PACIFIC JOURNAL OF MATHEMATICS
Vol. 160, No. 2, 1993

A CONTINUATION PRINCIPLE FOR PERIODIC SOLUTIONS OF FORCED MOTION EQUATIONS ON MANIFOLDS AND APPLICATIONS TO BIFURCATION THEORY

MASSIMO FURI AND MARIA PATRIZIA PERA

We give a continuation principle for forced oscillations of second order differential equations on not necessarily compact differentiable manifolds. A topological sufficient condition for an equilibrium point to be a bifurcation point for periodic orbits is a straightforward consequence of such a continuation principle. Known results on open sets of euclidean spaces as well as a recent continuation principle for forced oscillations on compact manifolds with nonzero Euler-Poincaré characteristic are also included as particular cases.

0. Introduction. Let M be a smooth (boundaryless) m-dimensional manifold in $\mathbb{R}^n$ and consider on M a time dependent T-periodic tangent vector field, i.e. a continuous map $f: \mathbb{R} \times M \to \mathbb{R}^n$ with the property that, for all $(t, q) \in \mathbb{R} \times M$, $f(t, q)$ is tangent to M at q and $f(t+T, q) = f(t, q)$. The map f may be interpreted as a (periodic) force acting on a mass point q (of mass 1) constrained on M. A forced (or harmonic) oscillation on M is a T-periodic solution of the motion problem associated to the force f.

In **[FP4]**, in the attempt to solve the conjecture about the existence of forced oscillations for the spherical pendulum (i.e. for the case $M = S^2$, the two dimensional sphere), we have studied the one-parameter motion problem associated to the force λf, $\lambda \geq 0$. In this context, we say that (λ, x) is a solution (pair) of the problem, if $\lambda \geq 0$ and $x: \mathbb{R} \to M$ is a forced oscillation corresponding to λf. Let us denote by X the set of all solution pairs. Since any point $q \in M$ is a rest point of the inertial problem (i.e. the motion problem with $\lambda = 0$), the constraint M may be regarded as a subset of X by means of the embedding $q \mapsto (0, q)$. With this in mind, we say that M is the manifold of trivial solutions of X and, consequently, any element of $X \backslash M$ will be a nontrivial solution (pair). We observe that in the non-flat case one may have nontrivial solutions even when $\lambda = 0$. Closed geodesics may be, in fact, T-periodic orbits if they have appropriate speed.

If we consider the standard C^1 metric structure on the space $C^1_T(M)$ of all T-periodic maps $x: \mathbb{R} \to M$ of class C^1, the main result in **[FP4]** can be stated as follows:

THEOREM 0.1. *Assume that the constraint M is compact with nonzero Euler-Poincaré characteristic. Then $X \backslash M$ contains an unbounded connected subset whose closure in $[0, \infty) \times C^1_T(M)$ meets M.*

In **[FP4]**, an element q of the trivial subset M of X was called a *bifurcation point* (for the forced motion problem considered above) if any neighborhood of q contains a nontrivial solution, that is, if q is in the closure of $X \backslash M$. So, as a consequence of the above result, one gets the existence of bifurcation points for a parametrized forced constrained system, provided that the constraint M is compact and $\chi(M)$, the Euler-Poincaré characteristic of M, is nonzero. We will show that a necessary condition for a point $q \in M$ to be a bifurcation point is that the average force $\overline{f}$ is zero at q; i.e.

$$\overline{f}(q) = \frac{1}{T} \int_0^T f(t, q)\, dt = 0.$$

Thus, Theorem 0.1 may also be regarded as an extension (or dynamical version) of the classical Poincaré-Hopf theorem, which asserts that any tangent vector field on a compact boundaryless manifold M, with $\chi(M) \neq 0$, vanishes somewhere. In fact, observe that a time independent tangent vector field f on M may be regarded as a periodic force (of any period).

Theorem 0.1 above was successfully used in **[FP5]** to give an affirmative answer to the conjecture about the forced spherical pendulum. However, since the problem on whether or not any compact constrained system M with $\chi(M) \neq 0$ has forced oscillations is still unsolved, we think that further investigations about one-parameter forced constrained systems may be of some interest. Our aim here is to extend to the noncompact case some of the results of **[FP4]**, including the one above. The interest of this is mainly related to the fact that open sets in $\mathbb{R}^m$ are noncompact manifolds. So, the above theorem does not apply to the flat case. Moreover, once the necessary condition for a point $q \in M$ to be a bifurcation point is fulfilled, the possibility of dealing with noncompact manifolds will give us the tools to restrict our attention to a neighborhood of q, in order to get sufficient conditions for bifurcation.

In other words, what we do here is the spirit of **[FP2]** where our effort was devoted to first order differential equations on noncompact manifolds.

One may argue that a motion equation on a manifold $M \subset \mathbb{R}^n$ is just a special first order differential equation on the tangent bundle

$$T(M) = \{(q, v) \in \mathbb{R}^n \times \mathbb{R}^n : q \in M, \ v \text{ is tangent to } M \text{ at } q\}.$$

However, a bifurcation problem associated to a force of the form λf, where, as above, $\lambda \geq 0$ and $f: \mathbb{R} \times M \to \mathbb{R}^n$ is a T-periodic tangent vector field on M, cannot be simply reduced to a problem of the form:

$$\dot{z}(t) = \lambda g(t, z(t)), \quad t \in \mathbb{R}, \ z(t) \in T(M),$$

studied in **[FP2]**. The reason is that the inertial motion problem does not correspond, in the phase space, to the trivial equation $\dot{z}(t) = 0$. Actually, as is well known, the motion problem of a mass point q acted on by a force λf has the following form on $T(M)$:

$$\dot{z}(t) = h(z(t)) + \lambda g(t, z(t)),$$

where the term h, which is a (nontrivial) tangent vector field on $T(M)$, is related to the geometry of M and linear only in the flat case. This makes the problem in the non-flat case hard to deal with and cannot be handled with the techniques developed in **[FP1]** and in **[Mar]**, where, for $\lambda = 0$, the problem was linear (with nontrivial kernel).

We point out that a very interesting continuation principle for equations of the above form (and not necessarily related to second order equations) is given in **[CMZ]**, where, roughly speaking, the equation is given on an open subset of a euclidean space and the existence of a branch of solution pairs (λ, z) is ensured provided that the Brouwer topological degree of h is (well defined and) nonzero.

What seems peculiar to us, and interesting for further investigations, in our situation, is the role of g (or, equivalently, of the force f) which is important for the existence of a bifurcating branch. In fact, as we shall see, it is just the Euler characteristic of the average force $\overline{f}$ which, when defined and nonzero, ensures the existence of a global bifurcating branch. To see the relation with well-known concepts we recall that the Euler characteristic of a tangent vector field coincides with the Brouwer degree in the flat case and with the Euler-Poincaré characteristic of the manifold in the compact boundaryless case.

1. Notation and preliminaries. In this section we recall some definitions and results that will be needed in the sequel.

The inner product of two vectors v and w in $\mathbb{R}^n$ will be denoted by $\langle v, w\rangle$ and $|v|$ will stand for the euclidean norm of v (i.e. $|v| = \langle v, v\rangle^{1/2}$).

Let M be an m-dimensional boundaryless smooth manifold in $\mathbb{R}^n$ and, for any $q \in M$, let $T_q(M) \subset \mathbb{R}^n$ and $T_q(M)^\perp \subset \mathbb{R}^n$ denote respectively the tangent space and the normal space of M at q. Let $T(M)$ denote the tangent bundle of M, i.e. the $2m$-differentiable submanifold

$$T(M) = \{(q, v) \in \mathbb{R}^n \times \mathbb{R}^n : q \in M,\ \ v \in T_q(M)\}$$

of $\mathbb{R}^n \times \mathbb{R}^n$, containing a natural copy of M, via the embedding $q \mapsto (q, 0)$.

Given $(q, v) \in T(M)$, $v \neq 0$, we shall denote by $k(q, v)$ the curvature (in $\mathbb{R}^n$) of the geodesic through q with velocity v (the Riemannian structure on M is the one inherited by the euclidean space $\mathbb{R}^n$).

Let J be a real interval and let $x: J \to \mathbb{R}^n$ be a C^2 curve on $\mathbb{R}^n$. We will denote by $\dot{x}: J \to \mathbb{R}^n$ and $\ddot{x}: J \to \mathbb{R}^n$ the velocity and the acceleration of x, respectively.

Given any $q \in M$ and any $w \in \mathbb{R}^n$, the vector w can be uniquely decomposed into a *normal component*, $w_n \in T_q(M)^\perp$, of w at q and a *parallel component*, $w_p \in T_q(M)$, of w at q (clearly, given $w \in \mathbb{R}^n$, this decomposition depends on the chosen element q of M). So, if $x: J \to M$ is a C^2 curve on M and $t \in J$, $\ddot{x}_n(t)$ and $\ddot{x}_p(t)$ will denote, respectively, the normal and the parallel component of $\ddot{x}(t)$ at $x(t) \in M$.

