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ELLIPTIC REPRESENTATIONS FOR Sp(2n) AND SO(n)

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Let G be a connected, reductive p-adic group and let G^e denote the set of regular elliptic elements of G. Let π be an irreducible, tempered representation of G with character Θ_π, and write Θ_π^e for the restriction of Θ_π to G^e. We say π is elliptic if Θ_π^e is non-zero. In this paper we will characterize the elliptic representations for the p-adic groups $\mathrm{Sp}(2n)$ and $\mathrm{SO}(n)$. We will show for $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ that every irreducible, tempered representation is either elliptic or can be irreducibly induced from an elliptic representation. We will then show that this fails for the groups $\mathrm{SO}(2n)$. In this case there are irreducible tempered representations which cannot be irreducibly induced and are not elliptic.

Introduction. For real reductive Lie groups, the elliptic representations are the discrete series and limits of discrete series representations. Knapp and Zuckerman [**K-Z**] classified the irreducible tempered representations by proving that every irreducible, tempered representation is either elliptic, or can be irreducibly induced from an elliptic representation of a proper parabolic subgroup in an essentially unique way. Thus the p-adic groups $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ behave in the same way as real groups. In the p-adic case, Kazhdan [**K**] proved that an irreducible tempered representation is elliptic just in the case that it is not a linear combination (in the Grothendieck group) of properly induced representations. Clozel [**C**] conjectured that an irreducible tempered representation is elliptic, if and only if, it cannot be realized as a full induced representation from a proper parabolic subgroup. The case of $\mathrm{SO}(2n)$ provides a counterexample to Clozel's conjecture.

Every irreducible tempered representation is a subrepresentation of a representation unitarily induced from a discrete series representation of a parabolic subgroup. Thus in order to classify elliptic representations it is necessary to know which irreducible constituents of reducible induced representations are elliptic. In [**A**], Arthur gives such a characterization in terms of the R-group corresponding to the induced representation. In this paper we will use Arthur's results to characterize the elliptic representations of the symplectic and special orthogonal groups where Goldberg [G] has computed the R-groups for all tempered representations unitarily induced from discrete series of proper parabolic subgroups.

In §1 we will review the theory of the R-group and the results of Arthur which will be needed in studying elliptic representations. In §2 we will use the results of Goldberg to characterize the elliptic, irreducible, tempered representations for the symplectic and odd special orthogonal groups. In this case we will see that an irreducible, tempered representation is either elliptic or is irreducibly induced from an elliptic representation of a proper parabolic subgroup. In §3 we will use results of Goldberg to treat the even special orthogonal groups, which are technically more difficult than the groups considered in §2. In this case there are examples of irreducible, tempered representations which are not elliptic, but cannot be irreducibly induced from any representation of a proper parabolic subgroup.

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1. Preliminaries. Let F be a locally compact, non-discrete, nonarchimedean local field of characteristic zero. Let G be the F-rational points of a connected, reductive algebraic group over F. Let G' denote the set of regular elements of G. Thus $x \in G'$ if $D_G(x) \neq 0$ where $D_G(x)$ is defined as in [HC, §15]. We say $x \in G$ is elliptic if it is contained in a Cartan subgroup which is compact modulo the center of G. Write G^e for the set of regular elliptic elements of G. Let $\mathscr{E}_t(G)$ denote the set of (equivalence classes of) irreducible, tempered representations of G and let $\mathscr{E}_2(G)$ denote the subset of $\mathscr{E}_t(G)$ consisting of square-integrable representations. Given any $\pi \in \mathscr{E}_t(G)$ we write Θ_π for the character of π and Θ_π^e for the restriction of Θ_π to G^e.

We say that $M \subseteq G$ is a Levi subgroup of G if there is a parabolic subgroup $P = MN$ of G so that M is a Levi component of P. Given $\sigma \in \mathscr{E}_t(M)$, we write $\mathrm{Ind}_P^G(\sigma)$ for the corresponding induced representation of G. (We will always use unitary induction.) Since the class of $\mathrm{Ind}_P^G(\sigma)$ is independent of P, we will also write $i_{G,M}(\sigma)$ for the corresponding equivalence class.

Let P be a parabolic subgroup of G with Levi component M and split component A and let $\mathfrak{a}$ denote the real Lie algebra of A. Let $W(G/A) = N_G(A)/M$. Then $W(G/A)$ acts on $\mathscr{E}_2(M)$. For each $w \in W(G/A)$, let $\mathscr{H}_w$ denote the representation space for $\mathrm{Ind}_P^G(w\sigma)$. Associated to each $w \in W(G/A)$, there is a meromorphic family of intertwining operators, $A(w, \nu, \sigma)$, $\nu \in \mathfrak{a}_{\mathbf{C}}^*$, defined by the standard integral formula. By normalizing with (scalar) meromorphic normalizing factors, we obtain intertwining operators $\mathscr{A}(w, \nu, \sigma)$ which are holomorphic on the unitary axis. Write $\mathscr{A}(w, \sigma) = \mathscr{A}(w, 0, \sigma)$. Now $\mathscr{A}(w, \sigma): \mathscr{H}_1 \to \mathscr{H}_w$ and satisfies the cocycle condition

