BIHOLOMORPHIC CONVEX MAPPINGS OF BALL IN $\mathbb{C}^n$

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Biholomorphic convex mappings from unit ball in $\mathbb{C}^n$ into $\mathbb{C}^n$ are studied in this paper. A Schwartz type lemma for the class of mappings and a necessary and sufficient condition under which a holomorphic mapping is biholomorphic and convex are established. The results are used to describe some characteristics of the image of the class of mappings.

1. Introduction. Since Loebe discovered as early as 1907 his "Verzerrugsatz", classical distortion theorems for families of univalent functions defined the unit disc in the complex plane $\mathbb{C}$ have developed systematically in depth and scope. For several variables, H. Cartan showed his interest in the field and conjectured [1] that the magnitude of the determinant of the complex Jacobian of a normalized biholomorphic mapping on the unit polydisc in $\mathbb{C}^2$ should have a finite upper and a positive lower bound. But, it was pointed out in [2] that the conjecture is not correct. That is the distortion theorem for general biholomorphic mappings in several variables does not hold. This suggests one has to find some of their subclasses for which the distortion theorem can still hold. In 1988 S. Gong, C. H. FitzGerald and R. W. Bernard [3] obtained first time the upper and lower bound of the magnitude of the determinant of the Jacobian of a normalized, convex and biholomorphic mapping from the unit ball $B = \{Z = (z^1, z^2) \in \mathbb{C}^2 | |Z| < 1\}$ into $\mathbb{C}^2$. After that Taishun Liu [4] generalized the result to the unit ball in $\mathbb{C}^n$ and X. A. Zheng [5] to the bounded symmetric domains.

It is worth pointing out that the estimate of the magnitude of the determinant of the Jacobian of a holomorphic mapping is one of the generalization of distortion theorems of one variable. The estimation of eigenvalues of the Hermitian matrix, the product of the complex Jacobian of a holomorphic mapping and its transpose conjugate, should also be considered to be another form of generalization of distortion theorems of one complex variable. In this contribution we will discuss the type of distortion theorems and its application to biholomorphic convex mappings.

The paper is arranged as follows: In §2 we will prove a Schwartz type lemma of biholomorphic convex mappings from the unit ball in $\mathbb{C}^n$ into $\mathbb{C}^n$. The lemma describes the type of distortion theorems given by us. In §3 we will give a necessary and sufficient condition under which a holomorphic mapping is biholomorphic and convex. In [6, 7] two necessary and sufficient conditions which are equivalent were also given. However, in the section we would quote a counter-example by Taishun Liu in [4] to illustrate that the conditions are not sufficient. In §4, as an application of the above results, we will study the geometric characteristics of the image of biholomorphic convex mappings.

In this paper summation convention is used.

2. A Schwartz type lemma of biholomorphic convex mappings. B denotes the unit ball in $\mathbb{C}^n$, that is

$$B = \left\{ Z = (z^1, \ldots, z^n) \in \mathbb{C}^n \,|\, |Z|^2 = \sum_{i=1}^{n} |z^i|^2 < 1 \right\}.$$

When $n = 1$ replace B by the symbol D. The Jacobian of a mapping f from B to $\mathbb{C}^n$ is denoted by J_f. So-called normalized holomorphic mapping means the holomorphic one with $f(0) = 0$ and $J_f(0) = I$, I is the unit matrix.

In this section we demonstrate a Schwartz type lemma of biholomorphic convex mappings from B into $\mathbb{C}^n$.

THEOREM 1. *Let $f: B \to \mathbb{C}^n$ be a biholomorphic convex mapping. Then*

$$\left(\frac{1 + |Z|}{1 - |Z|} \right)^2 g_{\alpha\bar{\beta}} \, dz^\alpha \, d\bar{z}^\beta \geq f^* \delta_{ij} \, dw^i \, d\bar{w}^j \geq \left(\frac{1 - |Z|}{1 + |Z|} \right)^2 g_{\alpha\bar{\beta}} \, dz^\alpha \, d\bar{z}^\beta$$

holds, where

$$g_{\alpha\bar{\beta}} = \frac{(1 - |Z|^2)\delta_{\alpha\beta} + \bar{z}^\alpha z^\beta}{(1 - |Z|^2)^2}$$

is the Bergman metric of the unit ball B in $\mathbb{C}^n$ and the inequalities are sharp.

The proof of the theorem depends on the following two lemmas.

LEMMA 1. *Suppose that $F: B \to \mathbb{C}^n$, $W = F(\xi)$ is a biholomorphic convex mapping and $\xi = F^{-1}(W)$ is the inverse mapping of $F(\xi)$.*

Then

$$4\delta_{\alpha\beta}|a| \frac{\partial \xi^\alpha}{\partial w^i} \frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}\bigg|_{\xi=0} v^i \overline{v}^j$$

$$\geq 2\left|\operatorname{Re}\left(\delta_{\alpha\beta}\frac{\partial^2 \xi^\alpha}{\partial w^k \partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}\frac{\partial w^k}{\partial z^l}\bigg|_{\xi=0} a^l v^i \overline{v}^j\right)\right|$$

holds for any $a = (a^1, \dots, a^n) \in B$ *and* $V = (v^1, \dots, v^n) \in \mathbb{C}^n$, *where* w^i ($i = 1, 2, \dots, n$) *are components of* W *and* ξ^α ($\alpha = 1, 2, \dots, n$) *ones of* ξ.

LEMMA 2. *Suppose that* $f\colon B \to \mathbb{C}^n$, $W = f(Z)$, *is a biholomorphic convex mapping with* $f(0) = 0$ *and* $Z = f^{-1}(W)$ *is the inverse mapping of* $f(Z)$. *Then*

$$4|Z| g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i}\frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j$$

$$\geq \left|2\operatorname{Re}\left\{ g_{\beta\overline{\gamma}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \overline{z}^\gamma}{\partial \overline{w}^j}\overline{z}^\alpha z^\beta v^i \overline{v}^j + |Z|^2 g_{\alpha\overline{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j \right. \right.$$

$$\left. \left. + (1-|Z|^2) g_{\alpha\overline{\beta}}\frac{\partial^2 z^\alpha}{\partial w^k \partial w^i}\frac{\partial \overline{z}^\beta}{\partial \overline{w}^j}\frac{\partial w^k}{\partial z^\gamma}z^\gamma v^i \overline{v}^j \right\}\right|$$

holds for any $Z \in B$, $V = (v^1, \dots, v^n) \in \mathbb{C}^n$.

Proof of Lemma 1. Since $F(B)$ is a convex set we have

$$\lambda F(\xi) + (1-\lambda)F(\xi_0) \in F(B)$$

for any $\lambda \in [0, 1]$ and $\xi, \xi_0 \in B$. Fix ξ_0 and define a holomorphic mapping ξ_λ from B to B by

$$\xi_\lambda = F^{-1}(\lambda F(\xi) + (1-\lambda)F(\xi_0)), \qquad \xi \in B.$$

From Schwartz's lemma of the unit ball in $\mathbb{C}^n$, it follows that

$$g_{\alpha\overline{\beta}}(\xi)v^\alpha \overline{v}^\beta \geq g_{\alpha\overline{\beta}}(\xi_\lambda)\frac{\partial \xi_\lambda^\alpha}{\partial \xi^k}\frac{\partial \overline{\xi}_\lambda^\beta}{\partial \overline{\xi}^l}v^k \overline{v}^l$$

or

$$(1) \quad g_{\alpha\overline{\beta}}(\xi)v^\alpha \overline{v}^\beta \geq \lambda^2 g_{\alpha\overline{\beta}}(\xi_\lambda)\frac{\partial \xi_\lambda^\alpha}{\partial w^i}(\xi_\lambda)\frac{\partial w^i}{\partial \xi^k}(\xi)\frac{\partial \overline{\xi}_\lambda^\beta}{\partial \overline{w}^j}(\xi_\lambda)\frac{\partial \overline{w}^j}{\partial \overline{\xi}^l}(\xi)v^k \overline{v}^l.$$