We recall that there exists a C^∞ map $r: T(M) \to \mathbb{R}^n$, called the *reactive force* (or force of constraint or inertial reaction), such that

$$\ddot{x}_n(t) = r(x(t), \dot{x}(t))$$

for any C^2 curve $x: J \to M$ on M. Such a reactive force, at $(q, v) \in T(M)$, belongs to $T_q(M)^\perp$ and, when the curvature $k(q, v)$ of the geodesic through q with velocity v is nonzero, is directed toward the center of curvature. Moreover, its norm equals $k(q, v)|v|^2$.

Lemma 1.1 below has an evident physical meaning and will be used several times in the next section.

LEMMA 1.1. *Let* $x\colon \mathbb{R} \to M$ *be a periodic* C^2 *curve on* M. *Let*

$$\begin{aligned} d &= \max\{|x(t_1) - x(t_2)|\colon t_1, t_2 \in \mathbb{R}\}, \\ u &= \max\{|\dot{x}(t)|\colon t \in \mathbb{R}\}, \\ F &= \max\{|\ddot{x}_p(t)|\colon t \in \mathbb{R}\}, \\ K &= \max\{k(x(t), \dot{x}(t))\colon t \in \mathbb{R}\}. \end{aligned}$$

Then $u^2 \le Fd/(1 - Kd)$, *provided that* $Kd < 1$.

Proof. Suppose first that the manifold M coincides with $\mathbb{R}^n$. Then $K = 0$ and $F = \max\{|\ddot{x}(t)|\colon t \in \mathbb{R}\}$, so the assertion of the Lemma reduces to the inequality

$$u^2 \le Fd.$$

Let $\mu\colon \mathbb{R}^n \to \mathbb{R}$ be any linear functional of unitary norm and let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be given by $\varphi(t) = \mu(x(t))$. Because μ is arbitrary, it suffices to show that

$$\dot{\varphi}^2(t) \le Fd, \quad \forall t \in \mathbb{R}.$$

Let $t \in \mathbb{R}$ be such that $\dot{\varphi}^2(t) \ne 0$ (if such a t does not exist, then the inequality is obvious). By the periodicity of x, there exists an open interval (a, b) containing t, such that $\dot{\varphi}(a) = \dot{\varphi}(b) = 0$ and $\dot{\varphi}(\tau) \ne 0$ for all $\tau \in (a, b)$. Therefore, φ is monotone in (a, b). Hence

$$|\varphi(t) - \varphi(a)| + |\varphi(b) - \varphi(t)| = |\varphi(b) - \varphi(a)| \le |x(b) - x(a)| \le d.$$

Now, without loss of generality, we may assume $\dot{\varphi}(\tau) > 0$ in (a, b) and $\varphi(t) - \varphi(a) \le d/2$, so that

$$\frac{1}{2}\dot{\varphi}^2(t) = \frac{1}{2}|\dot{\varphi}^2(t) - \dot{\varphi}^2(a)| = \left|\int_a^t \ddot{\varphi}(\tau)\dot{\varphi}(\tau)\,d\tau\right|.$$

Since $|\ddot{\varphi}(\tau)| = |\mu(\ddot{x}(\tau))| \le \max\{|\ddot{x}(t)|\colon t \in \mathbb{R}\} = F$, we obtain

$$\dot{\varphi}^2(t) \le 2F\left|\int_a^t \dot{\varphi}(\tau)\,d\tau\right| \le 2F(\varphi(t) - \varphi(a)) \le Fd.$$

Consider now the case $M \ne \mathbb{R}^n$. By the first part of the proof it follows that

$$\begin{aligned} u^2 &\le d\max\{|\ddot{x}(t)|\colon t \in \mathbb{R}\} \\ &\le d\max\{|\ddot{x}_n(t)|\colon t \in \mathbb{R}\} + d\max\{|\ddot{x}_p(t)|\colon t \in \mathbb{R}\}. \end{aligned}$$

Hence, by recalling that $\ddot{x}_n(t) = r(x(t), \dot{x}(t))$ and that

$$|r(x(t), \dot{x}(t))| = k(x(t), \dot{x}(t))|\dot{x}(t)|^2,$$

we have

$$\begin{aligned} u^2 &\leq d\max\{k(x(t), \dot{x}(t)) \colon t \in \mathbb{R}\}u^2 + d\max\{|\ddot{x}_p(t)| \colon t \in \mathbb{R}\} \\ &\leq Kdu^2 + Fd, \end{aligned}$$

which implies the assertion of the lemma, since we have assumed $Kd < 1$. □

A time dependent tangent vector field on M is a map $u\colon \mathbb{R} \times M \to \mathbb{R}^n$ such that $u(t, q) \in T_q(M)$ for each $t \in \mathbb{R}$ and $q \in M$. Analogously, a (time dependent) tangent vector field on $T(M)$ is a map $w\colon \mathbb{R} \times T(M) \to \mathbb{R}^n \times \mathbb{R}^n$ which assigns to any $(t, q, v) \in \mathbb{R} \times T(M)$ a vector

$$w(t, q, v) = (w_1(t, q, v), w_2(t, q, v)) \in \mathbb{R}^n \times \mathbb{R}^n$$

such that $w(t, q, v) \in T_{(q, v)}(T(M))$ for all $t \in \mathbb{R}$. We point out that a necessary condition for a vector $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$ to be tangent to $T(M)$ at (q, v) is that $v_1 \in T_q(M)$. Moreover, given $(q, v) \in T(M)$ and $v_1 \in T_q(M)$, there exists a unique vector, $n = n(q, v, v_1) \in \mathbb{R}^n$, normal to M at q, with the property that $(v_1, v_2) \in T_{(q, v)}(T(M))$ if and only if v_2 belongs to the linear manifold $n + T_q(M)$. It is known that the map n, which is defined on the differentiable manifold

$$\{(q, v, v_1) \in M \times \mathbb{R}^n \times \mathbb{R}^n \colon v, v_1 \in T_q(M)\},$$

is smooth and linear with respect to each one of the last two variables.

A (time dependent) tangent vector field on $T(M)$, $w = (w_1, w_2)$, is called a *second order* vector field if it satisfies the condition $w_1(t, q, v) = v$, identically (see e.g. **[BC]**, **[L]**). So, any second order tangent vector field on $T(M)$ can be represented in the form

$$w(t, q, v) = (v, n(q, v, v) + f(t, q, v)) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where $n(q, v, v) = r(q, v)$ is the reactive force at (q, v) and $f\colon \mathbb{R} \times T(M) \to \mathbb{R}^n$, called the *active force* on M, satisfies the condition $f(t, q, v) \in T_q(M)$, identically. Hence, a second order (time dependent) tangent vector field on $T(M)$ is uniquely determined by its *parallel part*, i.e. the active force f.

Given any active force $f\colon \mathbb{R} \times T(M) \to \mathbb{R}^n$, one can consider in M the second order differential equation of motion

$$\ddot{x}(t) = r(x(t), \dot{x}(t)) + f(t, x(t), \dot{x}(t)), \qquad x(t) \in M. \tag{1.1}$$

As previously observed, the "normal part" of the above equation, i.e.

$$\ddot{x}_n(t) = r(x(t), \dot{x}(t)),$$

is satisfied by any curve $x: J \to M$. Hence, (1.1) turns out equivalent to its parallel part

$$\ddot{x}_p(t) = f(t, x(t), \dot{x}(t)). \tag{1.2}$$

Moreover, (1.1) is clearly equivalent to the differential system

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = r(x(t), y(t)) + f(t, x(t), y(t)), \end{cases} \tag{1.3}$$

where $(x(t), y(t)) \in T(M)$, which, actually, represents a first order differential equation on $T(M)$, since the map

$$(t, q, v) \in \mathbb{R} \times T(M) \mapsto (v, r(q, v) + f(t, q, v))$$

is a tangent vector field on $T(M)$.

In the case when the active force is identically zero, (1.1) reduces to the inertial equation of motion

$$\ddot{x}(t) = r(x(t), \dot{x}(t)),$$

whose solutions are the geodesics of M. As usual, let us denote by

$$t \mapsto \exp_q(tv)$$

the geodesic through q with initial velocity v.

The following lemma shows that, given an interval of time $[0, T]$, the inertial motion is, in some sense, locally fixed point free, provided that the initial speed is nonzero and sufficiently small. This fact will be used in the sequel in order to apply the fixed point index theory to the translation operator along the orbits.