$$\mathscr{A}(w_1 w_2, \sigma) = \mathscr{A}(w_1, w_2\sigma)\mathscr{A}(w_2, \sigma)$$

for all $w_1, w_2 \in W(G/A)$. Define $W(\sigma) = \{w \in W(G/A): w\sigma \simeq \sigma\}$. Let V be the representation space of σ. Then for each $w \in W(\sigma)$ there is an intertwining operator $T(w): V \to V$ so that $T(w)(w\sigma)(m) = \sigma(m)T(w)$ for all $m \in M$. Now $\mathscr{A}'(w, \sigma) = T(w)\mathscr{A}(w, \sigma)$ gives a self-intertwining operator of $\mathrm{Ind}_P^G(\sigma)$ for all $w \in W(\sigma)$ and these span the commuting algebra $C(\sigma)$ of $\mathrm{Ind}_P^G(\sigma)$.

Given any reduced root $\beta \in \Phi(P, A)$, let M_β be the Levi subgroup of G with $M \subseteq M_\beta$ defined as in [HC, §13], and let $\mu_\beta(\sigma)$ be the Plancherel measure associated to the representation $i_{M_\beta, M}(\sigma)$. Let $\Delta' = \{\beta \in \Phi(P, A): \mu_\beta(\sigma) = 0\}$ and let $W(\Delta')$ be the subgroup of $W(G/A)$ generated by reflections in the roots of Δ'. Then $W(\Delta') = \{w \in W(\sigma): \mathscr{A}'(w, \sigma)$ is scalar$\}$. We can write $W(\sigma) = R \times_s W(\Delta')$, the semidirect product of R and $W(\Delta')$, where $R = \{w \in W(\sigma): w\beta > 0, \forall \beta \in \Delta'\}$. Then $\{\mathscr{A}'(w, \sigma): w \in R\}$ is a linear basis for the commuting algebra [S]. Further, given $w_1, w_2 \in R$, $\mathscr{A}'(w_1 w_2, \sigma) = \eta(w_1, w_2)\mathscr{A}'(w_1, \sigma)\mathscr{A}'(w_2, \sigma)$ where $\eta(w_1, w_2) \in \mathbf{C}^\times$ satisfies $T(w_1 w_2) = \eta(w_1, w_2)T(w_1)T(w_2)$. Thus $C(\sigma)$ is isomorphic as an algebra to the complex group algebra $\mathbf{C}[R]$ if and only if the intertwining operators $T(w)$, $w \in R$, can be chosen so that $T(w_1 w_2) = T(w_1)T(w_2)$ for all $w_1, w_2 \in R$.

Assume for simplicity in the remainder of this section that R is abelian and $C(\sigma) \simeq \mathbf{C}[R]$ as algebras. (This will be the case in our examples.) For each $w \in R$, define

$$\mathfrak{a}_w = \{H \in \mathfrak{a}: wH = H\}.$$

Let Z be the split component of G and let $\mathfrak{z}$ denote the real Lie algebra of Z. Then $\mathfrak{z} \subseteq \mathfrak{a}_w$ for all $w \in R$. Now a special case of Arthur's result is the following.

THEOREM 1.1 (*Arthur* [A, 2.1]). *Suppose that R is abelian and that $C(\sigma) \simeq \mathbf{C}[R]$. Then $i_{G,M}(\sigma)$ has an elliptic constituent $\Leftrightarrow$ all constituents of $i_{G,M}(\sigma)$ are elliptic $\Leftrightarrow$ there is $w \in R$ such that $\mathfrak{a}_w = \mathfrak{z}$.*

The irreducible constituents of $i_{G,M}(\sigma)$ can be described as follows. Let $\mathscr{H}$ be the representation space of $i_{G,M}(\sigma)$. Now given any unitary character $\kappa \in \widehat{R}$, let

$$\mathscr{H}_\kappa = \{v \in \mathscr{H} : \mathscr{A}'(r, \sigma)v = \kappa(r)v \text{ for all } r \in R\}.$$

Then $\mathscr{H} = \bigoplus_{\kappa \in \widehat{R}} \mathscr{H}_\kappa$ is exactly the decomposition of $\mathscr{H}$ into irreducibles. Let π_κ denote the irreducible representation of G on $\mathscr{H}_\kappa$.