The inequality (1) can be rewritten as

$$g_{\alpha\overline{\beta}}(\xi)\frac{\partial\xi^{\alpha}}{\partial w^{i}}\frac{\partial\overline{\xi}^{\beta}}{\partial\overline{w}^{j}}\bigg|_{\xi}v^{i}\overline{v}^{j}-\lambda^{2}g_{\alpha\overline{\beta}}(\xi_{\lambda})\frac{\partial\xi_{\lambda}^{\alpha}}{\partial w^{i}}\frac{\partial\overline{\xi}_{\lambda}^{\beta}}{\partial\overline{w}^{j}}\bigg|_{\xi_{\lambda}}v^{i}\overline{v}^{j}\geq 0.$$

For $\lambda\in[0,1)$ we have

$$\lim_{\lambda\to 1}\left[\frac{g_{\alpha\overline{\beta}}(\xi)\frac{\partial\xi^{\alpha}}{\partial w^{i}}\frac{\partial\overline{\xi}^{\beta}}{\partial\overline{w}^{j}}|_{\xi}v^{i}\overline{v}^{j}-\lambda^{2}g_{\partial\overline{\beta}}(\xi_{\lambda})\frac{\partial\xi_{\lambda}^{\alpha}}{\partial w^{i}}\frac{\partial\overline{\xi}^{\beta}}{\partial\overline{w}^{j}}|_{\xi_{\lambda}}v^{i}\overline{v}^{j}}{1-\lambda}\right]\geq 0.$$

Straightforward calculation yields

$$0\leq\lim_{\lambda\to 1}\frac{d}{d\lambda}\left[\lambda^{2}g_{\alpha\overline{\beta}}(\xi_{\lambda})\frac{\partial\xi_{\lambda}^{\alpha}}{\partial w^{i}}\frac{\partial\overline{\xi}_{\lambda}^{\beta}}{\partial\overline{w}^{j}}\bigg|_{\xi_{\lambda}}v^{i}\overline{v}^{j}\right]$$

$$=\lim_{\lambda\to 1}2\left\{\lambda g_{\alpha\overline{\beta}}\frac{\partial\xi_{\lambda}^{\alpha}}{\partial w^{i}}\frac{\partial\overline{\xi}_{\lambda}^{\beta}}{\partial\overline{w}j}\bigg|_{\xi_{\lambda}}v^{i}\overline{v}^{j}\right.$$

$$+\operatorname{Re}\left[\lambda^{2}g_{\alpha\overline{\beta}}\frac{\partial^{2}\xi_{\lambda}^{\alpha}}{\partial w^{k}\partial w^{i}}\frac{\partial\overline{\xi}_{\lambda}^{\beta}}{\partial\overline{w}^{j}}\bigg|_{\xi_{\lambda}}(w^{k}(\xi)-w^{k}(\xi_{0}))v^{i}\overline{v}^{j}\right.$$

$$\left.\left.+\lambda^{2}\frac{\partial\xi_{\lambda}^{\alpha}}{\partial w^{i}}\frac{\partial\overline{\xi}_{\lambda}^{\beta}}{\partial\overline{w}^{j}}\bigg|_{\xi_{\lambda}}v^{i}\overline{v}^{j}\frac{\partial g_{\alpha\overline{\beta}}}{\partial\xi_{\lambda}^{\gamma}}\frac{\partial\xi^{\gamma}}{\partial w^{k}}\bigg|_{\xi_{\lambda}}(w^{k}(\xi)-w^{k}(\xi_{0}))\right]\right\}.$$

Noting that $\xi_{\lambda}\to\xi$ as $\lambda\to 1$, we obtain at the point ξ

$$(2)\quad 2g_{\alpha\overline{\beta}}\frac{\partial\xi_{\lambda}^{\alpha}}{\partial w^{i}}\frac{\partial\overline{\xi}^{\beta}}{\partial\overline{w}^{j}}v^{i}\overline{v}^{j}$$

$$+2\operatorname{Re}\left\{g_{\alpha\overline{\beta}}\frac{\partial^{2}\xi^{\alpha}}{\partial w^{k}\partial w^{i}}\frac{\partial\overline{\xi}^{\beta}}{\partial\overline{w}^{j}}(w^{k}(\xi)-w^{k}(\xi_{0}))v^{i}\overline{v}^{j}\right.$$

$$\left.+\frac{\partial\xi^{\alpha}}{\partial w^{i}}\frac{\partial\overline{\xi}^{\beta}}{\partial\overline{w}^{j}}v^{i}\overline{v}^{j}\frac{\partial g_{\alpha\overline{\beta}}}{\partial\xi^{\gamma}}\frac{\partial\xi^{\gamma}}{\partial w^{k}}(w^{k}(\xi)-w^{k}(\xi_{0}))\right\}\geq 0.$$

The inequality (2) holds for any $\xi,\xi_{0}\in B$. Now fix ξ and take $\xi_{0}=\frac{t\xi}{|\xi|}$ in (2) where $t\in D$, D is the unit disc in $\mathbb{C}$. We then obtain

a nonnegative harmonic function defined on the unit disc by

$$H(t) = 2g_{\alpha\overline{\beta}}\frac{\partial \xi^\alpha}{\partial w^i}\frac{\partial \overline{\xi}^\beta_\lambda}{\partial \overline{w}^j}v^i\overline{v}^j$$

$$+ 2\operatorname{Re}\left\{g_{\alpha\overline{\beta}}\frac{\partial^2 \xi^\alpha}{\partial w^k\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}\left(w^k(\xi) - w^k\left(\frac{t\xi}{|\xi|}\right)\right)v^i\overline{v}^j\right.$$

$$\left. + \frac{\partial \xi^\alpha}{\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}v^i\overline{v}^j\frac{\partial g_{\alpha\overline{\beta}}}{\partial \xi^\gamma}\frac{\partial \xi^\gamma}{\partial w^k}\left(2^k(\xi) - w^k\left(\frac{t\xi}{|\xi|}\right)\right)\right\}.$$

It follows from Harnack's inequality that

$$\frac{1 - |t|}{1 + |t|}H(0) \le H(t) \le H(0)\frac{1 + |t|}{1 - |t|}.$$

Now taking $t = |\xi|$, we obtain

$$\frac{1 - |\xi|}{1 + |\xi|}H(0) \le H(|\xi|) \le H(0)\frac{1 + |\xi|}{1 - |\xi|}$$

or

(3) $$\frac{1 - |\xi|}{1 + |\xi|}H(|\xi|) \le H(0) \le H(|\xi|)\frac{1 + |\xi|}{1 - |\xi|},$$

where

$$H(|\xi|) = 2g_{\alpha\overline{\beta}}\frac{\partial \xi^\alpha}{\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}v^i\overline{v}^j$$

and

$$H(0) = 2g_{\alpha\overline{\beta}}\frac{\partial \xi^\alpha}{\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}v^i\overline{v}^j$$

$$+ 2\operatorname{Re}\left\{g_{\alpha\overline{\beta}}\frac{\partial^2 \xi^\alpha}{\partial w^k\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}(\xi)(w^k(\xi) - w^k(0))v^i\overline{v}^j\right.$$

$$\left. + \frac{\partial \xi^\alpha}{\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}v^i\overline{v}^j\frac{\partial g_{\alpha\overline{\beta}}}{\partial \xi^\gamma}\frac{\partial \xi^\gamma}{\partial w^k}(\xi)(w^k(\xi) - w(0))\right\}.$$

