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KNOTTING TRIVIAL KNOTS AND RESULTING KNOT TYPES

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Let (V, K) be a pattern (i.e. V is a standardly embedded solid torus in oriented S^3 and K is a knot in V) and f an orientation preserving emdedding from V into S^3 such that $f(V)$ is knotted.

In this paper answers to the following questions will be given depending upon whether the winding number of K_2 in V is zero or not.

(1) Suppose that K_1 is unknotted and K_2 is knotted in S^3. Can $f(K_1)$ be ambient isotopic to $f(K_2)$ in S^3 for some embedding $f: V \hookrightarrow S^3$?

(2) Suppose that K_1 and K_2 are both unknotted in S^3. How are (V, K_1) and (V, K_2) related if $f(K_1)$ is ambient isotopic to $f(K_2)$ in S^3 for some embedding $f: V \hookrightarrow S^3$?

1. Introduction. Let K be a knot in S^3, which is contained in a standardly embedded solid torus V ($\subset S^3$). Assume that K is not contained in a 3-ball in V. Let f be an orientation preserving embedding from V into S^3 such that $f(C)$ is knotted in S^3, here C denotes a core of V. Then we get a new knot $f(K)$ in S^3 called a satellite knot with a companion knot $f(C)$. The knot K is called a preimage knot and we call the pair (V, K) a pattern (see Figure 1 on the next page).

Throughout this paper for an embedding f from V into S^3, we assume that it is orientation preserving and $f(C)$ is knotted in S^3.

We concern ourselves with the following questions.

(1) Suppose that K_1 is unknotted and K_2 is knotted in S^3. Can $f(K_1)$ be ambient isotopic to $f(K_2)$ in S^3 for some embedding $f: V \hookrightarrow S^3$?

(2) Suppose that K_1 and K_2 are both unknotted in S^3. How are (V, K_1) and (V, K_2) related if $f(K_1)$ is ambient isotopic to $f(K_2)$ in S^3 for some embedding $f: V \hookrightarrow S^3$?

For two knots K_1 and K_2, we write $K_1 \cong K_2$ provided that there exists an orientation preserving self-homeomorphism of S^3 carrying K_1 to K_2 (or equivalently, K_1 and K_2 are ambient isotopic in S^3). For two patterns (V, K_1) and (V, K_2), if there exists an orientation preserving self-homeomorphism h of V sending *longitude* to

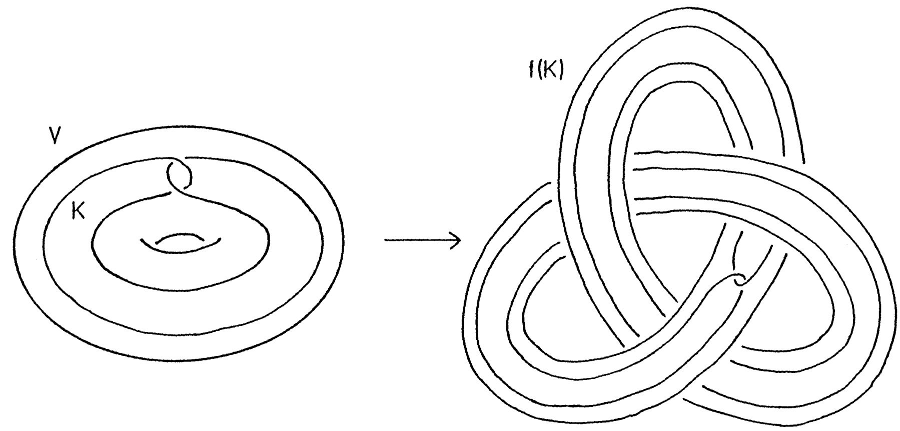


FIGURE 1

±longitude which satisfies $h(K_1) = K_2$, then we write $(V, K_1) \sim (V, K_2)$. In addition if the homeomorphism h sends *longitude* to *longitude*, then we write $(V, K_1) \cong (V, K_2)$. It is known that $(V, K_1) \cong (V, K_2)$ if and only if K_1 and K_2 are ambient isotopic in V. Throughout this paper *longitude* means *preferred longitude.*

The wrapping number of K in V—the minimal geometric intersection number of K with a meridian disk in V—is denoted by $\mathrm{wrap}_V(K)$, and the winding number of K in V—the algebraic intersection number of K with a meridian disk in V—is denoted by $\mathrm{wind}_V(K)$. (We may assume $\mathrm{wind}_V(K) \geq 0$ by considering an appropriate orientation of K.)

Now our main result is stated as follows.