Suppose that M' is a Levi subgroup of G with $M \subseteq M'$ which satisfies the compatibility condition of [A, §2] with respect to the choice of positive roots Δ' used to define R. Let $R' = R \cap W(M'/A)$. Then R' can be identified with the reducibility group for $i_{M',M}(\sigma)$. Now as above we can use the characters of R' to decompose $i_{M',M}(\sigma)$ into irreducible constituents $\tau_{\kappa'}$, $\kappa' \in \widehat{R}'$. For each $\kappa' \in \widehat{R}'$, define the subset $\widehat{R}(\kappa')$ of $\widehat{R}$ by

$$\widehat{R}(\kappa') = \{\kappa \in \widehat{R} : \kappa(r) = \kappa'(r), \ r \in R'\}.$$

Then another consequence of [A, 2.1] is the following.

LEMMA 1.2 (*Arthur*). *For each* $\kappa' \in \widehat{R}'$, *we have*

$$i_{G,M'}(\tau_{\kappa'}) = \bigoplus \sum_{\kappa \in \widehat{R}(\kappa')} \tau_\kappa.$$

In particular we see that the irreducible constituents π_κ of $i_{G,M}(\sigma)$ can be irreducibly induced from M' if and only if $R = R'$.

Define

$$\mathfrak{a}_R = \bigcap_{w \in R} \mathfrak{a}_w.$$

LEMMA 1.3. *Suppose that* R *is abelian and* $C(\sigma) \simeq \mathbf{C}[R]$. *Let* π *be an irreducible constituent of* $i_{G,M}(\sigma)$. *Then there are a proper Levi subgroup* M' *and* $\tau \in \mathscr{E}_t(M')$ *such that* $\pi = i_{G,M'}(\tau)$ *if and only if* $\mathfrak{a}_R \neq \mathfrak{z}$. *Further,* M' *and* τ *can be chosen so that* τ *is elliptic if and only if there is* $w_0 \in R$ *such that* $\mathfrak{a}_R = \mathfrak{a}_{w_0}$.

Proof. As in [A, §2], for each $w \in R$, there is a Levi subgroup L_w of G containing M which satisfies the compatibility condition and such that $\mathfrak{a}_w = \mathfrak{a}_{L_w}$, the split component of L_w. Thus there is a Levi subgroup M' containing M which satisfies the compatibility condition so that $\mathfrak{a}_{M'} = \mathfrak{a}_R$. Since every element of R centralizes $\mathfrak{a}_{M'}$ we have $R \subseteq W(M'/A)$. Thus as above, each irreducible constituent of

$i_{G,M}(\sigma)$ is of the form $i_{G,M'}(\tau)$ where τ is an irreducible constituent of $i_{M',M}(\sigma)$. Now if $\mathfrak{a}_R \neq \mathfrak{z}$, then M' is proper.

Conversely, if such an M' and τ exist, then they can be chosen so that $M \subseteq M'$ and τ is an irreducible constituent of $i_{M',M}(\sigma)$. Thus as above we must have $R' = R$. Thus $R \subseteq W(M'/A)$ so that $\mathfrak{a}_{M'} \subseteq \mathfrak{a}_R$. Thus if M' is proper we have $\mathfrak{a}_R \neq \mathfrak{z}$. Further, $i_{M',M}(\sigma)$ has elliptic constituents if and only if there is $w_0 \in R$ so that $\mathfrak{a}_{w_0} = \mathfrak{a}_{M'}$. But since $\mathfrak{a}_{M'} \subseteq \mathfrak{a}_R \subseteq \mathfrak{a}_w$ for all $w \in R$, this is true if and only if $\mathfrak{a}_{M'} = \mathfrak{a}_R = \mathfrak{a}_{w_0}$. □

2. Elliptic representations of $\mathrm{Sp}(2n)$ **and** $\mathrm{SO}(2n+1)$**.** Goldberg's results in this case can be summarized as follows. Let $G = \mathrm{Sp}(2n, F)$ or $\mathrm{SO}(2n+1, F)$. Since all our groups will be F-rational points of algebraic groups, we will drop the F's. Similarly we write $\mathrm{GL}(n)$ for $\mathrm{GL}(n, F)$. Then if $P = MN$ is a proper parabolic subgroup of G, there are $r \geq 1$, positive integers $m_1, m_2, \ldots, m_r$, and an $m \geq 0$, with $\sum_{i=1}^r m_i + m = n$, such that

$$M \simeq \mathrm{GL}(m_1) \times \cdots \times \mathrm{GL}(m_r) \times G(m),$$

where $G(0) = \{1\}$, while for $m > 0$ we have

$$G(m) = \begin{cases} \mathrm{Sp}(2m), & \text{if } G = \mathrm{Sp}(2n); \\ \mathrm{SO}(2m+1), & \text{if } G = \mathrm{SO}(2n+1). \end{cases}$$