The inequalities (3) can be rewritten as

$$\frac{4|\xi|}{1 - |\xi|}g_{\alpha\overline{\beta}}\frac{\partial \xi^\alpha}{\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}v^i\overline{v}^j$$

$$\ge 2\operatorname{Re}\left\{g_{\alpha\overline{\beta}}\frac{\partial^2 \xi^\alpha}{\partial w^k\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}v^i\overline{v}^j(w^k(\xi) - w^k(0))\right.$$

$$\left. + \frac{\partial \xi^\alpha}{\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}v^i\overline{v}^j\frac{\partial g_{\alpha\overline{\beta}}}{\partial \xi^\gamma}\frac{\partial \xi^\gamma}{\partial w^k}(w^k(\xi) - w^k(0))\right\}$$

$$\ge \frac{-4|\xi|}{1 + |\xi|}g_{\alpha\overline{\beta}}\frac{\partial \xi^\alpha}{\partial w^i}\frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j}v^i\overline{v}^j.$$

Taking $\xi = \nu a$, $\nu \in (0, 1)$, and letting $\nu \to 0$ in the inequalities above, we have

$$(4) \qquad 4|a|\delta_{\alpha\bar{\beta}}\frac{\partial\xi^\alpha}{\partial w^i}\frac{\partial\bar{\xi}^\beta}{\partial\overline{w}^j}v^i\overline{v}^j\bigg|_{\xi=0}$$

$$\geq \left|2\,\mathrm{Re}\left(\delta_{\alpha\bar{\beta}}\frac{\partial^2\xi^\alpha}{\partial w^k\partial w^i}\frac{\partial\bar{\xi}^\beta}{\partial\overline{w}^j}\frac{\partial w^k}{\partial\xi^l}a^l v^i\overline{v}^j\right)\right|\bigg|_{\xi=0}.$$

Lemma 1 is completely proved.

Proof of Lemma 2. Let $\varphi_a: B \to B$ be a holomorphic automorphism defined by

$$\varphi_a(\xi) = \frac{a - \xi}{1 - \xi\overline{a}'}A,$$

where $A = sI + \frac{\overline{a}'a}{1+s}$, $s = \sqrt{1 - |a|^2}$, $a \in B$. Denote $Z = \varphi_a(\xi)$. The inverse mapping of φ_a is as follows:

$$\xi = \varphi^{-1}(Z) = \frac{a - Z}{1 - Z\overline{a}'}A.$$

Let $F(\xi)$ in Lemma 1 be $f(\varphi_a(\xi))$. Straightforward calculation gives

$$(5) \qquad \frac{\partial^2\xi^\alpha}{\partial w^k\partial w^i}\bigg|_{\xi=0} = \frac{\partial^2\xi^\alpha}{\partial z^\mu\partial z^\nu}\frac{\partial z^\mu}{\partial w^k}\frac{\partial z^\nu}{\partial w^i} + \frac{\partial\xi^\alpha}{\partial z^\nu}\frac{\partial^2 z^\nu}{\partial w^k\partial w^i}$$

$$= \frac{\overline{a}^\mu A_\nu^\alpha + \overline{a}^\nu A_\mu^a}{(1 - |a|^2)^2}\frac{\partial z^\nu}{\partial w^k}\frac{\partial z^\mu}{\partial w^i} - \frac{A_\mu^\alpha}{1 - |a|^2}\frac{\partial^2 z^\mu}{\partial w^k\partial w^i},$$

$$\frac{\partial\bar{\xi}^\beta}{\partial\overline{w}^j}\bigg|_{\xi=0} = -\frac{\overline{A}_\sigma^\beta}{1 - |a|^2}\frac{\partial\overline{z}^\sigma}{\partial\overline{w}^j},$$

$$\frac{\partial w^k}{\partial\xi^l}\bigg|_{\xi=0} = (\overline{a}^l a^\tau - A_l^\tau)\frac{\partial w^k}{\partial z^\tau}.$$

Substituting (5) into the right side in (4) we have at $\xi = 0$ ($Z = a$)

$$(6) \quad 2\,\mathrm{Re}\left(\delta_{\alpha\bar{\beta}}\frac{\partial^2\xi^\alpha}{\partial w^k\partial w^i}\frac{\partial\bar{\xi}^\beta}{\partial\overline{w}^j}\frac{\partial w^k}{\partial\xi^l}\bigg|_{\xi=0}a^l v^i\overline{v}^j\right)$$

$$= -2\,\mathrm{Re}\left\{(\overline{a}^\mu g_{\nu\bar{\sigma}} + \overline{a}^\nu g_{\mu\bar{\sigma}})a^\nu\frac{\partial z^\mu}{\partial w^i}\frac{\partial\overline{z}^\sigma}{\partial\overline{w}^j}v^i\overline{v}^j\right.$$

$$\left.+ (1 - |a|^2)g_{\mu\bar{\sigma}}\frac{\partial^2 z^\mu}{\partial w^k\partial w^j}\frac{\partial\overline{z}^\sigma}{\partial\overline{w}^\gamma}\frac{\partial w^k}{\partial z^\tau}v^i\overline{v}^j\right\}.$$

Notice that the metric $(g_{\alpha\bar{\beta}})_{1\leq\alpha,\beta\leq n}$ on B is invariant under its holomorphic automorphism. That is to say

$$(7) \qquad \delta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial w^i}\frac{\partial\bar{\xi}^\beta}{\partial\overline{w}^j}\bigg|_{\xi=0}v^i\overline{v}^j = g_{\alpha\bar{\beta}}(a)\frac{\partial z^\alpha}{\partial w^i}\frac{\partial\overline{z}^\beta}{\partial\overline{w}^j}\bigg|_{Z=a}v^i\overline{v}^j.$$

Combining (6), (7) with (4), we obtain at $Z = a$.

$$(8) \quad 4|a|g_{\alpha\bar{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial\overline{z}^\beta}{\partial\overline{w}^j}v^i\overline{v}^j$$

$$\geq \left|2\,\mathrm{Re}\left\{\overline{a}^\alpha a^\beta g_{\beta\bar{\gamma}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial\overline{z}^\gamma}{\partial\overline{w}^j}\overline{z}^\alpha z^\beta v^i\overline{v}^j\right.\right.$$

$$+|a|^2 g_{\alpha\bar{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial\overline{z}^\beta}{\partial\overline{w}^j}v^i\overline{v}^j$$

$$\left.\left.+(1-|a|^2)g_{\alpha\bar{\beta}}\frac{\partial^2 z^\alpha}{\partial w^k\partial w^i}\frac{\partial\overline{z}^\beta}{\partial\overline{w}^j}\frac{\partial w^k}{\partial z^\gamma}a^\gamma v^i\overline{v}^j\right\}\right|.$$

The inequality (8) is valid for any $a \in B$. we complete the proof of Lemma 2.