THEOREM 1.1. *Let (V, K_i) $(i = 1, 2)$ be a pattern. Suppose that K_1 is unknotted in S^3 and $\mathrm{wind}_V(K_2) \neq 0$. If $f(K_1) \cong f(K_2)$ in S^3 for some embedding f from V into S^3, then $(V, K_1) \sim (V, K_2)$ holds.*

REMARK 1.2. (1) In this theorem the condition $\mathrm{wind}_V(K_2) \neq 0$ is essential. The example below (Figure 2) demonstrates the necessity of such a condition.

In this example K_1 is unknotted in S^3 and K_2 is knotted (and hence $(V, K_1) \not\sim (V, K_2)$), but $\mathrm{wind}_V(K_2) = 0$. From them, we can obtain the same knot $f(K_1) \cong f(K_2)$.

The modification to recognize that $f(K_1) \cong f(K_2)$ is given by Figure 3.

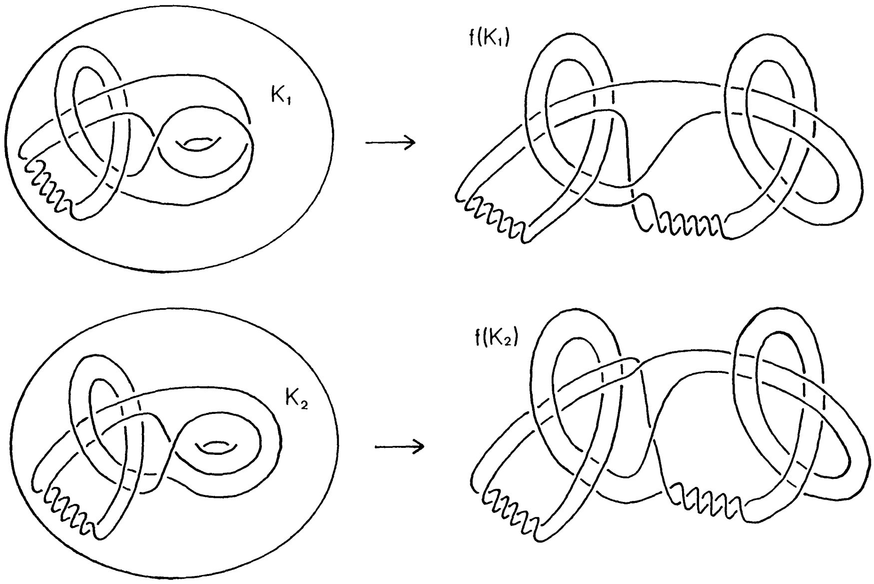


FIGURE 2

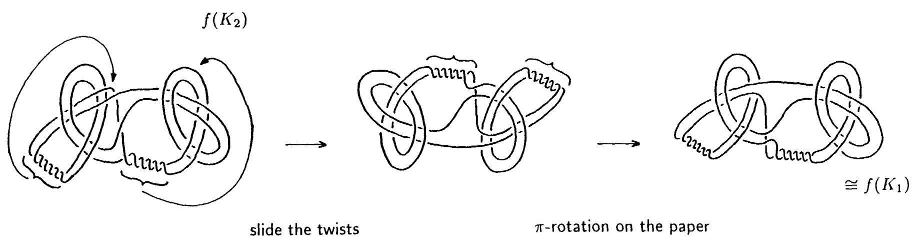


FIGURE 3

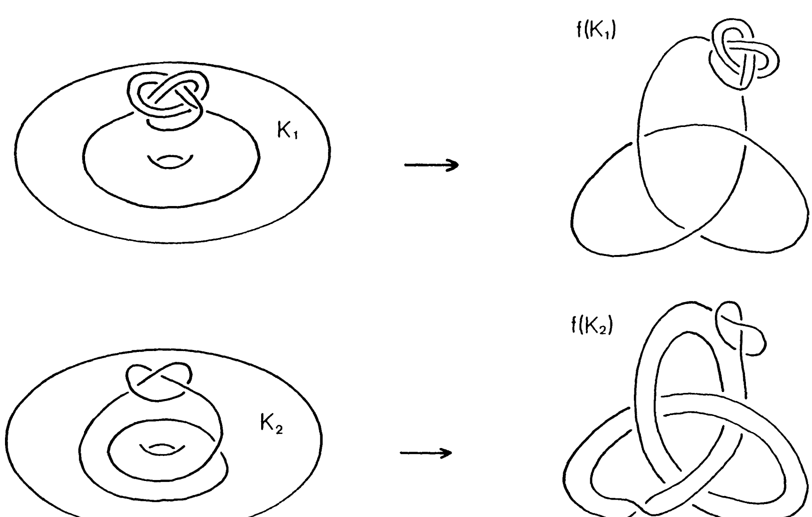


FIGURE 4

(2) If K_1 is knotted, even when $\mathrm{wind}_V(K_2) \neq 0$, it is easy to construct the example such that $(V, K_1) \not\cong (V, K_2)$ but $f(K_1) \cong f(K_2)$ in S^3 (see Figure 4).