Let A be the split component of M. Then $A \simeq (F^\times)^r$ where the ith copy of $F^\times$ corresponds to the scalar matrices in the subgroup $\mathrm{GL}(m_i)$, $1 \leq i \leq r$. Now if we use this identification to write each $a \in A$ as $a = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_i \in F^\times$, then $W(G/A)$ can be identified with a subgroup of the group of all permutations and sign changes of the λ_i, $1 \leq i \leq r$. Specifically, the permutation (ij) which interchanges λ_i and λ_j is in $W(G/A)$ just in case $m_i = m_j$ so that the corresponding scalar matrices are the same size. Let c_i be the sign change $\lambda_i \to \lambda_i^{-1}$. Then $c_i \in W(G/A)$ for all $1 \leq i \leq r$. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho \in \mathscr{E}_2(M)$. Here for $1 \leq i \leq r$, $\sigma_i \in \mathscr{E}_2(\mathrm{GL}(m_i))$ and $\rho \in \mathscr{E}_2(G(m))$. Now

$$(ij)\sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \sigma_j,$$
$$c_i c_j (ij)\sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \tilde{\sigma}_j,$$

and

$$c_i \sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \tilde{\sigma}_i$$

where $\tilde{\sigma}$ is the contragredient of σ. Set

$$(2.1)\qquad I(\sigma) = \{1 \le i \le r \colon \sigma_i \simeq \tilde{\sigma}_i \text{ and } i_{G(m+m_i),\, \mathrm{GL}(m_i)\times G(m)}(\sigma_i \otimes \rho) \text{ is reducible}\}.$$

Of course $\sigma_i \simeq \tilde{\sigma}_i$ is in fact a necessary condition for

$$i_{G(m+m_i),\, \mathrm{GL}(m_i)\times G(m)}(\sigma_i \otimes \rho)$$

to be reducible, since it is the condition that $\sigma_i \otimes \rho$ is ramified in $G(m+m_i)$.

THEOREM 2.2 (*Goldberg* **[G]**). *Suppose M and $\sigma \in \mathscr{E}_2(M)$ are as above. Let d be the number of inequivalent σ_i such that $i \in I(\sigma)$. Then $R \simeq \mathbf{Z}_2^d$ and is generated by d of the sign changes c_i, $i \in I(\sigma)$.*

PROPOSITION 2.3. *Suppose that M is any Levi subgroup of G and $\sigma \in \mathscr{E}_2(M)$. Then*

$$C(\sigma) \simeq \mathbf{C}[R].$$

Proof. Renumber indices so that $c_1, \dots, c_d$ are the generators of R. For $1 \le i \le d$, a representative $\overline{c}_i \in N_G(A)$ for c_i can be chosen so that

$$\overline{c}_i(m_1, \dots, m_r, m')\overline{c}_i^{-1} = (m_1, \dots, (m_i^t)^{-1}, \dots, m_r, m')$$

where $m = (m_1, \dots, m_r, m') \in \mathrm{GL}(m_1) \times \cdots \times \mathrm{GL}(m_r) \times G(m)$. For $1 \le i \le d$, let V_i be the representation space of σ_i, and define a representation σ_i^* on V_i by $\sigma_i^*(g) = \sigma_i((g^t)^{-1})$, $g \in \mathrm{GL}(m_i)$. Now since $c_i \in W(\sigma)$, we have $\sigma_i^* \simeq \sigma_i$. (In stating Theorem 2.2 we used the fact that $\sigma_i^* \simeq \tilde{\sigma}_i$.) Let $T_i \colon V_i \to V_i$ be an intertwining operator between σ_i and σ_i^*. Since $(\sigma_i^*)^* = \sigma_i$, $T_i^2 = r_i$ is a non-zero complex scalar. Thus we can normalize T_i so that $T_i^2 = 1$. Now $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho$ acts on $V = V_1 \otimes \cdots \otimes V_r \otimes V'$. Extend T_i to an endomorphism T_i^V of V by making it act trivially on every factor except V_i, where it acts by T_i. Then T_i^V intertwines $c_i\sigma$ and σ and $(T_i^V)^2 = 1$. Further, for $1 \le i \ne j \le d$, $T_i^V T_j^V = T_j^V T_i^V$ since they act on different factors of V. Thus if we define $T(c_i) = T_i^V$, then $c_i \mapsto T(c_i)$ extends uniquely to a group homomorphism. □

LEMMA 2.4. *For any M, σ as above, there exists $w_0 \in R$ so that $\mathfrak{a}_R = \mathfrak{a}_{w_0}$. Further, there is $w \in R$ such that $\mathfrak{a}_w = \{0\} \Leftrightarrow \mathfrak{a}_R = \{0\} \Leftrightarrow R \simeq \mathbf{Z}_2^r$.*

Proof. By Theorem 2.2, $R = \mathbf{Z}_2^d \subseteq S(G/A)$ where $S(G/A)$ denotes the subgroup of $W(G/A)$ generated by the block sign changes c_i, $1 \leq i \leq r$. Renumber indices so that it is generated by the sign changes $c_1, \ldots, c_d$. If we let $w_0 = c_1 \cdots c_d$, then $\mathfrak{a}_{w_0} \subseteq \mathfrak{a}_w$ for all $w \in R$ so that $\mathfrak{a}_R = \mathfrak{a}_{w_0}$. Now for $w \in S(G/A)$, $\mathfrak{a}_w = \{0\}$ if and only if $w = c_1 c_2 \cdots c_r$, and $c_1 c_2 \cdots c_r \in R$ if and only if $c_i \in R$ for all $1 \leq i \leq r$. Thus $\mathfrak{a}_R = \mathfrak{a}_{w_0} = \{0\}$ if and only if $R \simeq \mathbf{Z}_2^r$. □

Lemma 2.4 can be combined with Theorem 1.1 and Lemma 1.3 to obtain the following theorems.