Proof of Theorem 1. From the inequality (8) it follows that

$$(9) \quad 2|Z|(2-|Z|)g_{\alpha\bar{\beta}}\frac{\partial\overline{z}^\alpha}{\partial w^i}\frac{\partial\overline{z}^\beta}{\partial\overline{w}^j}v^i\overline{v}^j$$

$$\geq 2\,\mathrm{Re}\left\{\overline{z}^\alpha z^\beta g_{\beta\bar{\gamma}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial\overline{z}^\gamma}{\partial\overline{w}^j}v^i\overline{v}^j\right.$$

$$\left.+(1-|Z|^2)g_{\alpha\bar{\beta}}\frac{\partial^2 z^\alpha}{\partial w^k\partial w^i}\frac{\partial\overline{z}^\beta}{\partial\overline{w}^j}\frac{\partial w^k}{\partial z^\gamma}z^\gamma v^i\overline{v}^j\right\},$$

for any $V \in \mathbb{C}^n$. Using the definition of $g_{\beta\bar{\gamma}}$ we have

$$(10) \qquad\qquad z^\beta g_{\beta\bar{\gamma}} = \frac{z^\gamma}{(1-|Z|^2)^2}$$

and it is easy to see that the second term of the right side in (9)

$$(11) \qquad (1 - |Z|^2) g_{\alpha\overline{\beta}} \frac{\partial^2 z^\alpha}{\partial w^k \partial w^i} \frac{\partial \overline{\xi}^\beta}{\partial \overline{w}^j} \frac{\partial w^k}{\partial z^\gamma} z^\gamma v^i \overline{v}^j$$

$$= z^\gamma \frac{\partial}{\partial z^\gamma} \left\{ (1 - |Z|^2) g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}_\beta}{\partial \overline{w}^j} v^i \overline{v}^j \right\}$$

$$+ |Z|^2 g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j$$

$$- \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j \left[|Z|^2 g_{\alpha\overline{\beta}} + \frac{\overline{z}^\beta z^\alpha}{(1 - |Z|^2)^2} \right]$$

$$= z^\gamma \frac{\partial}{\partial z^\gamma} \left[(1 - |Z|^2) g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j \right]$$

$$- \frac{\overline{z}^\alpha z^\beta}{(1 - |Z|^2)^2} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j.$$

Substituting (10), (11) into (9) immediately gives

$$(12) \qquad 2|Z|(2 - |Z|) g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j$$

$$\geq 2 \operatorname{Re} \left\{ z^\gamma \frac{\partial}{\partial z^\gamma} \left[(1 - |Z|^2) g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j \right] \right\}.$$

Noting the Euclidean inner product $\langle dz^\alpha, dz^\beta \rangle = 2\delta_{\alpha\overline{\beta}}$, $\langle dz^\alpha, d\overline{z}^\beta \rangle = 0$, we easily see that the right side in (12)

$$(13) \qquad 2 \operatorname{Re} \left\{ z^\gamma \frac{\partial}{\partial z^\gamma} \left[(1 - |Z|^2) g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j \right] \right\}$$

$$= \frac{1}{2} \left\langle d \left[(1 - |Z|^2) g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j \right], \ d|Z|^2 \right\rangle.$$

By substituting (13) into (12), it follows that

$$(14) \qquad 2|Z|(2 - |Z|) g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j$$

$$\geq \frac{1}{2} \left\langle d \left[(1 - |Z|^2) g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j \right], \ d|Z|^2 \right\rangle.$$

The inequality (14) can be written as

$$(15) \qquad \frac{2(2 - |Z|)}{1 - |Z|^2} \geq \left\langle d \left\{ \lg \left[(1 - |Z|^2) g_{\alpha\overline{\beta}} \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} v^i \overline{v}^j \right] \right\}, \ d|Z| \right\rangle.$$

Hence, integrating both sides of (15) one obtains

$$\int_0^{|Z|} \frac{2(2-r)}{1-r^2}\, dr \geq \lg\left[(1-|Z|^2)g_{\alpha\bar{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \bar{z}^\beta}{\partial \overline{w}^j}v^i\bar{v}^j\right] - \lg|V|^2.$$

So

$$(16) \qquad \frac{(1+|Z|)^3}{(1-|Z|)}|V|^2 \geq (1-|Z|^2)g_{\alpha\bar{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \bar{z}^\beta}{\partial \overline{w}^j}v^i\bar{v}^j.$$

(16) implies that for any $Z \in B$ the following matrix

$$\frac{(1+|Z|)^3}{1-|Z|}I - (1-|Z|^2)\left(g_{\alpha\bar{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \bar{z}^\beta}{\partial \overline{w}^j}\right)_{1\leq i,j\leq n}$$

is positive semi-definite. So we get the following matrix inequality

$$(17) \qquad \left(\frac{1-|Z|}{1+|Z|}\right)^2 (g_{\alpha\bar{\beta}})_{1\leq \alpha,\beta\leq n} \leq \left(\delta_{ij}\frac{\partial w^i}{\partial z^\alpha}\frac{\partial \overline{w}^j}{\partial \bar{z}^\beta}\right)_{1\leq \alpha,\beta\leq n}.$$

Similarly, we can prove

$$(18) \qquad \left(\frac{1-|Z|}{1+|Z|}\right)^2 (g_{\alpha\bar{\beta}})_{1\leq \alpha,\beta\leq n} \geq \left(\delta_{ij}\frac{\partial w^i}{\partial z^\alpha}\frac{\partial \overline{w}^j}{\partial \bar{z}^\beta}\right)_{1\leq \alpha,\beta\leq n}.$$

Combining (17) with (18), we complete the proof of the inequalities in Theorem 1.

Now let us give an extreme example to illustrate that the inequalities in Theorem 1 are sharp. Set $n = 2$ and let

$$W = f(Z) = f^1(z^1, z^2), \quad f^1(z^1, z^2) = \left(\frac{z^1}{1-z^1}, \frac{z^2}{1-z^1}\right).$$

It is easy to check that the mapping is normalized, convex and biholomorphic and its Jacobian is

$$J_f(Z) = \begin{pmatrix} \dfrac{1}{(1-z^1)^2} & 0 \\ \dfrac{z^2}{(1-z^1)^2} & \dfrac{1}{1-z^1} \end{pmatrix}.$$

Take $V = (1, 0)$. Then

$$VJ_f(Z) = V\left(\frac{\partial W}{\partial Z}\right) = \left(\frac{1}{(1-z^1)^2}, 0\right)$$

and

$$(19) \quad VJ_f(Z)\overline{J_f(Z)}'\overline{V}'$$

$$= \frac{1}{(1-z^1-\bar{z}^1+|z^1|^2)^2} = \frac{1}{(1-2r_1\cos\theta_1+r_1^2)^2},$$

where $z^1 = r_1 e^{i\theta_1}$. On the other hand, we have

$$(20) \qquad \left(\frac{1 - |Z|}{1 + |Z|}\right)^2 V(g_{\alpha\bar{\beta}})_{\alpha,\beta=1,2} \overline{V}' = \frac{1 - |z^2|^2}{(1 + |Z|)^4},$$

$$(21) \qquad \left(\frac{1 + |Z|}{1 - |Z|}\right)^2 V(g_{\alpha\bar{\beta}})_{\alpha,\beta=1,2} \overline{V}' = \frac{1 - |z^2|^2}{(1 - |Z|)^4}.$$

If $z^2 = 0$, $z^1 = r_1$ ($\theta_1 = 0$), we obtain from (19) and (20)

$$(22) \qquad V J_f(Z)\overline{J_{f(Z)}}' \overline{V}' = \left(\frac{1 + |Z|}{1 - |Z|}\right)^2 V(g_{\alpha\bar{\beta}})_{\alpha,\beta=1,2} \overline{V}'.$$

If $z^2 = 0$, $z^1 = -r_1$ ($\theta_1 = \pi$), then from (19) and (21) we get

$$(23) \qquad V J_f(Z)\overline{J_f(Z)}' \overline{V}' = \left(\frac{1 - |Z|}{1 + |Z|}\right)^2 V(g_{\alpha\bar{\beta}})_{\alpha,\beta=1,2} \overline{V}'.$$

The two inequalities (22) and (23) imply that the inequalities in Theorem 1 are sharp.

3. Necessary and sufficient conditions of biholomorphic convex mappings. Denote $B_r = \{Z \in \mathbb{C}^n | |Z| < r < 1\}$ and the boundary of B_r by S_r. We will demonstrate the following theorem.