As consequences of the Theorem 1.1, we have Corollary 2.6 and Theorem 3.1. By these results together with Remark 1.2 and Theorem 3.3, we can answer the above questions depending upon whether $\mathrm{wind}_V(K_2) = 0$ or not.

Throughout this paper $N(X)$, ∂X and $\mathrm{int}\, X$ denote the tubular neighborhood of X, the boundary of X and the interior of X respectively.

2. Isotopy between satellite knots and equivalence of patterns. To prove Theorem 1.1 we prepare some lemmas and give a necessary condition for a pattern (V, K) so that K is unknotted in S^3.

The next lemma is well known and we omit the proof here (see [7]).

LEMMA 2.1. *Let W be a knotted solid torus in S^3 and K a knot in W with $\mathrm{wrap}_W(K) \neq 0$. Then K is knotted in S^3.*

Consider a nontrivial knot exterior E (i.e. E is homeomorphic to $S^3 - \mathrm{int}\, N(k)$ for some nontrivial knot k in S^3) embedded in V. Since ∂E is compressible in $V - \mathrm{int}\, N(K)$, otherwise $V = (V - \mathrm{int}\, E) \cup E$ has an incompressible torus ∂E, there exists an embedded disk D in $V - \mathrm{int}\, E$ such that ∂D is essential in $\partial(V - \mathrm{int}\, E) = \partial E$. Thus D is a meridian disk for the solid torus $W = S^3 - \mathrm{int}\, E$ and is contained in V. We call the disk D a meridian disk for E in V. The following lemma is a straightforward consequence of Lemma 2.1.

LEMMA 2.2. *Let (V, K) be a pattern such that K is unknotted in S^3 and E a nontrivial knot exterior embedded in $V - K$. Then the algebraic intersection number of K and a meridian disk for E in V is zero* (*see Figure* 5).

Now we shall prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $f(K_1) \cong f(K_2)$ in S^3 for two patterns (V, K_1) and (V, K_2). Then there is an orientation preserving homeomorphism h of S^3 carrying $f(K_1)$ to $f(K_2)$. It suffices to show that by an isotopy of S^3 which leaves $f(K_2) = h(f(K_1))$ fixed, we can modify the homeomorphism h so that $h(S^3 - \mathrm{int}\, f(V)) = S^3 - \mathrm{int}\, f(V)$. To do this we need the next lemma, which was proved by H. Schubert in [**8**], but since we rely heavily on this theorem, we give a proof here using torus decompositions.

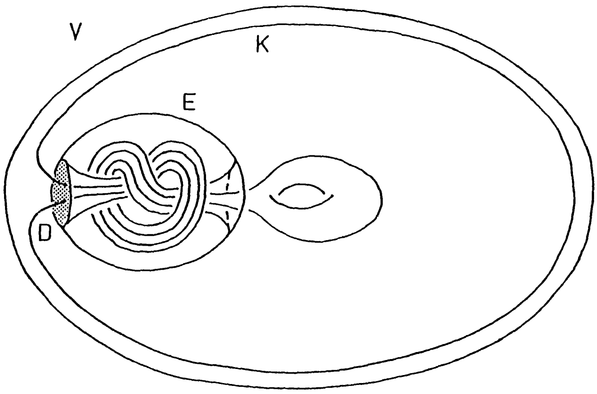


FIGURE 5

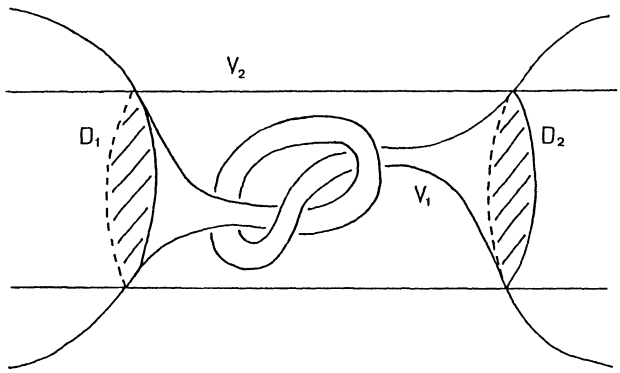


FIGURE 6

LEMMA 2.3 ([8]). *Let V_i $(i = 1, 2)$ be a knotted solid torus in S^3 which contains a knot K with $\mathrm{wrap}_{V_i}(K) \neq 0$. Then by an ambient isotopy of S^3 which leaves K fixed, V_2 can be deformed so that one of the following holds.*