THEOREM 2.5. *Let M be a Levi subgroup of G and let $\sigma \in \mathcal{E}_2(M)$. Then $i_{G,M}(\sigma)$ has an elliptic constituent $\Leftrightarrow$ all constituents of $i_{G,M}(\sigma)$ are elliptic $\Leftrightarrow$ $R \simeq \mathbf{Z}_2^r$.*

THEOREM 2.6. *Let $\pi \in \mathcal{E}_t(G)$. Then either π is elliptic or $\pi = i_{G,M}(\tau)$ for some proper Levi subgroup M of G and some elliptic $\tau \in \mathcal{E}_t(M)$.*

Suppose now that $R \simeq \mathbf{Z}_2^r$. For $\kappa \in \widehat{R}$, define $\varepsilon(\kappa) = \kappa(\prod_{i=1}^r c_i) = \pm 1$. Let $1 \in \widehat{R}$ denote the trivial character.

PROPOSITION 2.7. *For all $\kappa \in \widehat{R}$ we have $\Theta_\kappa^e = \varepsilon(\kappa)\Theta_1^e$.*

Proof. For $1 \leq i \leq r$, let M_i be the maximal parabolic subgroup containing M with $M_i \simeq \mathrm{GL}(m_i) \times G(n - m_i)$. Let R_i be the reducibility group for $i_{M_i,M}(\sigma)$. We can identify R_i with the subgroup of R generated by $\{c_j, 1 \leq j \leq r, j \neq i\}$. (Since $\Delta' = \varnothing$ there is no compatibility condition.) Then for each $\kappa_i \in \widehat{R}_i$, $\widehat{R}(\kappa_i) = \{\kappa_i(+), \kappa_i(-)\}$ where $\kappa_i(\pm)(c_j) = \kappa_i(c_j)$, $j \neq i$, and $\kappa_i(\pm)(c_i) = \pm 1$. Now using Lemma 1.2, for $\kappa_i \in \widehat{R}_i$ we have $i_{G,M_i}(\tau_{\kappa_i}) = \pi_{\kappa_i(+)} \oplus \pi_{\kappa_i(-)}$. Thus $\Theta_{\kappa_i(+)}^e = -\Theta_{\kappa_i(-)}^e$.

Now the proof is by induction on $s(\kappa)$, the number of indices $1 \leq i \leq r$ so that $\kappa(c_i) = -1$. It is trivial if $s(\kappa) = 0$ since $\varepsilon(1) = 1$. Assume that the lemma is proven for $\kappa \in \widehat{R}$ so that $s(\kappa) = s \geq 0$. Fix $\kappa \in \widehat{R}$ with $s(\kappa) = s + 1$. Then there is $1 \leq i \leq r$ so that $\kappa(c_i) = -1$. Let κ_i denote the restriction of κ to R_i. Then $\kappa = \kappa_i(-)$ and $s(\kappa_i(+)) = s$. Thus by the induction hypothesis we have $\Theta_{\kappa_i(+)}^e = \varepsilon(\kappa_i(+))\Theta_1^e$. But as above

$$\Theta_\kappa^e = -\Theta_{\kappa_i(+)}^e = -\varepsilon(\kappa_i(+))\Theta_i^e = \varepsilon(\kappa)\Theta_1^e. \qquad \square$$

3. Elliptic representations of $SO(2n)$. Let $G = SO(2n) = SO(2n, F)$. Then if $P = MN$ is a proper parabolic subgroup of G, as in §2 there are $r \geq 1$, positive integers $m_1, m_2, \ldots, m_r$ and an $m \geq 0$, $m \neq 1$, with $\sum_{i=1}^{r} m_i + m = n$, such that

$$M \simeq GL(m_1) \times \cdots \times GL(m_r) \times G(m),$$

where $G(0) = \{1\}$, while for $m \geq 2$ we have $G(m) = SO(2m)$.