LEMMA 1.2. *Let C be a compact subset of M. Then there exists $\varepsilon > 0$ such that $\exp_q(tv) \neq q$, for all $q \in C$, $t \in \mathbb{R}$ and $v \in T_q(M)$ with $0 < |tv| < \varepsilon$.*

Proof. Take $\varepsilon > 0$ and let

$$K(\varepsilon) = \sup\{k(q, v): (q, v) \in T(B(C, \varepsilon))\},$$

where $T(B(C, \varepsilon))$ is the tangent bundle of

$$B(C, \varepsilon) = \{q \in M: |q - p| < \varepsilon \text{ for some } p \in C\}.$$

Clearly, for ε sufficiently small, $K(\varepsilon)$ is finite and nondecreasing with ε. So, one may take ε such that $\varepsilon K(\varepsilon) < 2\pi$. Let $q \in C$ and observe that any geodesic $t \mapsto \exp_q(tv)$ such that $\exp_q(tv) = q$ has length $|tv|$. As is known, by a result due to H. A. Schwarz (see **[B]**,

§31, for references and related results), the length L of an arbitrary closed curve satisfies the inequality $L \geq 2\pi/K$, where K denotes the maximum curvature. Now, if $|tv| < \varepsilon$, the geodesic $s \mapsto \exp_q(sv)$, $0 \leq s \leq t$, lies entirely in $B(C, \varepsilon)$ and, consequently, cannot be closed, since its maximum curvature is less than or equal to $K(\varepsilon)$. □

Now let $g: M \to \mathbb{R}^n$ be a continuous tangent vector field on M. Then (see e.g. **[H]**, **[M]**, **[T]** and references therein), if the set $\{q \in M: g(q) = 0\}$ is compact, one can associate to g an integer $\chi(g)$, called the Euler characteristic of g, which, roughly speaking, counts (algebraically) the number of zeros of g. This integer, in the particular case when all zeros of g are isolated, is simply defined as the sum of the indices at these zeros. In the general case $\chi(g)$ is defined just taking sufficiently close approximations of g with only isolated zeros (the existence of such approximations is ensured by Sard's Lemma). The Poincaré-Hopf theorem asserts that, when M is compact, this integer equals $\chi(M)$, the Euler-Poincaré characteristic of M. On the other hand, in the particular case when M is an open subset of $\mathbb{R}^m$, $\chi(g)$ is just the Brouwer degree (with respect to zero) of the map $g: M \to \mathbb{R}^m$. Moreover, all standard properties of the Brouwer degree on open subsets of euclidean spaces, such as homotopy invariance, excision, additivity, existence, etc., are still valid in the more general context of differentiable manifolds. To see this one can use an equivalent definition of Euler characteristic of a vector field based on fixed point index theory given in **[FP2]**. To avoid any possible confusion we point out that in the literature there exists a different extension of the Brouwer degree to the context of differentiable manifolds (see e.g. **[M]** and references therein), called the Brouwer degree on manifolds. This second extension, roughly speaking, counts the (algebraic) number of solutions of an equation of the form $h(x) = y$, where $h: M \to N$ is a map of oriented manifolds of the same (finite) dimension and $y \in N$ (the assumption that $h^{-1}(y)$ is compact is needed for this degree to be defined). This dichotomy of notions in the context of manifolds or, according to the viewpoint, confusion of concepts in the flat case, is mainly due to the fact that in $\mathbb{R}^m$ an equation of the form $h(x) = y$ can be equivalently written as $h(x) - y = 0$ and a map from an open subset of $\mathbb{R}^m$ into $\mathbb{R}^m$ can be also viewed as a vector field.

In what follows, to remind that the Euler characteristic of a tangent vector field g on M, reduces, in the flat case, to the classical Brouwer degree (with respect to zero), $\chi(g)$ will be called the (*global*) *degree* of the vector field g and denoted by $\deg(g)$. As in the flat case, g

will be said to be *admissible* if the set of its zeros is compact. The case when M has boundary may also be considered if one takes the restriction of the vector field g to the interior $M \backslash \partial M$ of M. In particular, if M is a compact manifold with boundary and $g(q) \neq 0$ for all $q \in \partial M$, then g is admissible (and $\deg(g)$ is not necessarily equal to $\chi(M)$, unless g points outward along ∂M).

Observe also that no orientability of M is required for the global degree of a tangent vector field g to be defined. For example, the degree of an admissible tangent vector field on the boundaryless Möbius strip makes sense. Moreover, given any integer, it is not difficult to provide an example of an admissible tangent vector field on this noncompact nonorientable manifold whose degree is such an integer.

A homotopy $G: M \times [0, 1] \to \mathbb{R}^n$ of (tangent) vector fields on M is said to be admissible if the set

$$\{q \in M: G(q, \mu) = 0 \text{ for some } \mu \in [0, 1]\}$$

is compact. The homotopy invariance property of the degree asserts that $\deg(G(\cdot, \mu))$ does not depend on μ, provided that G is admissible.

Any open subset N of a manifold M is still a manifold, so the degree of the restriction of g to N makes sense provided that g is admissible on N, i.e. $\{q \in N: g(q) = 0\}$ is compact. The degree of such a restriction will be denoted by $\deg(g, N)$. The additivity property of the degree asserts that if N_1 and N_2 are open subsets of M and g is admissible in N_1, N_2 and $N_1 \cap N_2$, then

$$\deg(g, N_1 \cup N_2) = \deg(g, N_1) + \deg(g, N_2) - \deg(g, N_1 \cap N_2).$$

Lemma 1.3 below relates the degree of a tangent vector field on M with the degree of the associated second order vector field on the tangent bundle $T(M)$. As already observed M is considered as a submanifold of $T(M)$.

LEMMA 1.3. *Let $g: M \to \mathbb{R}^n$ be a tangent vector field on M and let $\hat{g}: T(M) \to \mathbb{R}^n \times \mathbb{R}^n$, $\hat{g}(q, v) = (v, r(q, v) + g(q))$, be the second order vector field associated to g. Then, given an open set U of $T(M)$, $\hat{g}$ is admissible on U if and only if g is admissible on $U \cap M$ and*

$$\deg(\hat{g}, U) = (-1)^m \deg(g, U \cap M), \qquad m = \dim M.$$

Proof. Since the reactive force $r(q, v)$ vanishes whenever v is zero, one has $\hat{g}(q, v) = (0, 0)$ if and only if $v = 0$ and $g(q) = 0$. So, as claimed, $\hat{g}$ is admissible on U if and only if g is admissible on $U \cap M$ and, due to the excision property of the global degree, we may suppose $U = T(U \cap M)$. Moreover, since $U \cap M$ is manifold, we may assume without loss of generality that $M = U \cap M$. Now join g, by means of an admissible homotopy, to a smooth vector field $h: M \to \mathbb{R}^n$ possessing only isolated zeros. Let $\hat{h}: T(M) \to \mathbb{R}^n \times \mathbb{R}^n$ be the second order tangent vector field associated to h as above. Clearly, the homotopy joining g and h induces an admissible homotopy between $\hat{g}$ and $\hat{h}$. Therefore, it suffices to prove that

$$\deg(h) = (-1)^m \deg(\hat{h}).$$

So, as previously observed, it is enough to compute the sum of the indices at the zeros of h and $\hat{h}$ respectively. To this end, let q be an isolated zero of h (thus, $(q, 0)$ is an isolated zero of $\hat{h}$). Without loss of generality we may also assume that q is nondegenerate; that is, the derivative dh_q, considered as a linear transformation of $T_q(M)$ to itself, is nonsingular. As it is known (see e.g. [**M**]), the index of h at q [resp. of $\hat{h}$ at $(q, 0)$] is the sign of the determinant of the derivative dh_q [resp. $d\hat{h}_{(q,0)}$]. Now, by reducing to $\mathbb{R}^m$ [resp. $\mathbb{R}^m \times \mathbb{R}^m$] by means of suitable diffeomorphisms, and by computing the determinants of the Jacobian matrices associated to the derivatives of the vector fields which correspond to h and $\hat{h}$ under such diffeomorphisms, it is not hard to verify the equality

$$\det dh_q = (-1)^m \det d\hat{h}_{(q,0)}.$$

Consequently, the assertion of the lemma holds locally in a neighborhood of any isolated zero and so, to complete the proof, it suffices to take the sum over the (finite number of) zeros of the computed indices. □

The following global connectivity result will play a crucial role in the next section. From the abstract point of view the locally compact metric space Y may represent, for example, the set of solutions of a given equation $\Psi(x) = 0$, where Ψ is a map from an open subset of Banach spaces E into a Banach space F (in many cases Ψ is a nonlinear Fredholm operator of index 1). The distinguished compact subset Y_0 of Y represents the set of trivial solutions of the given equation. Given $C \subset Y$, by a bifurcation point for the pair (Y, C) we mean a point of C which lies in the closure of the set $Y \backslash C$. Lemma 1.4 asserts that if any pair (Y, C), where C is a compact subset of Y containing Y_0, has a bifurcation point, then $Y \backslash Y_0$ contains a global bifurcating branch emanating from Y_0. That is, a connected component of $Y\backslash Y_0$ whose closure in Y intersects Y_0 and is not contained in any compact subset of Y.

