \\
&\le \left(|B|\int_{B\cap D} |\nabla_2 Ga|^2\, dx\right)^{1/2} + C \\
&\le \left(|B|\int_{D} |\Delta Ga|^2\, dx\right)^{1/2} + C \\
&\le \left(|B|\int_{B_1(P_a)} |a|^2\, dx\right)^{1/2} + C \\
&\le (|B|/|B_1(P_a)|)^{1/2} + C \le C'\,,
\end{aligned}$$

where C' depends only on M. This finishes the proof of the lemma. □

4. Main results.

Proof of Theorem 1. Define for each $1 \leq k, l \leq n$ the operator $T_{k,l}: L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ by

$$(T_{k,l}f)(x) = \begin{cases} D_{k,l}Gf(x) & \text{for } x \in D, \\ 0 & \text{for } x \in D^-, \end{cases}$$

where $D_{k,l}Gf$ denotes the distributional derivative of Gf in D with respect to the variables x_k and x_l. The linearity of this operator is immediate and by Theorem 2.5 it is bounded. It is also defined for atoms and the following simple considerations show that it is also bounded: $H^1_{at}(\mathbf{R}^n) \to L^1(\mathbf{R}^n)$. Suppose $f = \sum_j \lambda_j a_j \in H^1_{at}(\mathbf{R}^n)$. The partial sums converge to f in $L^1(\mathbf{R}^n)$ and consequently the Green potentials of the partial sums converge to Gf in $L^1_{\text{loc}}(\mathbf{R}^n)$. Hence, $D_{k,l}Gf = \sum_j \lambda_j D_{k,l}Ga_j$ as distributions in D. Since $D_{k,l}Ga_j \in L^1(D)$ by Lemma 3.1 and

$$\left\| \sum_{j=M}^{N} \lambda_j D_{k,l}Ga_j \right\|_{L^1(D)} \leq C \sum_{j=M}^{N} |\lambda_j|,$$

we see that the partial sums of the derivatives form a Cauchy sequence in $L^1(D)$. Therefore, $D_{k,l}Gf \in L^1(D)$ and

$$\|D_{k,l}Gf\|_{L^1(D)} \leq C\|f\|_{H^1_{at}(\mathbf{R}^n)}.$$

From the results in [**CW**, p. 596] we see that $T_{k,l}: L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ is well-defined and bounded. Suppose that $f \in L^p(\mathbf{R}^n)$. According to [**CW**] we can decompose f as $f = g + h \in L^2(\mathbf{R}^n) + H^1_{at}(\mathbf{R}^n)$ and then

$$T_{k,l}f = T_{k,l}g + T_{k,l}h = \begin{cases} D_{k,l}Gg + D_{k,l}Gh & \text{for } x \in D \\ 0 & \text{for } x \in D^- \end{cases} \in L^p(\mathbf{R}^n).$$

Hence, for $f \in L^p(D)$ extended by zero outside D, we get that $D_{k,l}Gf \in L^p(D)$ and

$$\|D_{k,l}Gf\|_{L^p(D)} = \|T_{k,l}f\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(D)}.$$

This finishes the proof of the theorem. □

An open subset Ω of $\mathbf{R}^n$ is said to be Lipschitz if its boundary is locally given as a Lipschitz function. That is, for every $x \in \partial\Omega$ there is a rectangular neighborhood V of x in $\mathbf{R}^n$ and a, with the usual coordinate system, isometric coordinate system $\{y_1, \ldots, y_n\}$

such that $V = \{(y_1, \dots, y_n) : -a_j < y_j < a_j, \ 1 \le j \le n\}$ and fulfilling the following properties. For every $y' = (y_1, \dots, y_{n-1}) \in V'$, $|\varphi(y')| \le a_n/2$, $\Omega \cap V = \{y = (y', y_n) \in V : y_n < \varphi(y')\}$, and $\partial\Omega \cap V = \{y = (y', y_n) \in V : y_n = \varphi(y')\}$. Here V' is the projection of V onto the first $n-1$ coordinates and φ is a Lipschitz function.

Via a patching argument we can now reduce the case of a bounded convex domain to the graph case.