Let A be the split component of M. Then $A \simeq (F^\times)^r$ and, as in §2, $W(G/A)$ can be identified with a subgroup of the group of all permutations and sign changes of the λ_i, $1 \leq i \leq r$. As before, the permutation (ij) is in $W(G/A)$ just in case $m_i = m_j$. Let $G' = O(2n)$. For $1 \leq i \leq r$, there is $\overline{c}_i \in N_{G'}(A)$ such that for $m = (m_1, \ldots, m_r, m') \in GL(m_1) \times \cdots \times GL(m_r) \times G(m)$, $\overline{c}_i m \overline{c}_i^{-1} = (m_1, \ldots, (m_i^t)^{-1}, \ldots, m_r, m')$. Thus conjugation by $\overline{c}_i$ gives the sign change c_i taking λ_i to λ_i^{-1}. Further, if $m \geq 2$ there is $\overline{c}' \in N_{G'}(A)$ so that $\overline{c}' m \overline{c}'^{-1} = (m_1, \ldots, m_r, c'm')$, where c' is an outer automorphism of $SO(2m)$ with $(c')^2 = 1$. Note that conjugation by $\overline{c}'$ acts trivially on A. Now if $1 \leq i \leq r$ and m_i is even, then $\overline{c}_i$ can be chosen to be in $N_G(A)$, so that conjugation by $\overline{c}_i$ gives the sign change of $c_i \in W(G/A)$. Further, if m_i is odd and $m \geq 2$, then $\overline{c}_i$ can be chosen so that $\overline{c}_i \overline{c}' \in N_G(A)$ and conjugation by $\overline{c}_i \overline{c}'$ gives the sign change $c_i \in W(G/A)$. If m_i is odd and $m = 0$, then the individual sign change c_i is not in $W(G/A)$, but for two such indices, $\overline{c}_i \overline{c}_j \in N_G(A)$ and gives the product $c_i c_j \in W(G/A)$. This makes the groups $SO(2n)$ more complicated than the groups $Sp(2n)$ and $SO(2n+1)$.

Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho \in \mathscr{E}_2(M)$. Here for $1 \leq i \leq r$, $\sigma_i \in \mathscr{E}_2(GL(m_i))$ and $\rho \in \mathscr{E}_2(G(m))$. Now as in §2 we have

$$(ij)\sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \sigma_j,$$
$$c_i c_j (ij)\sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \tilde{\sigma}_j.$$

Further, if m_i is even, then

$$c_i \sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \tilde{\sigma}_i.$$

If m_i is odd and $m \geq 2$, then

$$c_i \sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \tilde{\sigma}_i \quad \text{and} \quad c'\rho \simeq \rho.$$

Finally, if m_i, m_j are odd, then

$$c_i c_j \sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \tilde{\sigma}_i \quad \text{and} \quad \sigma_j \simeq \tilde{\sigma}_j.$$

Write $I_e = \{1 \leq i \leq r\colon m_i \text{ is even}\}$ and $I_o = \{1 \leq i \leq r\colon m_i \text{ is odd}\}$. Define

$$I_1 = \begin{cases} I_e \cup I_o, & \text{if } m \geq 2 \text{ and } c'\rho \simeq \rho; \\ I_e, & \text{otherwise.} \end{cases}$$

Define $I_2 = I_1^c$. Now set

$$I(\sigma) = I_1(\sigma) \cup I_2(\sigma)$$

where

$$(3.1\text{a}) \qquad I_1(\sigma) = \{i \in I_1\colon \sigma_i \simeq \tilde{\sigma}_i \text{ and } i_{G(m+m_i),\,\mathrm{GL}(m_i)\times G(m)}(\sigma_i \otimes \rho) \text{ is reducible}\}$$

and

$$(3.1\text{b}) \qquad I_2(\sigma) = \{i \in I_2\colon \sigma_i \simeq \tilde{\sigma}_i\}.$$

THEOREM 3.2 (*Goldberg* [G]). *Suppose M and $\sigma \in \mathscr{E}_2(M)$ are as above. For $j = 1, 2$, let d_j be the number of inequivalent σ_i such that $i \in I_j(\sigma)$, and let $d = d_1 + d_2$. If $d_2 = 0$, then $R \simeq \mathbf{Z}_2^d$, while if $d_2 > 0$, then $R \simeq \mathbf{Z}_2^{d-1}$. In either case, $R \subseteq S(G/A)$, the subgroup of $W(G/A)$ generated by sign changes.*

PROPOSITION 3.3. *Suppose that M is a Levi subgroup of G and that $\sigma \in \mathscr{E}_2(M)$. Then $C(\sigma) \simeq \mathbf{C}[R]$.*