LEMMA 1.4. *Let Y be a locally compact metric space and let Y_0 be a compact subset of Y. Assume that any compact subset of Y containing Y_0 has nonempty boundary. Then $Y\backslash Y_0$ contains a not relatively compact component whose closure (in Y) intersects Y_0.*

Proof. By the assumption it follows immediately that the space Y is not compact. Let us adjoin to Y a point $\{\infty\}$ and define a Hausdorff topology on $\widehat{Y} = Y \cup \{\infty\}$ by taking the complements of the compact sets as (open) neighborhoods of ∞. Our assertion is now equivalent to proving the existence of a connected subset of $\widehat{Y}\backslash(Y_0 \cup \{\infty\})$ whose closure contains $\{\infty\}$ and intersects Y_0. Suppose such a connected set does not exist. Then, since $\widehat{Y}$ is a compact Hausdorff space, by a well-known point set topology result (see e.g. [A] and references therein), Y_0 and $\{\infty\}$ are separated in $\widehat{Y}$, i.e. there exist two compact subsets C_0, C_∞ of $\widehat{Y}$ such that $Y_0 \subset C_0$, $\infty \in C_\infty$, $C_0 \cap C_\infty = \varnothing$, $C_0 \cup C_\infty = \widehat{Y}$. So, C_0 is a compact subset of Y containing Y_0 with empty boundary, a contradiction. Therefore, the existence of the required connected set is proved. □

2. Global continuation and bifurcation. Let $C^1_T(M)$ denote the metric subspace of the Banach space $C^1_T(\mathbb{R}^n)$ of all T-periodic C^1 maps $x\colon \mathbb{R} \to M$. Observe that this space is not necessarily complete, unless M is a closed submanifold $\mathbb{R}^n$. However, due to the fact that M is locally compact, one can prove that it is always locally complete.

Consider now the parametrized motion equation

$$\ddot{x}_p(t) = \lambda f(t, x(t), \dot{x}(t)), \qquad \lambda \geq 0, \tag{2.1}$$

where $f\colon \mathbb{R} \times T(M) \to \mathbb{R}^n$ is a T-periodic continuous active force on M. An element $(\lambda, x) \in [0, +\infty) \times C^1_T(M)$ will be called a *solution pair* of the above forced equation provided that x is a (clearly T-periodic) solution of (2.1).

Denote by X the subset of $[0, +\infty) \times C^1_T(M)$ of all solutions pairs of (2.1). It is not hard to show that X is closed in $[0, +\infty) \times C^1_T(M)$. Moreover, because of Ascoli's theorem, X is locally precompact (i.e. locally totally bounded). Therefore, since X is a closed subset of a locally complete space, it is in fact locally compact. This fact will turn out to be useful in the sequel.

Since any element q of M is an equilibrium point of (2.1) corresponding to the value $\lambda = 0$ of the parameter, the manifold M can be thought of as a subset of X just by considering the embedding which assigns to any $q \in M$ the trivial solution pair $(0, \hat{q})$, where $\hat{q}: \mathbb{R} \to C^1_T(M)$ denotes the constant map $t \mapsto q$. We point out that, despite the fact that $[0, +\infty) \times C^1_T(M)$ may not be closed in $[0, +\infty) \times C^1_T(\mathbb{R}^n)$, M is always closed in $[0, +\infty) \times C^1_T(M)$, as well as in X. We will say that M is the *manifold of the trivial solutions* of (2.1), or, simply, the *trivial manifold* of X. Consequently, $X \backslash M$ will be called the set of *nontrivial solutions* of (2.1), or the nontrivial subset of X. With this distinction in mind a trivial solution $q \in M$ will be called a *bifurcation point* of (2.1) if it lies in the closure of $X \backslash M$. We observe that this definition does not depend on where the closure is taken: in X, in $[0, +\infty) \times C^1_T(M)$ or in $[0, +\infty) \times C^1_T(\mathbb{R}^n)$.

Clearly, not all solution pairs of the form $(0, x)$ are necessarily trivial. In fact, any closed geodesic x whose speed u satisfies the condition $uT = jL$, where L is the length of x and j is any positive integer, is a T-periodic solution of (2.1) corresponding to $\lambda = 0$. However, in the slice

$$X_0 = \{x \in C^1_T(M): (0, x) \in X\}$$

of X, the trivial solutions are separated from the nontrivial ones (and, consequently, any nontrivial solution (λ, x) sufficiently close to a bifurcation point must have $\lambda > 0$). To see this observe that the solutions of the inertial equation $\ddot{x}_p(t) = 0$ are the geodesics of M, and there are no (nontrivial) closed geodesics too close to a given point $q \in M$, in a Riemannian manifold. This well-known fact can also be regarded as a consequence of Lemma 1.1, since, given any nonconstant closed geodesic γ, we must have $Kd \geq 1$, where K is the maximum of the curvature along γ and d is its diameter. By the definition of the embedding $M \hookrightarrow X$, however, any solution pair (λ, x) with $\lambda > 0$ is necessarily nontrivial.

We give now a necessary condition for a point $q \in M$ to be a bifurcation point.

THEOREM 2.1. *Let M be a boundaryless m-dimensional smooth manifold in $\mathbb{R}^n$ and let $f: \mathbb{R} \times T(M) \to \mathbb{R}^n$ be a T-periodic continuous active force on M. If $q \in M$ is a bifurcation point for the parametrized forced second order equation*

$$\ddot{x}_p(t) = \lambda f(t, x(t), \dot{x}(t)), \qquad \lambda \geq 0,$$

then the average force is zero at q. *That is*

$$\bar{f}(q) = \frac{1}{T}\int_0^T f(t, q, 0)\, dt = 0.$$

Proof. Let $q \in M$ be a bifurcation point for the equation (2.1). Then there exists a sequence $\{(\lambda_j, x_j)\}$ of nontrivial solution pairs such that $\lambda_j \to 0$, $x_j(t) \to q$ uniformly and $\dot{x}_j(t) \to 0$ uniformly. Integrating from 0 to T the equality

$$\ddot{x}_j(t) = r(x_j(t), \dot{x}_j(t)) + \lambda_j f(t, x_j(t), \dot{x}_j(t)), \qquad t \in \mathbb{R},$$

we get

$$0 = \dot{x}_j(T) - \dot{x}_j(0) = \int_0^T r(x_j(t), \dot{x}_j(t))\, dt + \lambda_j \int_0^T f(t, x_j(t), \dot{x}_j(t))\, dt.$$

Because of the uniform convergence of $\{x_j(t)\}$ and $\{\dot{x}_j(t)\}$ there exists a compact subset of $T(M)$ containing $(x_j(t), \dot{x}_j(t))$ for all $j \in \mathbb{N}$ and $t \in \mathbb{R}$. So, one can find two positive constants F and K such that

$$|f(t, x_j(t), \dot{x}_j(t))| \le F, \quad |r(x_j(t), \dot{x}_j(t))| \le K|\dot{x}_j(t)|^2$$

for all $t \in \mathbb{R}$ and $j \in \mathbb{N}$. Hence, we get

$$\lambda_j \left| \int_0^T f(t, x_j(t), \dot{x}_j(t))\, dt \right| \le \int_0^T K|\dot{x}_j(t)|^2, \qquad \forall j \in \mathbb{N}.$$

Let $d_j = \max\{|x_j(t_1) - x_j(t_2)|,\ t_1, t_2 \in \mathbb{R}\}$. Since $d_j \to 0$ as $j \to \infty$, one has $Kd_j < 1$ for j sufficiently large, and so, by Lemma 1.1,

$$|\dot{x}_j(t)|^2 \le \lambda_j F d_j/(1 - Kd_j), \quad \text{for all } t \in \mathbb{R}.$$

Therefore

$$\left| \int_0^T f(t, x_j(t), \dot{x}_j(t))\, dt \right| \le KTFd_j/(1 - Kd_j).$$

So, we finally obtain

$$\bar{f}(q) = \frac{1}{T}\int_0^T f(t, q, 0)\, dt = \lim_{j\to\infty} \frac{1}{T}\int_0^T f(t, x_j(t), \dot{x}_j(t))\, dt = 0. \quad \square$$

In what follows, by a *bifurcating branch* for (2.1) we mean a connected component of $X \backslash M$ whose closure in X (or, equivalently, in $[0, \infty) \times C^1_T(M)$) intersects M in a compact set. A *global bifurcating* *branch* is just a bifurcating branch which is not relatively compact in X. More generally, given any subset C of M, we say that a bifurcating branch *emanates* from C if its closure intersects some neighborhood of C in a nonempty compact subset of C, and by a *C-global bifurcating branch* we mean a bifurcating branch emanating from C whose closure in X is either noncompact or meets M outside C (or both). Observe that a subset of X which is not relatively compact cannot be contained in any bounded complete subset of $[0, \infty) \times C^1_T(M)$, since, because of Ascoli's theorem, any bounded subset of X is actually totally bounded. So, in particular, if M is compact or, more generally, closed in $\mathbb{R}^n$, the fact that $[0, \infty) \times C^1_T(M)$ is a complete metric space implies that any global bifurcating branch must be unbounded.

We are now in a position to state our main result on the existence of bifurcating branches.