(1) $\partial V_1 \cap \partial V_2 = \varnothing$

(2) *there exist meridian disks D_1 and D_2 of both V_1 and V_2 such that the closure of one component of $V_1 - \bigcup_{i=1}^2 D_i$ is a knotted 3-ball in the closure of some component of $V_2 - \bigcup_{i=1}^2 D_i$ (see Figure 6).*

Proof of Lemma 2.3. If $V_i-\operatorname{int} N(K)$ is homeomorphic to $S^1 \times S^1 \times I$ for $i = 1$ or 2, then K is a core of V_i and we can deform V_2 so that (1) in Lemma 2.3 holds. Now we assume that $V_i-\operatorname{int} N(K)$ is not homeomorphic to $S^1 \times S^1 \times I$ for $i = 1, 2$. Consider the torus decomposition of $S^3-\operatorname{int} V_i$ and $V_i-\operatorname{int} N(K)$ $(i = 1, 2)$ in the sense of Jaco-Shalen [4] and Johannson [5]. Then each piece is Seifert fibred or admits a complete hyperbolic structure of finite volume in its interior by Thurston's uniformization theorem [6]. Moreover, by

Theorem VI 3.4 in [4], the Seifert part is one of torus knot spaces, cable spaces and composing spaces. Let M_i be the piece in $S^3 - \operatorname{int} V_i$ which contains ∂V_i $(= \partial(S^3 - \operatorname{int} V_i))$ and N_i the piece in $V_i - \operatorname{int} N(K)$ which contains ∂V_i. We divide into two cases depending upon whether one of ∂V_i $(i = 1, 2)$ belongs to the minimal family of tori J_i defining a torus decomposition of $S^3 - \operatorname{int} N(K) = (S^3 - \operatorname{int} V_i) \cup (V_i - \operatorname{int} N(K))$ or not.

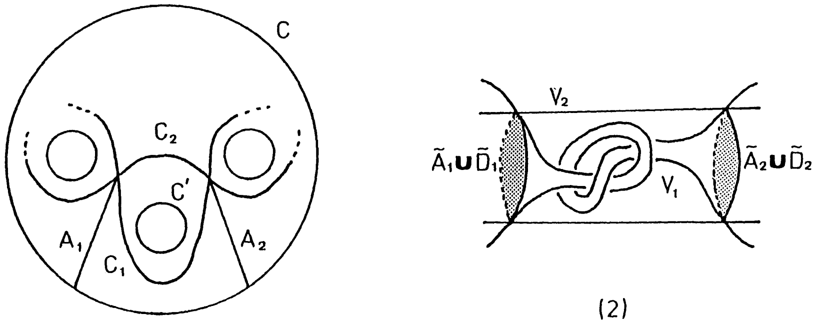
Case (A). At least one of $\partial V_i (i = 1, 2)$ belongs to $J_i (i = 1, 2)$.

In this case, by the uniqueness of the torus decomposition we can deform ∂V_2 so that $\partial V_1 \cap \partial V_2 = \varnothing$ or $\partial V_1 = \partial V_2$. If $\partial V_1 = \partial V_2$, then isotoping ∂V_2 slightly off ∂V_1 in the normal direction so that $\partial V_1 \cap \partial V_2 = \varnothing$. Thus (1) in Lemma 2.3 does hold.

Case (B). $\partial V_i (i = 1, 2)$ does not belong to J_i.

Then it turns out that $M_i \cup N_i$ is a composing space in $S^3 - \operatorname{int} N(K) = (S^3 - \operatorname{int} V_i) \cup (V_i - \operatorname{int} N(K))$. By the uniqueness of the torus decomposition, we can isotope $M_2 \cup N_2$ so that $M_2 \cup N_2$ is one of decomposing pieces defined by J_1. (Notationally we do not distinguish the original $M_2 \cup N_2$ and isotoped $M_2 \cup N_2$.) If $M_2 \cup N_2 \neq M_1 \cup N_1$, then (1) in Lemma 2.3 holds. Now suppose that $M_2 \cup N_2 = M_1 \cup N_1$. We note that ∂V_1 is a saturated torus (i.e. a union of fibres) in the composing space $M_1 \cup N_1$. Since a Seifert fibration of the composing space $M_1 \cup N_1$ is unique up to isotopy, we may modify ∂V_2 so that it is also saturated. Consider the orbit manifold of $M_1 \cup N_1$, which is a disk with holes. The image of ∂V_i in this orbit manifold is an essential circle C_i. If we can modify C_2 so that $C_1 \cap C_2 = \varnothing$, then we can also modify ∂V_2 so that $\partial V_1 \cap \partial V_2 = \varnothing$ and (1) in Lemma 2.3 holds. Assume we can not isotope C_2 so that $C_1 \cap C_2 = \varnothing$. Then isotope C_2 so that the number of points of $C_1 \cap C_2$ is minimal. Let T be the component of $\partial(M_1 \cup N_1)$ separating $\partial N(K)$ and $M_1 \cup N_1$. In S^3, T bounds a solid torus W containing K, whose meridian coincides with the regular fibre of $M_1 \cup N_1$. In the orbit manifold we can find arcs A_1 and A_2 joining a point in $C_1 \cap C_2$ and the boundary circle C which is the image of T (see Figure 7 (1)).