Proof. Suppose first that $m = 0$, or that $m \geq 2$ but $c'\rho \not\simeq \rho$. In this case $I_1 = I_e$ and $I_2 = I_o$. If $d_2 \leq 1$, then R is generated by d_1 sign changes in indices $i \in I_1(\sigma)$, and the proof is the same as that of Proposition 2.3. Assume that $d_2 \geq 2$. Renumber the indices so that $1, \dots, p = d_2 \in I_2(\sigma)$, $p+1, \dots, d = d_1 + d_2 \in I_1(\sigma)$, and $c_1c_p, c_2c_p, \dots, c_{p-1}c_p, c_{p+1}, \dots, c_d$ are a complete set of generators for $R \simeq \mathbf{Z}_2^{d-1}$. For each $1 \leq i \leq d$, we must have $\sigma_i \simeq \sigma_i^*$. As in Proposition 2.3, we can choose $T_i\colon V_i \to V_i$ intertwining σ_i^* and σ_i, so that $T_i^2 = 1$, $1 \leq i \leq d$, and extend them to endomorphisms T_i^V of $V = V_1 \otimes \cdots \otimes V_r \otimes V'$. Again, $(T_i^V)^2 = 1$ and $T_i^V T_j^V = T_j^V T_i^V$ for $1 \leq i, j \leq d$. Now we can define $T(c_ic_p) = T_i^V T_p^V$, $1 \leq i \leq p-1$, and $T(c_i) = T_i^V$, $p+1 \leq i \leq d$, and this extends to a group homomorphism.

In the case that $m \geq 2$ and $c'\rho \simeq \rho$, we have $I_1 = I_e \cup I_o$. Renumber indices so that $1, \dots, p \in I_1(\sigma) \cap I_o$, $p+1, \dots, d = d_1 \in I_1(\sigma) \cap I_e$, and $c_1, \dots, c_d$ are the generators of $R \simeq \mathbf{Z}_2^d$. Choose intertwining operators T_i, $1 \leq i \leq d$, as above, and also choose an intertwining operator $T'\colon V' \to V'$ which intertwines $c'\rho$ and ρ and satisfies $(T')^2 = 1$. Extend T' to an operator $(T')^V$ on V which acts non-trivially only on V'. Then define $T(c_i) = T_i^V(T')^V$, $1 \le i \le p$, and $T(c_i) = T_i^V$, $p+1 \le i \le d$. □

LEMMA 3.4. *There is $w_0 \in R$ such that $\mathfrak{a}_R = \mathfrak{a}_{w_0}$ if and only if d_2 is even or $d_2 = 1$. Further,*

$$\text{there is } w \in R \text{ such that } \mathfrak{a}_w = \{0\} \Leftrightarrow d = r \text{ and } d_2 \text{ is even}$$

and

$$\mathfrak{a}_R = \{0\} \Leftrightarrow d = r \text{ and } d_2 \neq 1.$$

Proof. We can write $\mathfrak{a} = \{(x_1, \dots, x_r)\colon x_i \in \mathbf{R}\}$ so that c_i corresponds to the sign change $x_i \mapsto -x_i$. Renumber indices so that $1, \dots, p = d_2 \in I_2(\sigma)$, $p+1, \dots, d \in I_1(\sigma)$, and R is generated by the elements c_ic_j, $1 \le i \neq j \le p$, and c_i, $p+1 \le i \le d$. Now if d_2 is even, we have $w_0 = c_1 \cdots c_d \in R$ and $\mathfrak{a}_R = \mathfrak{a}_{w_0} = \{(x_1, \dots, x_r)\colon x_1 = \cdots = x_d = 0\}$. If $d_2 = 1$ we have $w_0 = c_2 \cdots c_d \in R$ with $\mathfrak{a}_R = \mathfrak{a}_{w_0} = \{(x_1, \dots, x_r)\colon x_2 = \cdots = x_d = 0\}$. Finally, if $d_2 \ge 3$ is odd, then $\mathfrak{a}_R = \{(x_1, \dots, x_r)\colon x_1 = \cdots = x_d = 0\}$, but $\mathfrak{a}_w \neq \mathfrak{a}_R$ for any $w \in R$. □

Combining Lemma 3.4 with Theorem 1.1 and Lemma 1.3 we obtain the following.

THEOREM 3.5. *Let M be a Levi subgroup of G and $\sigma \in \mathcal{E}_2(M)$. Then $i_{G,M}(\sigma)$ has an elliptic constituent $\Leftrightarrow$ all constituents of $i_{G,M}(\sigma)$ are elliptic $\Leftrightarrow$ $d = r$ and d_2 is even.*

PROPOSITION 3.6. *Suppose that $d < r$ or that $d = r$ and $d_2 = 1$. Then each irreducible constituent of $i_{G,M}(\sigma)$ is of the form $i_{G,M'}(\tau)$ where M' is a proper Levi subgroup of G and $\tau \in \mathcal{E}_t(M')$. If d_2 is even or $d_2 = 1$ we can choose M' so that τ is elliptic.*

PROPOSITION 3.7. *Suppose that $d = r$ and $d_2 \ge 3$ is odd. Then each irreducible constituent of $i_{G,M}(\sigma)$ is a linear combination of representations induced from proper parabolic subgroups, but cannot be irreducibly induced. In fact, each irreducible constituent of $i_{G,M}(\sigma)$ is of the form $\sum_{i=1}^{d_2} c_i i_{G,M_i}(\tau_i)$ where the M_i are proper Levi subgroups of G, the $\tau_i \in \mathcal{E}_t(M_i)$ are elliptic, and the c_i are non-zero complex numbers.*