THEOREM 2.2. *Let M be a boundaryless m-dimensional smooth manifold in $\mathbb{R}^n$ and let $f\colon \mathbb{R}\times T(M) \to \mathbb{R}^n$ be a T-periodic continuous active force on M. Denote by $\overline{f}\colon M \to \mathbb{R}^n$ the autonomous tangent vector field*

$$\overline{f}(q) = \frac{1}{T}\int_0^T f(t, q, 0)\, dt.$$

Let N be an open subset of M. Assume that the global degree $\deg(\overline{f}, N)$ *of $\overline{f}$ on N is defined and nonzero. Then the parametrized forced second order equation*

$$\ddot{x}_p(t) = \lambda f(t, x(t), \dot{x}(t)), \qquad \lambda \geq 0,$$

has an N-global bifurcating branch.

The proof of the above theorem needs some preliminary lemmas. Assume for the moment the force f to be smooth and consider the family of equations

$$\ddot{x}_p(t) = \lambda(\mu f(t, x(t), \dot{x}(t)) + (1-\mu)\overline{f}(x(t))), \qquad \mu \in [0, 1],\ \lambda \geq 0. \tag{2.2}$$

Denote

$$D = \{(\lambda, q, v) \in [0, \infty) \times T(M)\colon \text{ the solution } x(\cdot) \text{ of (2.2) satisfying } x(0) = q,\ \dot{x}(0) = v \text{ is defined in } [0, T] \text{ for all } \mu \in [0, 1]\}$$

and let $H: D \times [0, 1] \to T(M)$ be the operator which associates to any (λ, q, v, μ) the values $x(T)$ and $\dot{x}(T)$ of the solution $x(\cdot)$ of (2.2) with initial conditions (q, v). It can be shown (see e.g. [L]) that D is an open set (clearly containing M) and that H is smooth in $D \times [0, 1]$.

The following lemma shows that if U is a suitable open subset of $T(M)$ and $\lambda > 0$ is sufficiently small, then the Poincaré T-translation operator $H_\mu^\lambda: U \to T(M)$, given by $H_\mu^\lambda(q, v) = H(\lambda, q, v, \mu)$, is defined and its fixed point index does not depend on μ. A detailed exposition of the fixed point index theory can be found, for instance, in **[Br]**, **[G]** and **[N]**.

LEMMA 2.1. *Let U be a relatively compact open subset of $T(M)$ which satisfies the following properties*:

(1) $\overline{f}(q) \neq 0$ *for all* $(q, 0)$ *in the boundary* ∂U *of* U;
(2) $\exp_q(Tv)$ *is defined for all* $(q, v) \in \overline{U}$ *and* $\exp_q(Tv) \neq q$ *if* $v \neq 0$.

Then there exists $\varepsilon > 0$ such that, for all $\lambda \in (0, \varepsilon)$, the homotopy $\{H_\mu^\lambda\}_{\mu \in [0,1]}$ is well defined in U and admissible for the fixed point index theory. Consequently, for any $\lambda \in (0, \varepsilon)$ the two T-translation operators H_0^λ and H_1^λ have the same index in U.

Proof. Since U is relatively compact and D is open, the assumption (2) implies, in particular, that for λ sufficiently small the slice

$$D_\lambda = \{(q, v) \in T(M): (\lambda, q, v) \in D\}$$

contains the closure, $\overline{U}$, of U. So, for such λ's, the homotopy

$$H^\lambda: (q, v; \mu) \mapsto H(\lambda, q, v, \mu)$$

is defined in $\overline{U} \times [0, 1]$. Let us show that there exists $\varepsilon > 0$ such that, for $0 < \lambda < \varepsilon$, H^λ is an admissible homotopy (for the fixed point index theory). More precisely, we shall prove that there exists $\varepsilon > 0$, such that there are no fixed points of $H^\lambda(\cdot, \cdot, \mu)$ on the boundary, ∂U, of U, for all $\mu \in [0, 1]$ and $\lambda \in (0, \varepsilon)$. Assume this is not the case. Hence there exists a sequence $\{(\lambda_j, q_j, v_j, \mu_j)\}$ in $D \times [0, 1]$ such that $\lambda_j \to 0$, $\lambda_j > 0$, $(q_j, v_j) \in \partial U$, $\mu_j \in [0, 1]$ and

$$H(\lambda_j, q_j, v_j, \mu_j) = (q_j, v_j).$$

Without loss of generality, we may assume $(q_j, v_j) \to (q_0, v_0) \in \partial U$ and $\mu_j \to \mu_0$. Since, by assumption (2), the solutions of the inertial equation $\ddot{x}_p(t) = 0$ are defined in the whole interval $[0, T]$, by well-known properties of differential equations the sequence $\{x_j\}$ of the solutions of the problems

$$\ddot{x}_p(t) = \lambda_j(\mu_j f(t, x(t), \dot{x}(t)) + (1-\mu_j)\overline{f}(x(t))), \qquad x(0) = q_j, \dot{x}(0) = v_j, j \in \mathbb{N},$$

converges in $C^1_T(M)$ to the solution $x(\cdot)$ of the inertial equation $\ddot{x}_p(t) = 0$ starting from (q_0, v_0). Now, if $v_0 = 0$, since $\{x_j\}$ converges to x in $C^1_T(M)$, as in the proof of Theorem 2.1, we obtain $\overline{f}(q_0) = 0$, contradicting the assumption (1). Otherwise, if $v_0 \neq 0$, the pair (q_0, v_0) turns out to be a starting point of a T-periodic nontrivial closed geodesic, and this is again impossible because of the hypothesis (2). □

The following result shows that if $U \subset T(M)$ is a suitable open set and $\lambda > 0$ is sufficiently small, then the fixed point index of the T-translation operator associated to the autonomous equation

$$\ddot{x}_p(t) = \lambda \overline{f}(x(t)), \qquad x(t) \in M,$$

is related to the global degree of the average force $\overline{f}$ on the open subset $U \cap M$ of M.

LEMMA 2.2. *Let U be as in Lemma 2.1. Then there exists $\delta > 0$ such that, for all $\lambda \in (0, \delta)$,*

$$\operatorname{ind}(H_0^\lambda, U) = (-1)^m \deg(\overline{f}, U \cap M),$$

where m is the dimension of M.

Proof. The proof of Lemma 2.1 shows that as $\lambda \to 0^+$ the fixed points of H_0^λ approach the compact set $Y_0 = \{q \in U \cap M : \overline{f}(q) = 0\}$. So, by the excision properties of the index and the global degree, we may assume that U is an open set of the type

$$U = \{(q, v) \in T(M) : q \in V, \ |v| < \rho\},$$

where V is a relatively compact subset of M and $\rho > 0$.

Consider in M the second order autonomous differential equation

$$\ddot{x}_p(t) = \overline{f}(x(t)), \qquad x(t) \in M, \tag{2.3}$$

where $\overline{f}: M \to \mathbb{R}^n$ is the average force vector field associated to f.

Denote

$$A = \{(\tau, q, v) \in \mathbb{R} \times T(M) : \text{ the solution of (2.3) which satisfies } x(0) = q, \ \dot{x}(0) = v \text{ is continuable at least to } t = \tau\},$$

and let $A_\tau = \{(q, v) \in T(M): (\tau, q, v) \in A\}$ be the slice of A at τ.

Let $\Phi_\tau: A_\tau \to T(M)$ be the translation operator which associates to any $(q, v) \in A_\tau$ the values $x(\tau)$ and $\dot{x}(\tau)$ of the solution $x(\cdot)$ of (2.3) satisfying $x(0) = q$, $\dot{x}(0) = v$. It can be shown (see e.g. **[L]**) that A is an open set and, since f is smooth, so is the flow operator $\Phi: A \to T(M)$, given by $(\tau, q, v) \mapsto \Phi_\tau(q, v)$.

We shall prove first that, for λ sufficiently small, the two operators H_0^λ and $\Phi_{\sqrt{\lambda}T}$ are both defined on U and have the same fixed point index. To accomplish this, given $\lambda > 0$, let $x(\cdot)$ be the solution of

$$\begin{cases} \ddot{x}(t) = r(x(t), \dot{x}(t)) + \overline{f}(x(t)), & 0 \le t \le \sqrt{\lambda}T, \\ x(0) = q, \\ \dot{x}(0) = v/\sqrt{\lambda}. \end{cases}$$

By setting $z(s) = x(s\sqrt{\lambda})$ we obtain

$$\begin{cases} \ddot{z}(s) = r(z(s), \dot{z}(s)) + \lambda\overline{f}(z(s)), & 0 \le s \le T, \\ z(0) = q, \\ \dot{z}(0) = v, \end{cases}$$

so $(z(T), \dot{z}(T)) = (x(\sqrt{\lambda}T), \sqrt{\lambda}\dot{x}(\sqrt{\lambda}T))$. This shows that if $(q, v) \in D_\lambda$ then $(q, v/\sqrt{\lambda}) \in A_{\sqrt{\lambda}T}$ and $H_0^\lambda = \sigma_\lambda \circ \Phi_{\sqrt{\lambda}T} \circ \sigma_\lambda^{-1}$, where $\sigma_\lambda: T(M) \to T(M)$ denotes the diffeomorphism $\sigma_\lambda(q, v) = (q, \sqrt{\lambda}v)$. Now, given $\varepsilon > 0$ as in Lemma 2.1, if $\lambda \in (0, \varepsilon)$ the index of H_0^λ in U is well defined. Consequently, by the commutativity property of the index, $\operatorname{ind}(\Phi_{\sqrt{\lambda}T}, \sigma_\lambda^{-1}(U))$ is well defined too and

$$\operatorname{ind}(\Phi_{\sqrt{\lambda}T}, \sigma_\lambda^{-1}(U)) = \operatorname{ind}(H_0^\lambda, U).$$