These arcs A_1 and A_2 are corresponding to saturated annuli $\widetilde{A}_1$ and $\widetilde{A}_2$. From $\widetilde{A}_j$ and meridian disk $\widetilde{D}_j$ of W, we can construct a meridian disk $\widetilde{A}_j \cup \widetilde{D}_j$ of both V_1 and V_2. Finally consider the boundary circle C' of the orbit manifold depicted in Figure 7 (1), which corresponds to the torus boundary T'. Since T' bounds a



(1)

FIGURE 7

nontrivial knot exterior in $V_2-\ \mathrm{int}\, V_1$ (see Figure 7 (2)), we get just a situation for (2) in Lemma 2.3, and this completes the proof. □

Let us study the relationship between two solid tori $h(f(V))$ and $f(V)$ using Lemma 2.3. By an ambient isotopy of S^3 which leaves $f(K_2) = h(f(K_1))$ fixed we can deform $h(f(V))$ into the position such that either $\partial h(f(V)) \cap \partial f(V) = \varnothing$ or there exist meridian disks D_1 and D_2 of both $f(V)$ and $h(f(V))$ such that one component of the closure of $f(V) - \bigcup_{i=1}^{2} D_i$ is a knotted 3-ball in some component of the closure of $h(f(V)) - \bigcup_{i=1}^{2} D_i$.

LEMMA 2.4. *We can deform* $h(f(V))$ *into the position such that* $\partial h(f(V)) \cap \partial f(V) = \varnothing$.

Proof of Lemma 2.4. If not, the second situation in the above occurs. Then we get a solid torus W' ($\supset h(f(K_1)) = f(K_2)$) in $\mathrm{int}\, h(f(V))$, whose core $C_{W'}$ satisfies $\mathrm{wrap}_{h(f(V))}(C_{W'}) = 1$ and is not a core of $h(f(V))$. It follows that the solid torus V also contains a knotted solid torus W ($\supset K_1$) in its interior such that $\mathrm{wrap}_V(C_W) = 1$ for the core C_W of W. Since $\mathrm{wrap}_W(K_1) \neq 0$, K_1 cannot be unknotted in S^3 by Lemma 2.1 and this contradicts the assumption. Hence we can deform $h(f(V))$ so that $\partial h(f(V)) \cap \partial f(V) = \varnothing$. □

Now we have following three possibilities.

(1) $h(S^3-\ \mathrm{int}\, f(V)) \subset \mathrm{int}\,(S^3-\ \mathrm{int}\, f(V))$

(2) $\mathrm{int}\, h(S^3-\ \mathrm{int}\, f(V)) \supset S^3-\ \mathrm{int}\, f(V)$

(3) $h(S^3-\ \mathrm{int}\, f(V)) \subset \mathrm{int}\, f(V)$

In (1) (or (2), resp.), assume $(S^3 - i\,\mathrm{int}\, f(V)) - \mathrm{int}(h(S^3 - \mathrm{int}\, f(V)))$ (or $h(S^3-\ \mathrm{int}\, f(V))-\ \mathrm{int}\,(S^3-\ \mathrm{int}\, f(V))$, resp.) is homeomorphic to $S^1 \times S^1 \times I$. Then by an isotopy which leaves $f(K_2)$ fixed, we may modify h so that $h(S^3 - \operatorname{int} f(V)) = S^3 - \operatorname{int} f(V)$ and also $h(f(V)) = f(V)$ with $h(f(K_1)) = f(K_2)$. For homological reasons, $h(\mathit{longitude}) = \pm\mathit{longitude}$. Hence $f^{-1} \circ h \circ f : V \to V$ is an orientation preserving homeomorphism carrying K_1 to K_2 and *longitude* to $\pm$*longitude*. So we get $(V, K_1) \sim (V, K_2)$.