REMARKS. Such representations exist. For example, suppose $G = SO(6)$, $M \simeq GL(1)^3$, and $\sigma \simeq \chi_1 \otimes \chi_2 \otimes \chi_3$ where the χ_i, $1 \leq i \leq 3$, are distinct characters of $F^\times$ with $\chi_i^2 = 1$. Note also that SO(6) is locally isomorphic to SL(4). In fact a non-elliptic representation which cannot be irreducibly induced can also be constructed in the principal series of SL(4). All of the above results on R-groups are equally valid for the real Lie groups $SO(2n, \mathbf{R})$. On the other hand, representations of the type described in Proposition 3.7 cannot exist for the real case. This is because the only odd integer m such that $GL(m, \mathbf{R})$ has discrete series is $m = 1$. Now there are only two distinct characters χ of $\mathbf{R}^\times$ with $\chi^2 = 1$.

Proof of Proposition 3.7. In this case, by Lemma 3.4, $\mathfrak{a}_R = \{0\}$. Thus by Lemma 1.3, the constituents cannot be irreducibly induced. The fact that each irreducible constituent of $i_{G,M}(\sigma)$ is a linear combination of representations induced from proper parabolic subgroups follows from a theorem of Kazhdan **[K]** since we know from Theorem 3.5 that the irreducible constituents are not elliptic. However since this is the first example in which non-elliptic representations are not irreducibly induced, it is interesting to show that directly.

In this case we again have $I_1(\sigma) = I_e$, $I_2(\sigma) = I_o$, and $R \simeq \mathbf{Z}_2^{r-1}$. Write $p = d_2$, and suppose that $m_1, \ldots, m_p$ are odd and $m_{p+1}, \ldots, m_r$ are even. Then R can be generated by $s_1 = c_1c_2$, $s_2 = c_2c_3, \ldots, s_{p-1} = c_{p-1}c_p$, $s_{p+1} = c_{p+1}, \ldots, s_r = c_r$.

For $1 \leq i \leq p$, let M_i be the Levi subgroup of G so that $M \subset M_i$ and $M_i \simeq GL(m_i) \times G(n - m_i)$. Fix i and define d', d_1', d_2' as in Theorem 3.2 with respect to $i_{M_i,M}(\sigma)$. Then $d_1' = d_1$ and $d_2' = d_2 - 1$. Thus $d' = r - 1$ and $d_2' > 0$ is even, so that $R' \simeq \mathbf{Z}_2^{r-2}$ and every irreducible constituent of $i_{M_i,M}(\sigma)$ is elliptic. Let $S = \{s_1, \ldots, s_{p-1}, s_{p+1}, \ldots, s_r\}$ be the set of generators of R. Then R_i has generators

$$S_i = \begin{cases} S\backslash\{s_1\}, & \text{if } i = 1; \\ (S\backslash\{s_{i-1}, s_i\}) \cup \{s_{i-1}s_i\}, & \text{if } 2 \leq i \leq p-1; \\ S\backslash\{s_{p-1}\}, & \text{if } i = p. \end{cases}$$

As in the proof of Proposition 2.7, for $\kappa, \kappa' \in \widehat{R}$ and $\kappa_i \in \widehat{R}_i$, $\pi_\kappa \oplus \pi_{\kappa'} = i_{G,M_i}(\tau_{\kappa_i})$ if $\kappa|_{R_i} = \kappa'|_{R_i} = \kappa_i$ and $\kappa \neq \kappa'$.

Now fix $\kappa_0 \in \widehat{R}$ and define κ_i, $1 \leq i \leq p-1$ by $\kappa_i(s_j) = \kappa_0(s_j)$, $j \neq i$, and $\kappa_i(s_i) = -\kappa_0(s_i)$. Define $\kappa_p = \kappa_0$. Then for $1 \leq i \leq p$, $\kappa_{i-1} \neq \kappa_i$, but κ_{i-1} and κ_i have the same restriction to R_i. Now since p

is odd we can write

$$\pi_{\kappa_0} = \sum_{i=1}^{p} \frac{(-1)^{i+1}}{2}(\pi_{\kappa_{i-1}} + \pi_{\kappa_i}),$$

and this expresses π_{κ_0} as a linear combination of properly induced representations of the desired form. □

Suppose that we are in the situation that $d = r$ and d_2 is even so that $\prod_{i=1}^r c_i \in R$. For $\kappa \in \widehat{R}$, define $\varepsilon(\kappa) = \kappa(\prod_{i=1}^r c_i)$. Then the following can be proven in the same way as Proposition 2.7.

PROPOSITION 3.8. *Suppose that $d = r$ and d_2 is even. Then $\Theta_\kappa^e = \varepsilon(\kappa)\Theta_1^e$ for all $\kappa \in \widehat{R}$.*

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