THEOREM 2. *If* $f\colon B \to \mathbb{C}^n$ *is a holomorphic mapping with* $f(0) = 0$ *and* $|\det J_f(Z)| > 0$, *then the following statements are equivalent.*

(i) f *is a biholomorphic convex mapping.*

(ii) *If* $Z \in B$ *and* $b = (b^1, \ldots, b^n) \in \mathbb{C}^n$ *satisfy* $\mathrm{Re}(b^\alpha \bar{z}^\alpha) = 0$, *then*

$$(24) \qquad |b|^2 + \mathrm{Re}\left(b^\beta b^\gamma \frac{\partial w^i}{\partial z^\beta} \frac{\partial \overline{w}^j}{\partial z^\gamma} z^\alpha \frac{\partial^2 z^\alpha}{\partial w^i \partial w^j}\right) \geq 0.$$

Proof. First prove (ii) from (i).

Since $f(B)$ is a convex set, the mapping

$$Z_\lambda = f^{-1}(\lambda f(Z_1) + (1 - \lambda) f(Z_2)),$$

$Z_1, Z_2 \in B$ and $0 \leq \lambda \leq 1$, defines a holomorphic mapping from $B \times B \to B$. Denote Carathéodory distance of the two points P and Q of a domain Ω in $\mathbb{C}^n$ by $\mathbb{C}_\Omega(P, Q)$. Writing $F(Z_1, Z_2) = Z_\lambda$ we see [8]

$$C_B(F(Z_1, Z_2), 0) \leq C_{B \times B}((Z_1, Z_2), (0, 0))$$
$$= \max\{C_B(Z_1, 0), C_B(Z_2, 0)\}$$

by the contraction property of Carathéodory distance for holomorphic mappings. Recalling the Carathéodory distance on the unit ball in $\mathbb{C}^n$

$$C_B(Z, 0) = \frac{1}{2} \lg \frac{1 + |Z|}{1 - |Z|}, \qquad Z \in B,$$

then one has

$$|Z_\lambda| \le \max\{|Z_1|, |Z_2|\}.$$

It implies that for any $\lambda \in (0, 1)$ and $Z_1, Z_2 \in B_r$, $Z_\lambda \in B_r$. Thus $f(B_r)$ is also a convex set. Since, moreover, f is biholomorphic the image $f(S_r)$ (or $\partial f(B_r)$) of S_r under the mapping f is a real convex hypersurface. The second fundamental form S of the hypersurface is positive semi-definite. In other words, for any tangent vector U to $f(S_r)$, we have

$$(25) \qquad S(U, U) \ge 0.$$

Note that $f(S_r)$ is an isothermal surface of the function $r^2 = |Z|^2 \circ f^{-1}$ and

$$\text{grad}\, r^2 = \frac{\partial r^2}{\partial w^i} \frac{\partial}{\partial \overline{w}^i} + \frac{\partial r^2}{\partial \overline{w}^i} \frac{\partial}{\partial w^i} = \overline{z}^\alpha \frac{\partial z^\alpha}{\partial w^i} \frac{\partial}{\partial \overline{w}^i} + z^\alpha \frac{\partial \overline{z}^\alpha}{\partial \overline{w}^i} \frac{\partial}{\partial w^i}$$

is a normal vector to $f(S_r)$. Denote the normal vector by ν. $\frac{\nu}{|\nu|}$ is a unit normal vector. Assume that $U = a^i \frac{\partial}{\partial w^i} + \overline{a}^i \frac{\partial}{\partial \overline{w}^i}$ with

$$\text{Re}(a^i \overline{z}^\alpha (\partial z^\alpha / \partial w^i)) = 0.$$

Then we have

$$S(U, U) = \left\langle \widetilde{D}_U \frac{\nu}{|\nu|}, U \right\rangle,$$

where $\widetilde{D}$ is the connection in R^{2n} (directional derivative). Via $\langle \nu, U \rangle = 0$ we have

$$\left\langle \widetilde{D}_U \frac{\nu}{|\nu|}, U \right\rangle = \frac{1}{|\nu|} \langle \widetilde{D}_U \nu, U \rangle.$$

Hence, the above inequality (25) can be rewritten as

$$(26) \qquad \frac{1}{|\nu|} \left(a^i \overline{a}^j \frac{\partial^2 r^2}{\partial w^i \partial \overline{w}^j} + \text{Re} \left(a^i a^j \frac{\partial^2 r^2}{\partial w^i \partial w^j} \right) \right) \ge 0.$$

Now set $b = a \frac{\partial Z}{\partial w}$ or $a^i = b^\beta (\partial w^i / \partial z^\beta)$. From $\text{Re}(a^i z^{\overline{z}^\beta} (\partial z^\beta / \partial w^i)) = 0$, it follows that $\text{Re}(b^\beta \overline{z}^\beta) = 0$. Substituting $a^i = b^\beta (\partial w^i / \partial z^\beta)$ into (26) we can get (24).

Now prove (i) from the condition (ii). We divide the proof into three steps.

In the first step, we show that if the mapping f is injective on B_μ $(0 < \mu < 1)$, then $f(B_\mu)$ is a convex set. In fact, since f is an injection on B_μ, $f(S_r)$ is a real hypersurface in $\mathbb{C}^n$ for any $r < \mu$. If $U = a^i(\partial/\partial w^i) + \overline{a}^i(\partial/\partial \overline{w}^i)$ is a real tangent vector to $f(S_r)$, the second fundamental form is

$$(27) \quad S(U, U) = \frac{1}{|\nu|} \left(a^i \overline{a}^j \frac{\partial^2 r^2}{\partial w^i \partial \overline{w}^j} + \operatorname{Re} \left(a^i a^j \frac{\partial^2 r^2}{\partial w^i \partial w^j} \right) \right)$$

$$= \frac{1}{|\nu|} \left(a^i \overline{a}^j \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\alpha}{\partial \overline{w}^j} + \operatorname{Re} \left(a^i a^j z^\alpha \frac{\partial^2 z^\beta}{\partial w^i \partial w^j} \right) \right).$$

Set $b = a\frac{\partial Z}{\partial W}$. We have $\operatorname{Re}(b^\beta \overline{z}^\beta) = 0$. Recalling the condition (ii) we get

$$S(U, U) \geq 0.$$

Hence, $f(S_r)$ is a real convex hypersurface for any $r < \mu$, and then we conclude that $f(B_\mu)$ is a convex set.

In the second step, we prove if the mapping f is injective on B_μ, then f is also injective on $\overline{B}_\mu$. Since $f(B_\mu)$ is a convex set $f(B_\mu)$ is starlike with respect to the origin. Therefore, we have

$$\langle dr^2|_{B_\mu}, \, d\rho^2 \rangle \geq 0$$

thanks to the necessary condition for starlike mappings [9], where $\rho^2 = |W|^2$. If the statement is not true then there are at least two distinct points $X, Y \in \overline{B}_\mu$ such that $f(X) = f(Y)$. We know, for all $0 \leq s \leq 1$, $sf(X)$, $sf(Y) \in f(\overline{B}_\mu)$. Because f is a holomorphic immersion one can obtain the curve $X(s)$ with $X(1) = X$ which falls in B by the method of analytic continuation such that $f(X(s)) = sf(X)$. That is $X(s) = f^{-1}(sf(X))$ is a univalent component of the inverse images of the segment $sf(X)$. Since

$$\frac{dr^2(X(s))}{ds} = \frac{1}{s} \langle dr^2(X(s)), d\rho \rangle|_{W=sf(X)} \geq 0$$

for $0 \leq s \leq 1$, we have

$$r^2(X(s)) \leq r^2(X(1)) = r^2(X) = \mu.$$

Thus

$$X(s) \in \overline{B}_\mu.$$

Suppose that $Y(s)$ is another univalent component of the inverse images of the segment $sf(X)$, but $Y(1) = Y$. A similar discussion shows $Y(s) \in \overline{B}_\mu$. Let