Let us prove now that, for $\lambda > 0$ small, one has

$$\operatorname{ind}(\Phi_{\sqrt{\lambda}T}, \sigma_\lambda^{-1}(U)) = \operatorname{ind}(\Phi_{\sqrt{\lambda}T}, U).$$

By the choice of U we get $U \subset \sigma_\lambda^{-1}(U)$ for all $0 < \lambda < 1$. So, by the excision property of the index it is enough to show that for λ sufficiently small the fixed points of $\Phi_{\sqrt{\lambda}T}$ which are in $\sigma_\lambda^{-1}(U)$ are actually in U. Assume this is not the case. Then there exists a sequence $\{(q_j, w_j)\}$ of fixed points for $\Phi_{\sqrt{\lambda_j}T}$ in $\sigma_{\lambda_j}^{-1}(U)$ with $\lambda_j \to 0^+$ and $|w_j| \ge \rho > 0$. Let x_j denote the solution of (2.3) starting from (q_j, w_j) and observe that $z_j(s) = x_j(s\sqrt{\lambda_j})$ is a T-periodic solution of the differential equation

$$\ddot{z}(s) = r(z(s), \dot{z}(s)) + \lambda_j\overline{f}(z(s)), \qquad 0 \le s \le T,$$

starting from $(q_j, v_j) = (q_j, \sqrt{\lambda_j}w_j) \in U$.

Since U is a relatively compact subset of $T(M)$, we may assume, without loss of generality, that $(q_j, v_j) \to (q, v) \in \overline{U}$. Now, due to the assumption (2) of Lemma 2.1, the solution of the inertial equation starting from (q, v) is defined in the compact interval $[0, T]$. So the sequence $\{z_j\}$ converges in $C^1_T(M)$ to the solution of the inertial equation

$$\ddot{z}(s) = r(z(s), \dot{z}(s))$$

starting from (q, v). This implies $v = 0$, since otherwise (q, v) would be a starting point of a nontrivial T-periodic geodesic, contradicting the assumption (2) of Lemma 2.1. Consequently, the sequences $\{z_j\}$ and $\{\dot{z}_j\}$ are uniformly bounded. So, there exist positive constants K and F such that, for all $j \in \mathbb{N}$ and $s \in [0, 1]$, $k(z_j(s), \dot{z}_j(s)) \leq K$ and $|\overline{f}(z_j(s))| \leq F$. Let

$$d_j = \max\{|z_j(t_1) - z_j(t_2)|, t_1, t_2 \in [0, 1]\}.$$

Since $d_j \to 0$ as $j \to \infty$, one has $d_j < 1/K$ for j sufficiently large, and, so, by Lemma 1.1,

$$(\sqrt{\lambda}_j|w_j|)^2 \leq \max|\dot{z}_j(s)|^2 \leq \lambda_j F d_j/(1 - K d_j).$$

Therefore, $w_j \to 0$ as $j \to \infty$, contradicting the assumption $|w_j| \geq \rho > 0$. Thus, as claimed, for $\lambda > 0$ sufficiently small, we have

$$\operatorname{ind}(H_0^\lambda, U) = \operatorname{ind}(\Phi_{\sqrt{\lambda}T}, U).$$

Let $\hat{f}: T(M) \to \mathbb{R}^n \times \mathbb{R}^n$, $\hat{f}(q, v) = (v, r(q, v) + \overline{f}(q))$, denote the second order tangent vector field associated to $\overline{f}$ and observe that the set of zeros of $\hat{f}$ in U coincides with the set of zeros of $\overline{f}$ in $U \cap M$; so, by assumption, it is a compact set. Therefore, recalling the equivalent definition of global degree of a vector field given in **[FP2]**, there exists $\alpha > 0$ such that $\operatorname{ind}(\Phi_\tau, U)$ is defined for $0 < \tau < \alpha$ and

$$\operatorname{ind}(\Phi_\tau, U) = \deg(-\hat{f}, U).$$

The assertion now follows directly from Lemma 1.3, noting that, since the manifold $T(M)$ is even-dimensional, one has $\deg(-\hat{f}, U) = \deg(\hat{f}, U)$. □

In order to prove Theorem 2.2 we need, in the smooth case, a preliminary finite dimensional investigation (Lemma 2.3 below) of the structure of the set of *starting points* of equation (2.1), i.e. of those elements $(\lambda, q, v) \in [0, \infty) \times T(M)$ for which the solution $x: \mathbb{R} \to M$ of (2.1) corresponding to the value λ of the parameter and satisfying the initial condition $x(0) = q$, $\dot{x}(0) = v$ is T-periodic. In what follows, such a solution will be simply called the *solution starting from* (λ, q, v). Let us denote by $S \subset [0, \infty) \times T(M)$ the set of starting points of (2.1). Observe that S may not be a closed subset of $[0, \infty) \times T(M)$. This is due to the fact that the solution starting from a given point (λ, q, v) of $[0, \infty) \times T(M)$ is not necessarily defined on the whole interval $[0, T]$. However, it turns out that S is locally closed and, consequently, also locally compact, since so is $[0, \infty) \times T(M)$. To see this observe that the solutions starting from the points of S are defined for all $t \in \mathbb{R}$; so, because of known properties of ordinary differential equations, any $(\lambda_0, q_0, v_0) \in S$ admits an open neighborhood B in $[0, \infty) \times T(M)$ such that the solution starting from any point (λ, q, v) in B is defined for all $t \in [0, T]$. The fact that S is locally compact will be crucial in the proof of Lemma 2.3 below, which, in some sense, represents a finite dimensional version of Theorem 2.2. As in the case of solution pairs, the manifold M can be regarded as a closed subset of S (the embedding now is the map $q \mapsto (0, q, 0)$). In this context M is the manifold of the *trivial starting points* and $S \backslash M$ the set of nontrivial ones. Notice also that if a point $q \in M$ belongs to the closure in S of $S \backslash M$, then the continuous dependence of the solutions on the data implies that q is a bifurcation point for the equation (2.1). Conversely, since the map which to any (λ, x) in $[0, \infty) \times C^1_T(M)$ assigns the triple $(\lambda, x(0), \dot{x}(0))$ is continuous, any bifurcation point for (2.1) is in the closure of $S \backslash M$.

LEMMA 2.3. *Let f be a smooth active force on M and let N be an open subset of M with the property that the degree,* $\deg(\bar{f}, N)$, *of the average force $\bar{f}$ on N is defined and nonzero. Then the set $S \backslash M$ of the nontrivial starting points of* (2.1) *contains a connected component whose closure in S intersects N and either is noncompact or hits M outside N.*

Proof. Let Y_0 denote the subset of N consisting of the zeros of the average force $\bar{f}$ and put $Y = (S \backslash M) \cup Y_0$. Observe that Y_0 is a nonempty compact set (recall that $\bar{f}$ is admissible on N and $\deg(\bar{f}, N) \neq 0$). Hence, since S is locally compact and M is closed in S, Y is a nonempty locally compact space.

Let us show that the assertion follows if we can apply Lemma 1.4 to Y and Y_0. Assume, in fact, that $Y \backslash Y_0$ has a connected component Σ whose closure in Y is noncompact and meets Y_0. If also the closure of Σ in S is noncompact we are done. Otherwise, since Σ is not relatively compact in Y, some point q of the closure of Σ in S must belong to the subset $S\backslash Y$ of M. Consequently, q is a bifurcation point for the equation (2.1). This implies, by Theorem 2.1, $\overline{f}(q) = 0$. So q is not in N, since otherwise q would belong to the subset Y_0 of Y.

Assume Lemma 1.4 does not apply to the pair (Y, Y_0). So Y_0 admits a compact open neighborhood C in Y. Consequently, in $[0, \infty) \times T(M)$, we can find an open set W such that $W \cap Y = C$. Let $Z \subset M$ denote the set of zeros of $\overline{f}$ and observe that $Z\backslash Y_0$ is closed in $[0, \infty) \times T(M)$. Therefore we can assume $W \cap Z = Y_0$. Now, the solutions of (2.1) starting from the points of C are defined for any $t \in \mathbb{R}$. Therefore, by well-known properties of ordinary differential equations, we can suppose (shrinking W if necessary) that all solutions of (2.1) starting (at $t = 0$) from any point of W are defined in the whole interval $[0, T]$. In other words we may assume W contained in the open domain E of the operator $H_1: E \to T(M)$ which associates to any $(\lambda, q, v) \in E$ the values $x(T)$ and $\dot{x}(T)$ of the solution $x(\cdot)$ of (2.1) with initial conditions (q, v). Moreover, clearly W can be taken with the additional property that the slice

$$W_\lambda = \{(q, v) \in T(M): (\lambda, q, v) \in W\}, \qquad \lambda \geq 0,$$

is relatively compact and independent of λ in a right neighborhood $[0, \delta)$ of $\lambda = 0$. Finally, by Lemma 1.2 and shrinking W if necessary, one can assume that the slice $U = W_\lambda$, $\lambda \in [0, \delta)$, satisfies the assumptions (1) and (2) of Lemma 2.1.