Proof of Theorem 2. Every bounded and convex domain is a Lipschitz domain. Accordingly, for each $x \in \partial\Omega$ there is a coordinate neighborhood V_x and a Lipschitz function, φ_x, describing the boundary locally. There is no restriction to assume that $\varphi(0) = 0$ and that $\Omega \subset \{y : y_n > 0\}$. Let x_0 be a point on the negative y_n-axis and let Ω_x be the shadow domain of Ω from x_0. That is, let for each $\theta \in S^{n-1}$, the line through x_0 with directional vector θ be denoted by ℓ_θ. Let x_θ be the point nearest to x_0 in the set $\ell_\theta \cap \overline{\Omega}$, if this set is non-empty. Then $\Omega_x = \bigcup_\theta \{x : x \in \ell_\theta, \ |x - x_0| \ge |x_\theta - x_0|\}$ where the union is taken over all θ such that $\ell_\theta \cap \overline{\Omega} \neq \varnothing$. Now, Ω_x is a convex domain above a Lipschitz graph and $\Omega \subset \Omega_x$. Taking the coordinate neighborhood V_x smaller if necessary, we can arrange it so that $(\partial\Omega_x \backslash \partial\Omega) \cap V_x = \varnothing$. In virtue of the compactness of $\partial\Omega$ we can cover it by a finite number of the V_x's, $\{V_j\}$, $j = 1, \dots, N$. Let V_0 be an open, compactly contained subset of Ω with a smooth boundary and such that $\{V_j\}$, $j = 0, \dots, N$, cover Ω. Let $\{\Theta_j\}$ be a partition of unity on $\overline{\Omega}$, subordinate to the cover $\{V_j\}_{j=0}^N$ so that $G_\Omega f = \sum_j \Theta_j G_\Omega f$. From Remark 1.2 follows that $h_j \equiv \Delta(\Theta_j G_\Omega f) \in L^p(\Omega)$. We first consider the case $j > 0$. We have that $\Omega \subset \Omega_j \equiv \Omega_{x_j}$ for each $j = 1, \dots, N$ and Ω_j is a convex domain above a Lipschitz graph. Let G_j be the Green function for Ω_j. We claim that $G_j h_j$ composed with a rotation and a translation, which we surpress notationally below, equals $\Theta_j G_\Omega f$. Taking the claim for granted we see that $\nabla_2 \Theta_j G_\Omega f \in L^p(\Omega)$, by Theorem 1. Moreover,

$$\begin{aligned} \|\nabla_2 \Theta_j G_\Omega f\|_{L^p(\Omega)} &\le C\|\nabla_2 G_j h_j\|_{L^p(\Omega_j)} \le C\|h_j\|_{L^p(\Omega_j)} = C\|h_j\|_{L^p(\Omega)} \\ &\le C(\|(\Delta\Theta_j)G_\Omega f\|_{L^p(\Omega)} \\ &\qquad + 2\|\nabla\Theta_j \cdot \nabla G_\Omega f\|_{L^p(\Omega)} + \|\Theta_j f\|_{L^p(\Omega)}) \\ &\le C\|f\|_{L^p(\Omega)}, \end{aligned}$$

where C is independent of $1 \le j \le N$ and $f \in L^p(\Omega)$. For the case

$j = 0$ the estimate

$$\|\nabla_2 \Theta_0 G_\Omega f\|_{L^p(\Omega)} = \|\nabla_2 \Theta_0 G_\Omega f\|_{L^p(V_0)} \le C\|f\|_{L^p(\Omega)},$$

is a standard result using uniqueness and existence in $W^{2,p}(\Omega_0)$, a priori estimates and an approximation technique with L^2 functions. That is, we use a standard local regularity theorem. Summing up we have

$$(*) \qquad \|\nabla_2 G_\Omega f\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}.$$

We next consider the dependence of the constant C in $(*)$. The constant in Remark 1.2 actually depends on $|\Omega|$. To get the constant in $(*)$ independent of this quantity we argue as follows. First we prove Theorem 2 under the additional condition that $|\Omega| = 1$. Then the general version of the theorem is a consequence of a rescaling since the inequality scales in the proper way. This is of course not true for the estimate of Remark 1.2.

It remains to prove the claim. For notational convenience we drop the index j, denote Ω_j by D and prove that $Gh(y) = (\Theta G_\Omega f)(Ry)$ where R is a translation and rotation. Since $(\partial\Omega_j \backslash \partial\Omega) \cap V_j = \varnothing$ the function $(\Theta_j G_\Omega f) \circ R$ is well-defined in D. Let $f_j \in C_0^\infty(\Omega)$ converge to f in $L^p(\Omega)$. Then $(\Theta G_\Omega f_j) \circ R \in H_0^1(D)$, $\nabla_2((\Theta G_\Omega f_j) \circ R) \in L^2(D)$ and $g_j \equiv -\Delta((\Theta G_\Omega f_j) \circ R) \in L^2(D)$. Hence, the odd reflection of $(\Theta G_\Omega f_j) \circ R$, in the boundary ∂D, is a weak solution of $Lu = \widetilde{g_j}$ in $\mathbf{R}^n$. By Theorem 2.5, $L\widetilde{Gg_j} = \widetilde{g_j}$ since Gg_j solves $-\Delta Gg_j = g_j$ weakly in D. Consequently, the difference of the solutions is a solution of the homogenuous equation in $\mathbf{R}^n$, and it has a limit zero at infinity. Hence, it is bounded. Therefore, $Gg_j = (\Theta G_\Omega f_j) \circ R$ in D. It is easily seen that $G_\Omega f_j \to G_\Omega f$ in $L^p(\Omega)$ and the same is true for the gradients. Hence, $(\Theta G_\Omega f_j) \circ R$ converge to $(\Theta G_\Omega f) \circ R$ in $L^p(D)$ and also, g_j converge to $g \equiv -\Delta((\Theta G_\Omega f) \circ R)$ in $L^p(D)$. The functions $g - g_j$ have support in Ω. We extend them by zero outside Ω. Since $|G(g-g_j)(y)|$ is estimated by the Newton potential, $(N * |g - g_j|)(y)$, it is easy to see that $Gg_j \to Gg$ in $L^p_{\mathrm{loc}}(D)$. Since $\operatorname{supp} Gg_j \subset \overline{\Omega}$ we have that $\operatorname{supp} Gg \subset \overline{\Omega}$ and as a consequence $Gg_j \to Gg$ in $L^p(D)$. The claim is proved. □

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