$$\mathscr{R} = \{s \in [0, 1] | X(s) = Y(s)\}.$$

If the set $\mathscr{R}$ is nonempty then the supremum s^* of $\mathscr{R}$ exists. Because $\mathscr{R}$ is a closed set $s^* \in \mathscr{R}$ and $s^* < 1$ due to $X(1) \neq Y(1)$. This implies that for any small positive ε satisfying $s^* < s^* + \varepsilon \leq 1$ we have $X(s^* + \varepsilon) \neq Y(s^* + \varepsilon)$. But

$$f(X(s^* + \varepsilon)) = f(Y(s^* + \varepsilon)) = (s^* + \varepsilon)f(X).$$

This is contrary to f being locally biholomorphic at $X(s^*)$. If $\mathscr{R}$ is empty then $X(0) \neq Y(0)$. In other words, at this case we have at least two points $X(0)$ and $Y(0)$ such that $f(X(0)) = f(Y(0)) = 0$. Since f is an injection on B_μ, at least one of the two points $X(0)$ and $Y(0)$ must be a boundary point of $\overline{B}_\mu$. Suppose that $X(0) \in \partial \overline{B}_\mu$. Let $B(X(0), \delta)$ be the open ball central at $X(0)$ with radius δ, which is so small that $B(X(0), \delta) \cap \mathscr{U}_0$ is empty, where $\mathscr{U}_0 \subset B_\mu$ is the neighborhood of the origin in B such that f is biholomorphic on it. Because of the open mapping theorem $f(B(X(0), \delta) \cap B_\mu)$ is an open set and $f(\mathscr{U}_0)$ an open set including the origin of $\mathbb{C}^n$. So the origin is also a boundary point of the open set $f(B(X(0), \delta) \cap B_\mu)$ and then $\{f(B(X(0), \delta) \cap B_\mu)\} \cap \{f(\mathscr{U}_0)\}$ is not empty. This implies that for any $W \in \{f(B(X(0), \delta) \cap B_\mu)\} \cap \{f(\mathscr{U}_0)\}$ it has two distinguished inverse images on B_μ. It is impossible due to the fact that f is injective on B_μ. So, f is also an injection on $\overline{B}_\mu$.

In the third step, we demonstrate $\mathscr{A} = [0, 1]$ where $\mathscr{A} = \{t \in [0, 1] | f$ is injective on $B_t\}$. Since $f(0) = 0$, $J_f(0) \neq 0$, the set $\mathscr{A}$ is nonempty. We claim that $\mathscr{A}$ is a closed set. In fact, if $0 < t_1 \in \mathscr{A}$ then all $t \leq t_1$ fall in $\mathscr{A}$. Therefore, to prove $\mathscr{A}$ is a closed set it is sufficient only to prove that if $t^* > t$ and all t fall in $\mathscr{A}$ then t^* is also in $\mathscr{A}$. If the result is not true, there exist at least two distinct points X_1, X_2 in B_{t^*} such that $f(X_1) = f(X_2)$. Because of $|X_1| < t^*$, $|X_2| < t^*$, one can find a t^{**} satisfying $|X_1| < t^{**} < t^*$, $|X_2| < t^{**} < t^*$. The formulas above imply $X_1, X_2 \in B_{t^{**}}$. But $t^{**} \in \mathscr{A}$. This yields $f(X_1) \neq f(X_2)$. It is impossible. So $\mathscr{A}$ is a closed set. Finally, we need to prove the set $\mathscr{A}$ is also an open set. For the end, only to verify that if f is a one to one mapping on B_t then there is a positive ε small enough such that f is also a one to one mapping on $B_{t+\varepsilon}$. If not, there is a sequence $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that one can find two sequences $\{X_n\}$, $\{Y_n\}$ satisfying the following conditions:

$$X_n, Y_n \in B_{t+\varepsilon_n}, \quad X_n \neq Y_n, \quad f(X_n) = f(Y_n)$$

for all $n = 1, 2, \ldots$. Obviously,

$$\lim_{n \to \infty} |Y_n| = \lim_{n \to \infty} |Y_n| = t.$$

It is easy to see that $\{X_n\}$, $\{Y_n\}$ are bounded sequences. So there exist two subsequences $\{X_{n_k}\}$, $\{Y_{n_k}\}$ of $\{X_n\}$, $\{Y_n\}$ which converge to X and Y respectively, that is

$$\lim_{k \to \infty} X_{n_k} = X, \quad \lim_{k \to \infty} Y_{n_k} = Y.$$

The two subsequences satisfy

$$f(X_{n_k}) = f(Y_{n_k}), \qquad X_{n_k} \neq Y_{n_k}, \quad \text{for } k = 1, 2, \ldots,$$

and

$$\lim_{k \to \infty} |X_{n_k}| = \lim_{k \to \infty} |Y_{n_k}| = t.$$

Obviously, $X, Y \in \partial B_t$. If $X \neq Y$ that is contrary to the result in the second step. If $X = Y$ it implies that there are two points denoted still by $\{X_{n_k}\}$, $\{Y_{n_k}\}$ in any neighborhood of $X = Y$ such that $X_{n_k} \neq Y_{n_k}$, but $f(X_{n_k}) = f(Y_{n_k})$. It is obviously impossible due to the assumption of f being local biholomorphic.

Combining the above argument we obtain $\mathscr{A} = [0, 1]$. Hence f is biholomorphic and f is convex from the argument of the first step.

It is worth noting when $n = 1$, the inequality (24) has the following form:

$$|b|^2 + \text{Re}\left(b^2 \left(\frac{dw}{dz}\right)^2 \overline{z} \frac{d^2 z}{dw^2}\right) \geq 0, \qquad \text{Re}(b\overline{z}) = 0.$$

Because of

$$\frac{d^2 w}{dz^2} \left(\frac{dz}{dw}\right)^2 = -\frac{dw}{dz} \frac{d^2 z}{dw^2},$$

we have

$$(28) \quad |b|^2 - \text{Re}\left(\overline{z} b^2 \frac{dw}{dz} \left(\frac{dz}{dw}\right)^2 \frac{d^2 w}{dz^2}\right) = |b|^2 - \text{Re}\left(\overline{z} b^2 \frac{d^2 w}{dz^2} \frac{dz}{dw}\right)$$

$$= |b|^2 + \text{Re}\left(\overline{z} b^2 \frac{f''}{f'}\right) \geq 0.$$

Furthermore, from $\text{Re}(b\overline{z}) = b\overline{z} - \overline{b}z = 0$, $\overline{z}b = -\overline{b}z$, the inequality (28) can be rewritten as

$$|b|^2 \left(1 + \text{Re} \frac{z f''}{f'}\right) \geq 0.$$

It just is the necessary and sufficient condition of univalent convex functions defined on the unit disc in the complex plane $\mathbb{C}$.

In [6] K. Kikuchi stated the following theorem: Let f be a holomorphic mapping from the ball B into $\mathbb{C}^n$ with $f(0) = 0$ and $\det J_f(Z) \neq 0$, $Z \in B$. Then f is biholomorphic and convex if and only if

$$(29) \qquad \mathrm{Re}[|ZA|^2 + \overline{Z}J_f^{-1}(Z)\tilde{A}] \geq 0,$$

where

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \quad a_j \geq 0, \qquad j = 1, 2, \ldots, n,$$

$$\tilde{A} = \begin{pmatrix} ZA\dfrac{\partial^2 f_1}{\partial Z' \partial Z}(Z)A'Z' \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ ZA\dfrac{\partial^2 f_n}{\partial Z' \partial Z}(Z)A'Z' \end{pmatrix}$$

with $ZA \neq 0$. The necessary and sufficient condition is equivalent to the one given by Suffridge in [7].