Given any $\lambda \geq 0$, let $H_1^\lambda: W_\lambda \to T(M)$ denote the Poincaré T-translation operator of (2.1). Clearly, if $\lambda \neq 0$, the compact set

$$C_\lambda = \{(q, v) \in W_\lambda: (\lambda, q, v) \in C\}$$

coincides with the fixed point set of H_1^λ. So, the generalized homotopy property of the index implies that the fixed point index, $\operatorname{ind}(H_1^\lambda, W_\lambda)$, of H_1^λ on the (possibly empty) open set W_λ must be independent of λ. Consequently, since C_λ is empty for λ sufficiently large, this index must be identically zero.

On the other hand, by Lemmas 2.1 and 2.2, when λ is positive and sufficiently small we get

$$\operatorname{ind}(H_1^\lambda, W_\lambda) = \operatorname{ind}(H_1^\lambda, U) = \operatorname{ind}(H_0^\lambda, U) = (-1)^m \deg(\overline{f}, U \cap M).$$

But $\deg(\overline{f}, U \cap M) = \deg(\overline{f}, N) \neq 0$, since, by the choice of W

one has

$$Z \cap U = Z \cap N = Y_0 .$$

This contradiction shows that, in fact, the pair of spaces (Y, Y_0) satisfies the assumptions of Lemma 1.4 and, consequently, our assertion holds true. □

We can now give the

Proof of Theorem 2.2. Assume first that f is smooth. As previously, let us denote by X the locally compact closed subset of $[0, +\infty) \times C^1_T(M)$ of all solution pairs of equation (2.1) and by $S \subset [0, \infty) \times T(M)$ the locally compact set of starting points of (2.1). Define $h: X \to S$ by $h(\lambda, x) = (\lambda, x(0), \dot{x}(0))$. Clearly h is continuous, onto, and, since f is smooth, it is also one-to-one. Moreover, the continuous dependence property from the data of the solutions of differential equations, ensures the continuity of its inverse h^{-1}. Also observe that, according to the identifications $M \hookrightarrow X$ and $M \hookrightarrow S$, h is the identity on M. Let us denote by Σ a connected component of $S \backslash M$ as in Lemma 2.3. Then $\Gamma = h^{-1}(\Sigma)$ is clearly an N-global bifurcating branch.

We now need to remove the smoothness assumption on f. Let Y_0 denote the set of zeros of $\overline{f}$ in N and put $Y = (X \backslash M) \cup Y_0$. As in the proof of Lemma 2.3 we need only to show that the pair (Y, Y_0) satisfies the hypothesis of Lemma 1.4. Assume the contrary. So, we can find a compact relatively open subset C of Y containing Y_0. Consequently, there exists a bounded open subset W of $[0, +\infty) \times C^1_T(M)$ whose intersection with Y coincides with C and such that $\partial W \cap Y = \varnothing$. The fact that $[0, +\infty) \times C^1_T(M)$ is locally complete permits us to choose W with complete closure. Moreover, we can clearly suppose $W \cap M$ to be relatively compact with closure contained in N.

By a well-known approximation result on manifolds (see e.g. **[H]**), we may take a sequence $\{f_j\}$ of T-periodic smooth active forces approximating f on $\mathbb{R} \times T(M)$ uniformly. For each $j \in \mathbb{N}$, let

$$\overline{f}_j(q) = \frac{1}{T} \int_0^T f_j(t, q, 0)\, dt$$

be the autonomous mean force vector field associated to f_j. Clearly, the sequence $\{\overline{f}_j\}$ converges uniformly to the vector field $\overline{f}$ on M. Consequently, since the closure of $W \cap M$ is a compact subset of N

and $Y_0 \subset W \cap M$, for $j \in \mathbb{N}$ large enough, the zeros of the homotopy $G_j \colon W \cap M \times [0, 1] \to \mathbb{R}^n$, given by

$$(q, \tau) \mapsto \tau \overline{f}_j(q) + (1 - \tau)\overline{f}(q),$$

lie in a compact subset of $W \cap M$. So $\deg(\overline{f}_j, W \cap M)$ is well defined and, by the homotopy invariance property, equals $\deg(\overline{f}, W \cap M)$, which is clearly nonzero, since it coincides, by the excision, with $\deg(\overline{f}, N)$. Hence, by the first part of the proof, any equation

$$\ddot{x}(t) = r(x(t), \dot{x}(t)) + \lambda f_j(t, x(t), \dot{x}(t))$$

admits a $W \cap M$-global bifurcating branch Γ_j. Moreover, since W is bounded and with complete closure, any Γ_j must intersect the complement of W in $[0, \infty) \times C^1_T(M)$. Hence, in particular, for each j, there exists $(\lambda_j, x_j) \in \Gamma_j \cap \partial W$. Now, by definition of solution pair, any function x_j satisfies the corresponding equation

$$\ddot{x}(t) = r(x(t), \dot{x}(t)) + \lambda_j f_j(t, x(t), \dot{x}(t)).$$

Therefore, the sequence $\{\ddot{x}_j\}$ is uniformly bounded in $C^0_T(M)$, so that, by Ascoli's theorem, we may assume $x_j \to x_0$ in $C^1_T(M)$. Without loss of generality, we may also assume $\{\lambda_j\}$ converging to λ_0. Hence, the sequence $\{\ddot{x}_j(t)\}$ converges to the function

$$r(x_0(t), \dot{x}_0(t)) + \lambda_0 f(t, x_0(t), \dot{x}_0(t))$$

uniformly in $\mathbb{R}$. This implies that $x_j \to x_0$ in $C^2_T(M)$ and that x_0 is a T-periodic solution of the differential equation

$$\ddot{x}(t) = r(x(t), \dot{x}(t)) + \lambda_0 f(t, x(t), \dot{x}(t)).$$

Thus, (λ_0, x_0) is a solution pair of (2.1) which clearly belongs to Y if either $\lambda_0 > 0$ or $\lambda_0 = 0$ and x_0 is a nontrivial geodesic (recall that M is identified with the set of trivial geodesics). Otherwise, if $\lambda_0 = 0$ and $x_0 \in M$, as in the proof of Theorem 2.1, we get $x_0 \in Y_0$. Therefore, in any case, $(\lambda_0, x_0) \in \partial W \cap Y$, which is a contradiction. □

The following consequence of Theorem 2.2 shows that given a constrained system M acted on by a T-periodic force f, if the global degree of the averaged force $\overline{f}$ associated to f is defined and nonzero, then, for sufficiently small values of λ, the system admits forced oscillations and these oscillations do not disappear provided that they remain in a compact subset of $T(M)$. This fact is well known in the flat case and is contained in the abstract continuation principle given by J. Mawhin in [**Maw**].

COROLLARY 2.1. *Let* $f: \mathbb{R} \times T(M) \to \mathbb{R}^n$ *be a* T*-periodic active force on* M *such that the global degree* $\deg(\overline{f})$ *of the mean force vector field* $\overline{f}: M \to \mathbb{R}^n$ *is well defined and nonzero. Then there exists* $\varepsilon > 0$ *such that the equation*

$$\ddot{x}_p(t) = \lambda f(t, x(t), \dot{x}(t))$$

admits forced oscillations for all $\lambda \in (0, \varepsilon)$. *Assume moreover that there exist a compact subset* C *of* M *and a constant* $R > 0$ *such that all possible* T*-periodic solutions* $x(\cdot)$ *of the above equation, which correspond to some* $\lambda \in (0, 1]$, *are such that* $x(t) \in C$, $|\dot{x}(t)| \le R$, $\forall t \in \mathbb{R}$. *Then*

$$\ddot{x}_p(t) = f(t, x(t), \dot{x}(t))$$

admits a forced oscillation.

Proof. By applying Theorem 2.2 we conclude that the equation

$$\ddot{x}_p(t) = \lambda f(t, x(t), \dot{x}(t)), \qquad \lambda \ge 0,$$

has a global bifurcating branch Γ of solution pairs $(\lambda, x) \in [0, \infty) \times C^1_T(M)$ which, by Theorem 2.1, emanates from the set $\{q \in M: \overline{f}(q) = 0\}$. The existence of $\varepsilon > 0$ with the required property is due to the fact that Γ is connected and cannot be "vertical", i.e. cannot be entirely contained in $\{0\} \times C^1_T(M)$. In fact, as already observed, nontrivial solution pairs (λ, x) sufficiently close to a bifurcation point must have $\lambda > 0$.