Let us consider the case where $h(S^3 - \operatorname{int} f(V)) \subset \operatorname{int}(S^3 - \operatorname{int} f(V))$ and $(S^3 - \operatorname{int} f(V)) - \operatorname{int}(h(S^3 - \operatorname{int} f(V)))$ is not homeomorphic to $S^1 \times S^1 \times I$. Then using the homeomorphism $h|_{S^3 - \operatorname{int} f(V)}$ from $S^3 - \operatorname{int} f(V)$ to $h(S^3 - \operatorname{int} f(V))$, we get mutually nonparallel incompressible tori $\{h^n(\partial f(V))\}$ in $S^3 - \operatorname{int} f(V)$ for any positive integer n. This contradicts Haken's finiteness theorem. The similar argument can be applied in the case where $\operatorname{int} h(S^3 - \operatorname{int} f(V)) \supset S^3 - \operatorname{int} f(V)$ and $h(S^3 - \operatorname{int} f(V)) - \operatorname{int}(S^3 - \operatorname{int} f(V))$ is not homeomorphic to $S^1 \times S^1 \times I$, and again we get a contradiction.

Let us consider the case (3). The assumption implies that the nontrivial knot exterior $E = h^{-1}(S^3 - \operatorname{int} f(V))$ is contained in $\operatorname{int} f(V)$. Since $\operatorname{wind}_V(K_2) \neq 0$, we have $\operatorname{wind}_{f(V)}(f(K_2)) \neq 0$ and so we have $\operatorname{wind}_{h^{-1}(f(V))}(h^{-1}(f(K_2))) \neq 0$. It follows that $\operatorname{wind}_{(S^3 - \operatorname{int} E)}(f(K_1))$ is also not zero. On the other hand since K_1 is unknotted in S^3, by Lemma 2.2, the algebraic intersection number of K_1 and a meridian disk for the nontrivial knot exterior $f^{-1}(E)$ in V must be zero. Hence we get $\operatorname{wind}_{(S^3 - \operatorname{int} E)}(f(K_1)) = 0$. This is a contradiction. □

In Theorem 1.1, if $f(C)$ is a noninvertible knot, where C is a core of V, then more precisely we have the following.

THEOREM 2.5. *Let (V, K_i) be a pattern. Suppose that K_1 is unknotted in S^3, and* $\operatorname{wind}_V(K_2) \neq 0$. *If $f(K_1) \cong f(K_2)$ in S^3 for some embedding f from V into S^3 such that $f(C)$ is noninvertible, then $(V, K_1) \cong (V, K_2)$, that is K_1 and K_2 are ambient isotopic in V.*

Proof. In the proof of Theorem 1.1, we have an orientation preserving homeomorphism h of S^3 satisfying $h(f(V)) = f(V)$ and $h(f(K_1)) = f(K_2)$. For homological reasons, $h(\mathit{longitude}) = \pm\mathit{longitude}$. In addition since $f(C)$ is noninvertible, we get $h(\mathit{longitude}) = \mathit{longitude}$ (see 3.19. Proposition in **[1]**). It follows that $f^{-1} \circ h \circ f : V \to V$ is an orientation preserving homeomorphism carrying K_1 to K_2 and *longitude* to *longitude*. Thus we conclude $(V, K_1) \cong (V, K_2)$. □

As an application of Theorem 1.1, we have the following corollary.

COROLLARY 2.6. *Let* (V, K_i) *be a pattern. Suppose that* K_1 *is unknotted and* K_2 *is knotted in* S^3 *and* $\mathrm{wind}_V(K_2) \neq 0$. *Then for any embedding* f *from* V *into* S^3, $f(K_1) \ncong f(K_2)$ *in* S^3.

Proof. If $f(K_1) \cong f(K_2)$ for some embedding f from V into S^3, then $(V, K_1) \sim (V, K_2)$ must hold by Theorem 1.1. Extending the orientation preserving homeomorphism h of V to that of S^3, we get $K_1 \cong K_2$. This is a contradiction. □

Concluding this section, we give the following proposition which is an implicit corollary of Soma's sum formula for the Gromov invariants **[10]**. We denote the Gromov invariant of X by $\|X\|$.

PROPOSITION 2.7. *Let* (V, K_i) $(i = 1, 2)$ *be a pattern such that* $\|V - \mathrm{int}\, N(K_1)\| \neq \|V - \mathrm{int}\, N(K_2)\|$. *Then* $f(K_1) \ncong f(K_2)$ *for any embedding* f *from* V *into* S^3.

So we see that, with no conditions on K_1 and K_2, if $f(K_1) \cong f(K_2)$ then the Gromov invariants of their complements in V are the same.