We quote a counter-example given by Taishun Liu in [4] to show that the condition (29) is not sufficient to conclude the section. Set

$$f(Z) = \left(\frac{z_1}{1 - z_1}, \frac{z_2}{1 - z_2}, \ldots, \frac{z_n}{1 - z_n} \right).$$

Clearly, the mapping is biholomorphic and a simple calculation gives

$$(30) \qquad J_f^{-1}(Z) = \begin{pmatrix} (1 - z_1)^2 & 0 & \cdots & 0 \\ 0 & (1 - z_2)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (1 - z_n)^2 \end{pmatrix}$$

and $\partial^2 f_j/\partial Z' \partial Z$ is the matrix of which the unique non-zero element is $\frac{1}{(1-z_j)^3}$ at j column and j row.

Putting (30) into the left side of (29), we have

$$\mathrm{Re}[|ZA|^2 + \overline{Z}J_f^{-1}(Z)\tilde{A}] = \mathrm{Re}\left[\sum_{j=1}^{n} a_j^2 |z_j|^2 \left(\frac{1 + z_j}{1 + z_j} \right) \right] > 0.$$

Hence, according to the result by Kikuchi, the mapping

$$f(Z) = \left(\frac{z_1}{1 - z_1}, \frac{z_2}{1 - z_2}, \ldots, \frac{z_n}{1 - z_n} \right)$$

should be convex. That is to say for $0 < r < 1$

$$\frac{1}{n}\sum_{k=1}^{n} f(re_k) \in \{f(B)\}$$

where e_k is the unit vector of which kth component equals to 1. So

$$f^{-1}\left[\frac{1}{n}\sum_{k=1}^{n} f(re_k)\right] \in B.$$

But one has

$$(31) \qquad f^{-1}\left[\frac{1}{n}\sum_{k=1}^{n} f(re_k)\right] = \left(\frac{r}{r+n(1-r)}, \cdots, \frac{r}{r+n(1-r)}\right).$$

When r is close to 1 the right side of (31) does not belong to B. Therefore, the mapping f is not convex.

4. Characteristics of the image of biholomorphic convex mappings. As an application of the theorems in §§1 and 2, we now consider characteristics of the image of biholomorphic convex mappings from B to $\mathbb{C}^n$.

THEOREM 3. *Suppose that* $f: B \to \mathbb{C}^n$ *is a normalized biholomorphic convex mapping. Then*

$$(1) \qquad \frac{(1-\mu)^5}{\mu(1+\mu)^3} \leq \textit{principal curvature of } f(S_\mu) \leq \frac{(1+\mu)^4}{\mu(1-\mu)^4},$$

$$(2) \qquad \omega_{2n-1}\frac{1-\mu}{2\mu(1+\mu)}\int_0^\mu \frac{(1-t)^{n+1}t^{2n-1}}{(1+t)^{3n+3}}\,dt \leq \textit{volume of } f(S_\mu)$$

$$\leq n\omega_{2n-1}\frac{1+\mu}{2\mu(1-\mu)}\int_0^\mu \frac{(1+t)^{n+1}t^{2n-1}}{(1-t)^{3n+3}}\,dt,$$

where ω_{2n-1} *is the volume of the unit sphere in* R^{2n}.

Proof. Fix $Z \in B - \{0\}$ and $b \in \mathbb{C}^n$ fulfilling $\text{Re}(b^\beta \bar{z}^\beta) = 0$. Let $\widetilde{Z} = \frac{tZ}{|Z|}$ and $\check{b} = \frac{tb}{|Z|}$, $t \in D$. Then $\widetilde{Z} \in B$ and $\text{Re}(\check{b}^\beta \bar{\widetilde{z}}^\beta) = 0$. Since $W = f(Z)$ is a normalized biholomorphic convex mapping it follows from Theorem 2 that

$$\frac{|t|^2|b|^2}{|Z|^2} + \text{Re}\left\{|t|^2\frac{t\bar{z}^\alpha b^\beta b^\gamma}{|Z|^3}\left(\frac{\partial w^i}{\partial z^\beta}\frac{\partial w^j}{\partial z^\gamma}\frac{\partial^2 z^\alpha}{\partial w^i \partial w^j}\right)\Big|_{\widetilde{z}}\right\} \geq 0$$

or

$$(32) \qquad \text{Re}\left\{|b|^2 + \frac{t\bar{z}^\alpha b^\beta b^\gamma}{|Z|}\left(\frac{\partial w^i}{\partial z^\beta}\frac{\partial w^j}{\partial z^\gamma}\frac{\partial^2 z^\alpha}{\partial w^i \partial w^j}\right)\Big|_{\widetilde{z}}\right\} \geq 0.$$

The left side of (32) is a real-valued and nonnegative harmonic function. From Harnack's inequality we have

$$(33)\ |b|^2 \left(\frac{1-|t|}{1+|t|}\right) \le |b|^2 + \mathrm{Re}\left\{ \frac{t\overline{z}^\alpha b^\beta b^\gamma}{|Z|} \left(\frac{\partial w^i}{\partial z^\beta} \frac{\partial w^j}{\partial z^\gamma} \frac{\partial^2 z^\alpha}{\partial w^i \partial w^j} \right)\bigg|_{\overline{z}} \right\}$$

$$\le |b|^2 \left(\frac{1+|t|}{1-|t|}\right).$$

If we take $t = |Z|$, the formulae (33) becomes as follows:

$$|b|^2 \frac{1-|Z|}{1+|Z|} \le |b|^2 + \mathrm{Re}\left\{ \overline{z}^\alpha b^\beta b^\gamma \left(\frac{\partial w^i}{\partial z^\beta} \frac{\partial w^j}{\partial z^\gamma} \frac{\partial^2 z^\alpha}{\partial w^i \partial w^j} \right) \right\} \le |b|^2 \frac{1+|Z|}{1-|Z|}$$

for any $Z \in B - \{0\}$ with $\mathrm{Re}(b^\beta \overline{z}^\beta) = 0$. Now taking $b = a(\frac{\partial Z}{\partial W})$ and noting (27) we have

$$(34)\qquad a^i \frac{\partial z^\beta}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} \overline{a}^j \left(\frac{1-|Z|}{1+|Z|}\right) \le |\nu| S(U, U)$$

$$\le a^i \frac{\partial z^\beta}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} \overline{a}^j \left(\frac{1+|Z|}{1-|Z|}\right),$$

where

$$|\nu| = \sqrt{ \overline{z}^\alpha \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^i} z^\beta }, \qquad U = a^i \frac{\partial}{\partial w^i} + \overline{a}^i \frac{\partial}{\partial \overline{w}^i}.$$

From Theorem 1, it follows that

$$\delta_{ij} \ge \left(\frac{1-|Z|}{1+|Z|}\right)^2 \left[\frac{1}{1-|Z|^2} \frac{\partial z^\alpha}{\partial \overline{w}^i} \frac{\partial \overline{z}^\alpha}{\partial \overline{w}^j} + \frac{1}{(1-|Z|^2)^2} \overline{z}^\alpha \frac{\partial z^\alpha}{\partial \overline{w}^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} z^\beta \right],$$

$$\delta_{ij} \le \left(\frac{1+|Z|}{1-|Z|}\right)^2 \left[\frac{1}{1-|Z|^2} \frac{\partial z^\alpha}{\partial \overline{w}^i} \frac{\partial \overline{z}^\alpha}{\partial \overline{w}^j} + \frac{1}{(1-|Z|^2)^2} \overline{z}^\alpha \frac{\partial z^\alpha}{\partial \overline{w}^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} z^\beta \right].$$