It remains to show that Γ contains a solution of the form $(1, x)$. Suppose not. Thus, the given a priori bounds on the possible forced oscillations of the parametrized equation

$$\ddot{x}_p(t) = \lambda f(t, x(t), \dot{x}(t)), \qquad \lambda \in (0, 1],$$

imply that Γ is contained in a bounded complete subset of $[0, +\infty) \times C^1_T(M)$. This, as already observed, is impossible since Γ is a global bifurcating branch. □

In the particular case when the constraint M is compact with nonzero Euler-Poincaré characteristic and the active force f is the sum of an applied force g of the form $(t, q, v) \mapsto g(t, q)$ plus a frictional force $(t, q, v) \mapsto -\varepsilon v$, $\varepsilon > 0$, from Corollary 2.1 (as well as from Corollary 2.2 below) one gets the existence of forced oscillations (for any value of the parameter $\lambda > 0$). In fact, in this case, by the Poincaré-Hopf Theorem, $\deg(\overline{f}) = \chi(M) \neq 0$. Moreover, the speed of the periodic orbits of (2.1) cannot exceed $\lambda G/\varepsilon$, where

$$G = \max\{|g(t, q)|: t \in \mathbb{R}, \ q \in M\}.$$

This result for dissipative systems was independently obtained by V. Benci and M. Degiovanni in **[BD]** and by M. Furi and M. P. Pera in **[FP3]**.

Another consequence of Theorem 2.2 is the following global result obtained in **[FP4]**.

COROLLARY 2.2. *Let M be compact and assume that the Euler-Poincaré characteristic $\chi(M)$ of M is non-zero. Then the equation* (2.1) *admits an unbounded bifurcating branch.*

Proof. Observe that, since M is a compact boundaryless manifold, then $\deg(\overline{f}) = \chi(M) \neq 0$. So Theorem 2.2 applies yielding the existence of a global bifurcating branch. Now, to get the assertion it suffices to observe that $[0, \infty) \times C_T^1(M)$ is complete and, consequently, Γ must be necessarily unbounded. □

The following straightforward consequence of Theorem 2.2 is in the spirit of a result obtained, for the flat case, by M. Martelli in **[Mar]**. In particular, it gives, in terms of the derivative of the average force $\overline{f}$ at an isolated zero q of $\overline{f}$, a sufficient condition for the existence of a q-global bifurcating branch.

COROLLARY 2.3. *Let q be an isolated zero for the average force $\overline{f}$ and assume that the index of $\overline{f}$ at q is different from zero. Then* (2.1) *has a q-global bifurcating branch. In particular, this occurs if $\overline{f}$ is differentiable at q and $d\overline{f}_q$ is nondegenerate* (*as a linear endomorphism of $T_q(M)$*).

In what follows, given an isolated zero q of $\overline{f}$, we will denote by $C(q)$ the connected component of $\{q\} \cup (X \backslash M)$ containing q. Observe that Corollary 2.3 implies that if the index of $\overline{f}$ at q in nonzero, then $C(q)$ is noncompact. It may happen, however, that for some pair of isolated zeros q_1 and q_2, of index j_1 and j_2 respectively, the union $C(q_1) \cup C(q_2)$ is a compact connected set (clearly, joining q_1 with q_2). The following result shows that, in this case, one has $j_1 + j_2 = 0$.

COROLLARY 2.4. *Let Y_0 be a finite family of isolated zeros for $\overline{f}$. Assume that the union $C(Y_0)$ of all sets $C(q)$, with $q \in Y_0$, is compact. Then the sum of the indices of $\overline{f}$ at the points of Y_0 is zero.*

Proof. Let Z denote the set of zeros of $\overline{f}$ and let N be an open subset of M such that $N \cap Z = Y_0$. Since the sum of the indices of $\overline{f}$ at the points of Y_0 equals the degree of $\overline{f}$ in N, we have to show that $\deg(\overline{f}, N) = 0$. Assume the contrary. Then (2.1) admits an N-global bifurcating branch Γ. Due to Theorem 2.1, the closure of Γ in X must contain a point of Y_0, so $\Gamma \subset C(Y_0)$. Consequently, the closure of Γ is a compact subset of $C(Y_0)$, and this is impossible, since, in this case, the closure of Γ could not meet $M \backslash N$ (observe that, by definition, $C(Y_0)$ does not intersect $M \backslash Y_0$). □

Results analogous to the previous ones can be obtained, with only minor changes in the proofs, for equations of the form

$$\ddot{x}_p(t) = \lambda f_1(\lambda, t, x(t), \dot{x}(t)), \qquad \lambda \geq 0,$$

with $f_1: [0, \infty) \times \mathbb{R} \times T(M) \to \mathbb{R}^n \times \mathbb{R}^n$ a T-periodic continuous tangent vector field depending on the parameter $\lambda \geq 0$, provided that in all statements one replaces the vector field $\overline{f}$ by $\overline{f}_1: M \to \mathbb{R}^n$ given by

$$\overline{f}_1(q) = \frac{1}{T} \int_0^T f_1(0, t, q, 0)\, dt.$$

Observe for instance that any smooth vector field $(\lambda, t, q, v) \mapsto h(\lambda, t, q, v)$ such that $h(0, t, q, v) = 0$ for all $(t, q, v) \in \mathbb{R} \times T(M)$ can be written in the form $\lambda f_1(\lambda, t, q, v)$ by taking

$$f_1(\lambda, t, q, v) = \int_0^1 \frac{\partial h}{\partial \lambda}(s\lambda, t, q, v)\, ds.$$

REFERENCES

[A] J. C. Alexander, *A primer on connectivity*, Proc. Conf. on Fixed Point Theory 1980, ed. E. Fadell and G. Fournier, Lecture Notes in Math., vol. 886, Springer Verlag, Berlin, 1981, 455–483.

[BD] V. Benci and M. Degiovanni, *Periodic solutions of dissipative dynamical systems*, Proc. Conf. on "Variational Problems", Berestycki, Coron and Ekeland editors, Birkhaüser, Basel, New York, 1990.

[B] W. Blaschke, *Vorlesungen uber Differentialgeometrie* I, 3d. ed., Berlin 1929; New York, 1945.

[BC] F. Brickell and R. S. Clark, *Differentiable Manifolds*, Van Nostrand Reinhold Company, London, 1979.

[Br] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, Glenview, Illinois, 1971.

[CMZ] A. Capietto, J. Mawhin and F. Zanolin, *A continuation approach to superlinear periodic boundary value problems*, J. Differential Equations, **88** (1990), 347–395.

[FP1] M. Furi and M. P. Pera, *Co-bifurcating branches of solutions for nonlinear eigenvalue problems in Banach spaces*, Ann. Mat. Pura Appl., **135** (1983), 119–132.

[FP2] ——, *A continuation principle for forced oscillations on differentiable manifolds*, Pacific J. Math., **121** (1986), 321–338.

[FP3] ——, *On the existence of forced oscillations for the spherical pendulum*, Boll. Un. Mat. Ital., **4-B** (1990), 381–390.

[FP4] ——, *A continuation principle for the forced spherical pendulum*, Fixed Point Theory and Applications, Théra M. A. and Baillon J.-B. Editors, Pitman Research Notes in Math. 252, Longman Scientific & Technical, 1991, 141–154.

[FP5] ——, *The forced spherical pendulum does have forced oscillations*, Delay Differential Equations and Dynamical Systems, S. Busenberg and M. Martelli Editors, Lecture Notes in Math., vol. 1475, Springer Verlag, Berlin, 1991, 176–183.

[G] A. Granas, *The Leray-Schauder index and the fixed point theory for arbitrary ANR's*, Bull. Soc. Math. de France, **100** (1972), 209–228.

[H] M. W. Hirsh, *Differential Topology*, Graduate Texts in Math., Vol. 33, Springer Verlag, Berlin, 1976.

[L] S. Lang, *Introduction to Differentiable Manifolds*, John Wiley & Sons Inc., New York, 1966.

[Mar] M. Martelli, *Large oscillations of forced nonlinear differential equations*, Contemp. Math., vol. 21, Amer. Math. Soc., Providence, RI, 1983, 151–157.

[Maw] J. Mawhin, *Equivalence theorems for nonlinear operator equations and coincidence degree for some mappings in locally convex topological vector spaces*, J. Differential Equations, **12** (1972), 610–636.

[M] J. W. Milnor, *Topology from the Differentiable Viewpoint,* Univ. Press of Virginia, Charlottesville, 1965.

[N] R. D. Nussbaum, *The Fixed Point Index and Some Applications*, Sem. Math. Sup., Vol. 94, Université de Montréal, 1985.

[T] A. J. Tromba, *The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray-Schauder degree*, Adv. in Math., **28** (1978), 148–173.

Received December 2, 1991 and in revised form, March 24, 1992.

Università di Firenze
Via S. Marta, 3
50139 Florence, Italy

and

Università di Siena
Via del Capitano, 15
53100 Siena, Italy