3. Classification of satellite knots constructed from trivial knots. As a special case of Theorem 1.1, we have

THEOREM 3.1. *Let* (V, K_i) *be a pattern and* K_i *a trivial knot in* S^3 $(i = 1, 2)$. *Suppose that* $\mathrm{wind}_V(K_1) \neq 0$ *or* $\mathrm{wind}_V(K_2) \neq 0$. *If* $f(K_1) \cong f(K_2)$ *in* S^3 *for some embedding* f *from* V *into* S^3, *then* $(V, K_1) \sim (V, K_2)$ *holds.*

The winding numbers and the wrapping numbers of knots in a solid torus are elementary invariants for them. Particularly for a faithful (i.e. sending *longitude* to *longitude*) embedding $f : V \hookrightarrow S^3$, winding number of K in V has an important role for Alexander polynomial of $f(K)$, as is shown by Seifert's formula **[9]** (**[1]**). However if K is unknotted and $f(C)$ has a trivial Alexander polynomial, then $f(K)$ has also a trivial one independent of $\mathrm{wind}_V(K)$. Moreover when $K_1 \cong K_2$ and $\mathrm{wind}_V(K_1) = \mathrm{wind}_V(K_2)$, $f(K_1)$ and $f(K_2)$ have the same Alexander polynomial.

As a consequence of Theorem 3.1, we have the following result for satellite knots constructed from trivial knots.

COROLLARY 3.2. *Suppose K_i is a trivial knot contained in a standardly embedded solid torus V in S^3 $(i = 1, 2)$.*

(1) *If* $\mathrm{wind}_V(K_1) \neq \mathrm{wind}_V(K_2)$, *then* $f(K_1) \not\cong f(K_2)$ *in* S^3 *for any embedding f from V into S^3.*

(2) *When* $\mathrm{wind}_V(K_1) = \mathrm{wind}_V(K_2) \neq 0$, *if* $\mathrm{wrap}_V(K_1) \neq \mathrm{wrap}_V(K_2)$, *then* $f(K_1) \not\cong f(K_2)$ *in* S^3 *for any embedding f from V into S^3.*

In the case $\mathrm{wind}_V(K_1) = \mathrm{wind}_V(K_2) = 0$, we have

THEOREM 3.3. *For any faithful embedding f from V into S^3 (i.e. f sends a longitude of V to a longitude of $f(V)$), there exist patterns (V, K_1) and (V, K_2) such that both K_1 and K_2 are unknotted in S^3, which satisfy the following properties:*

(1) $\mathrm{wind}_V(K_1) = \mathrm{wind}_V(K_2) = 0$ *and* $(V, K_1) \not\cong (V, K_2)$.

(2) $f(K_1) \cong f(K_2)$ *in* S^3.

Proof. Let us consider a 3-components Brunnian link $L = k \cup L_1 \cup L_2$ depicted in Figure 8.

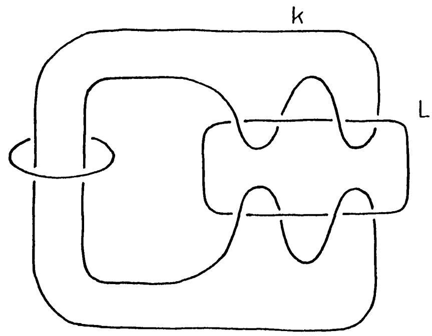


FIGURE 8

We denote the meridian-longitude pair of L_i by (m_i, l_i) $(i = 1, 2)$. Let t be a knot ambient isotopic to $f(C)$, where C is a core of V, and (m, l) a meridian-longitude pair of t. To obtain the required pattern, remove a tubular neighborhood $N(L_i)$ and glue the knot exterior $E(t) = S^3 - \mathrm{int}\, N(t)$ so that $m_i = l$ and $l_i = m$. Then, for $i = 1, 2$, the result $(S^3 - \mathrm{int}\, N(L_i)) \cup_{\substack{m_i = l \\ l_i = m}} (S^3 - \mathrm{int}\, N(t))$ is again S^3, and we have new knots K_{3-i} and $\widetilde{L}_{3-i}$ as the images of k and L_{3-i} respectively. It is easy to see that both K_{3-i} and $\widetilde{L}_{3-i}$ are unknotted in S^3. Thus the exterior V of $\widetilde{L}_{3-i}$ containing K_{3-i} forms a pattern (V, K_{3-i}). In this way we get two patterns (V, K_1) and (V, K_2). By the construction, for the faithful embedding $f : V \hookrightarrow S^3$, $f(K_1) \cong f(K_2)$ does hold in S^3. In fact, $f(K_1) \cong f(K_2)$ can be described as the knot obtained from k in Figure 8 by simultaneously replacing a neighborhood of a meridian disk of each of L_1 and L_2 by a tube knotted according to the given knot t.