Therefore, the inequalities

$$(35)\quad |a|^2 \ge \frac{1}{(1+|Z|)^4} \left[(1-|Z|^2) a^i \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\alpha}{\partial \overline{w}^j} \overline{a}^j + \left| a^i \overline{z}^\alpha \frac{\partial z^\alpha}{\partial w^i} \right|^2 \right],$$

$$|a|^2 \le \frac{1}{(1-|Z|)^4} \left[(1-|Z|^2) a^i \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\alpha}{\partial \overline{w}^j} \overline{a}^j + \left| a^i \overline{z}^\alpha \frac{\partial z^\alpha}{\partial w^i} \right|^2 \right]$$

hold for any $Z \in B$. By the Schwartz inequality

$$0 \le \left| a^i \overline{z}^\alpha \frac{\partial z^\alpha}{\partial w^i} \right|^2 \le |Z|^2 \left| a^i \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\alpha}{\partial \overline{w}^j} \overline{a}^j \right|$$

(34) can be read as

$$(36) \quad \frac{1 - |Z|}{(1 + |Z|)^3} a^i \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\alpha}{\partial \overline{w}^j} \overline{a}^j \le |a|^2 \le \frac{1}{(1 - |Z|)^4} a^i \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\alpha}{\partial \overline{w}^j} \overline{a}^j.$$

Substituting (36) into (34), we obtain

$$(37) \quad |\nu| S(U, U) \ge \frac{(1 - |Z|)^5}{1 + |Z|} |U|^2,$$

$$|\nu| S(U, U) \le \left(\frac{1 + |Z|}{1 - |Z|} \right)^4 |U|^2.$$

On the other hand, again using Theorem 1 we get

$$\left(\frac{1 + |Z|}{1 - |Z|} \right)^2 (g_{\alpha\overline{\beta}}) \ge \left(\frac{\partial w^i}{\partial z^\alpha} \frac{\partial \overline{w}^i}{\partial \overline{z}^\beta} \right) \ge \left(\frac{1 - |Z|}{1 + |Z|} \right)^2 (g_{\alpha\overline{\beta}}).$$

This yields

$$\left(\frac{1 - |Z|}{1 + |Z|} \right)^2 (g^{\alpha\overline{\beta}}) \le \left(\frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^i} \right) \le \left(\frac{1 + |Z|}{1 - |Z|} \right)^2 (g^{\overline{\beta}\alpha}),$$

where $(g^{\overline{\beta}\alpha})$ is the inverse matrix of $(g_{\alpha\overline{\beta}})$. Hence

$$\frac{(1 - |Z|)^3}{1 + |Z|} (\delta_{\beta\alpha} - \overline{z}^\beta z^\alpha) \overline{z}^\beta z^\alpha \le \overline{z}^\alpha \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^j} z^\beta$$

$$\le \frac{(1 + |Z|)^3}{1 - |Z|} (\delta_{\beta\alpha} - \overline{z}^\beta z^\alpha) \overline{z}^\beta z^\alpha.$$

So, we obtain

$$(38) \quad |Z|^2 (1 - |Z|)^4 \le |\nu|^2 \le |Z|^2 (1 + |Z|)^4.$$

Putting (38) into (37), we have

$$\frac{(1 - \mu)^5}{\mu (1 + \mu)^3} \le \text{principal curvature of } f(S_\mu) \le \frac{(1 + \mu)^4}{\mu (1 - \mu)^4}.$$

To yield the volume of the hypersurface $f(S_\mu)$ of $(2n - 1)$-dimension, we consider the function $r^2 = |Z|^2 \circ f^{-1}$. Denote $\Delta = 4\partial^2 / \partial w^i \partial \overline{w}^i$. Then we have

$$(39) \quad \int_{f(B_\mu)} \frac{\Delta r^2}{1 - r^2} dW = - \int_{f(B_\mu)} \frac{|\nabla r^2|}{(1 - r^2)^2} dW$$

$$+ \frac{1}{1 - \mu^2} \int_{\partial f(B_\mu)} \frac{\partial r^2}{\partial n} d\sigma,$$

where dW is the volume element of $\mathbb{C}^n = \mathscr{R}^{2n}$, $d\sigma$ that of $\partial f(B_\mu)$ and $n = \frac{\nu}{|\nu|}$ the unit normal vector. Because of

$$\frac{\partial r^2}{\partial n} = \frac{\nu}{|\nu|} r^2 = \frac{2}{|\nu|}\left(\overline{z}^\alpha \frac{\partial z^\alpha}{\partial w^i} \frac{\partial \overline{z}^\beta}{\partial \overline{w}^i} z^\beta\right) = 2|\nu|$$

(39) becomes

$$(40) \qquad \int_{f(B_\mu)} \frac{(1-r^2)\Delta r^2 + |\nabla r^2|^2}{(1-r^2)^2}\, dW = \frac{2}{1-\mu^2} \int_{f(S_\mu)} |\nu|\, d\sigma.$$

On the other hand, we have

$$\Delta r^2 = 4\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \overline{z}^\beta}{\partial \overline{w}^i}, \qquad |\nabla r^2|^2 = 4\overline{z}^\alpha \frac{\partial z^\alpha}{\partial w^i}\frac{\partial \overline{z}^\beta}{\partial \overline{w}^i} z^\beta.$$

Hence

$$(41) \qquad \frac{(1-r^2)\Delta r^2 + |\nabla r^2|^2}{(1-r^2)^2} = g_{\alpha\overline{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \overline{z}^\beta}{\partial \overline{w}^i}.$$

Putting (41) into (40) we obtain

$$(42) \qquad \int_{f(B_\mu)} g_{\alpha\overline{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \overline{z}^\beta}{\partial \overline{w}^i}\, dW = \frac{2}{1-\mu^2}\int_{f(S_\mu)} |\nu|\, d\sigma.$$

From Theorem 1 we have

$$(43) \qquad \left(\frac{1+|Z|}{1-|Z|}\right)^2 g_{\alpha\overline{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \overline{z}^\beta}{\partial \overline{w}^i} \geq n \geq \left(\frac{1-|Z|}{1+|Z|}\right)^2 g_{\alpha\overline{\beta}}\frac{\partial z^\alpha}{\partial w^i}\frac{\partial \overline{z}^\beta}{\partial \overline{w}^i}.$$

Substituting (43) into (42), we obtain

$$(44) \qquad n\int_{f(B_\mu)}\left(\frac{1-r}{1+r}\right)^2 dW \leq \frac{2}{1-\mu^2}\int_{f(S_\mu)} |\nu|\, d\sigma$$

$$\leq n\int_{f(B_\mu)}\left(\frac{1+r}{1-r}\right)^2 dW.$$

Combining (39) with (44), we obtain

$$\frac{n(1-\mu)}{2\mu(1+\mu)}\int_{f(B_\mu)}\left(\frac{1-r}{1+r}\right)^2 dW$$

$$\leq \int_{f(S_\mu)} d\sigma \leq \frac{n(1+\mu)}{2\mu(1-\mu)}\int_{f(B_\mu)}\left(\frac{1+r}{1-r}\right)^2 dW$$

or

$$(45) \qquad \int_{f(S_\mu)} d\sigma \geq \frac{n(1-\mu)}{2\mu(1+\mu)} \int_{B_\mu} \left(\frac{1-|Z|}{1+|Z|}\right)^2 |\det J_f(Z)|^2 \, dZ \, ,$$

$$\int_{f(S_\mu)} d\sigma \leq \frac{n(1+\mu)}{2\mu(1-\mu)} \int_{B_\mu} \left(\frac{1+|Z|}{1-|Z|}\right)^2 |\det J_f(Z)|^2 \, dZ \, ,$$

where dZ is the volume element of $\mathbb{C}^n = \mathscr{R}^{2n}$. Finally, again using Theorem 1 we obtain

$$(46) \qquad \frac{(1+|Z|)^{n-1}}{(1-|Z|)^{3n+1}} \geq |\det J_f(Z)|^2 \geq \frac{(1-|Z|)^{n-1}}{(1+|Z|)^{3n+1}}.$$

Putting (46) into (45) concludes Theorem 3.

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