From now on we prove $(V, K_1) \not\cong (V, K_2)$ by showing $\mathrm{wrap}_V(K_1) \neq \mathrm{wrap}_V(K_2)$. Clearly $\mathrm{wrap}_V(K_1) \le 2$ and $\mathrm{wrap}_V(K_2) \le 4$. Since $\mathrm{wind}_V(K_2)=0$, $\mathrm{wrap}_V(K_2)$ must be even. Now we assume $\mathrm{wrap}_V(K_2) = 2$. Then there exists a disk D_2 in

$$V = (S^3 - \mathrm{int}N(L_1)) \cup_{\substack{m_1=l \\ l_1=m}} (S^3 - \mathrm{int}\,N(t)) - \mathrm{int}\,N(L_2)$$

such that $D_2 \cap K_2 = D_2 \cap k$ consists of two points and $\partial D_2 = l_2$. Extending D_2, we may assume $\partial D_2 = L_2$. Let D_k be the disk depicted in Figure 9 (1), such that $\partial D_k = k$. We remark that $D_k \cap L_2$ consists of four points p_1, p_2, q_1, q_2 (see Figure 9 (1)).

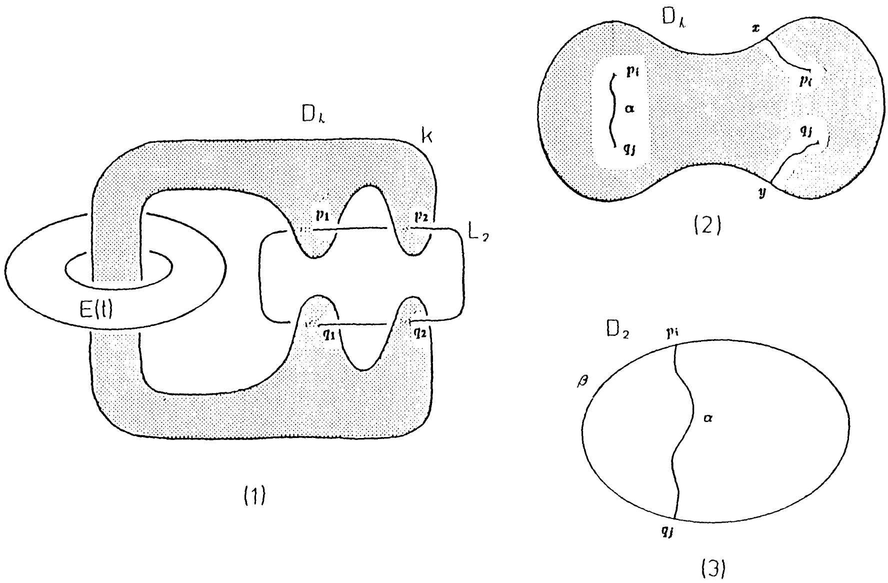


FIGURE 9

From the assumption we see that the boundary of arc components of $D_2 \cap D_k$ in D_k consists of six points p_1, p_2, q_1, q_2, x, y (see Figure 9 (2)). Considering the orientations, there exists an arc component α of $D_2 \cap D_k$ joining p_i and q_j for some i, j (see Figure 9 (2)(3)). Let β be an arc of L_2 connecting p_i and q_j, and D a disk in D_2 bounded by $\alpha \cup \beta$. Then $\alpha \cup \beta$ clearly has winding number one in the solid torus $S^3 -\ \mathrm{int}\,N(L_1)$, which is knotted as $N(t)$ in $S^3 \supset V$.

This contradicts, via Lemma 2.1, that $\alpha \cup \beta = \partial D$. Hence we can conclude $\mathrm{wrap}_V(K_2) \neq 2$, and applying the same argument we get also $\mathrm{wrap}_V(K_2) \neq 0$. It follows that $\mathrm{wrap}_V(K_2) = 4$. We see $\mathrm{wrap}_V(K_1) = 2$ easily as follows. If $\mathrm{wrap}_V(K_1) \neq 2$, then $\mathrm{wrap}_V(K_1) = 0$. However this means $\mathrm{wrap}_V(K_2) = 0$, thus $\mathrm{wrap}_V(K_1) = 2$.

In this way we get the required patterns. □

This result can be generalized to

COROLLARY 3.4. *For any knot K in S^3 and any faithful embedding f from V into S^3, there exist patterns (V, K_1) and (V, K_2) such that $K_i \cong K$ in S^3 $(i = 1, 2)$, which satisfy the following properties:*

(1) $(V, K_1) \not\sim (V, K_2)$.

(2) *$f(K_1) \cong f(K_2)$ in S^3 for the embedding f from V into S^3.*

Proof. Let (V, k_1) and (V, k_2) be the patterns constructed in Theorem 3.3 depending upon the embedding f. Since k_i $(i = 1, 2)$ is trivial in S^3, we can locally replace an unknotted arc of k_i by a knotted arc (with a suitable direction) so that the resulting knot K_i represents K in S^3. Then it follows from the choice of (V, k_1) and (V, k_2) that (V, K_1) and (V, K_2) are the required patterns. □

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