BERGMAN AND HARDY SPACES WITH SMALL EXPONENTS

Kehe Zhu

We show that for each $0 < p < 1$ the dual space of the Hardy and weighted Bergman space on the open unit ball is isomorphic to the Bloch space (with equivalent norms) under certain volume integral pairing.

1. Introduction. We present a new approach to an old problem, namely, the problem of describing the continuous linear functionals on the Bergman and Hardy spaces with $0 < p < 1$. We restrict our attention to the open unit ball in $\mathbf{C}^n$, even though our approach has the potential to generalize to bounded symmetric domains.

Let B_n be the open unit ball in $\mathbf{C}^n$ with boundary ∂B_n. Let $H(B_n)$ denote the space of all holomorphic functions in B_n. For $0 < p < +\infty$ and $\alpha > -1$ we let

$$L_a^p(B_n, dv_\alpha) = H(B_n) \cap L^p(B_n, dv_\alpha)$$

denoted the weighted Bergman space, where

$$dv_\alpha(z) = C_\alpha(1 - |z|^2)^\alpha \, dv(z).$$

Here dv is volume measure on B_n and C_α a normalizing constant so that dv_α has total mass 1. For $f \in L_a^p(B_n, dv_\alpha)$ we write

$$\|f\|_{\alpha,p} = \left[\int_{B_n} |f(z)|^p \, dv_\alpha(z) \right]^{1/p}.$$

A linear functional F on $L_a^p(B_n, dv_\alpha)$ is bounded if there exists a constant $C > 0$ such that $|F(f)| \le C\|f\|_{\alpha,p}$ for all f in $L_a^p(B_n, dv_\alpha)$. The dual space of $L_a^p(B_n, dv_\alpha)$, denoted $L_a^p(B_n, dv_\alpha)^*$, consists of all bounded linear functionals on $L_a^p(B_n, dv_\alpha)$. For each $0 < p < +\infty$ the space $L_a^p(B_n, dv_\alpha)^*$ is a Banach space with the norm

$$\|F\| = \sup\{|F(f)| : \|f\|_{\alpha,p} \le 1\}.$$

Note that $L_a^p(B_n, dv_\alpha)$ itself is not a Banach space when $0 < p < 1$.

For $0 < p < +\infty$ we let $H^p(B_n)$ denote the Hardy space consisting of holomorphic functions f in B_n such that

$$\|f\|_{H^p} = \sup_{0<r<1} \left[\int_{\partial B_n} |f(rw)|^p \, d\sigma(w) \right]^{1/p} < +\infty,$$

where $d\sigma$ is the normalized Euclidean measure on ∂B_n. For $0 < p < +\infty$ we let $H^p(B_n)^*$ denote the dual space of $H^p(B_n)$ consisting of linear functionals F on $H^p(B_n)$ such that

$$\|F\| = \sup\{|F(f)| : \|f\|_{H^p} \leq 1\} < +\infty.$$

Again $H^p(B_n)^*$ is a Banach space with the above norm for each $0 < p < +\infty$, even though $H^p(B_n)$ itself is not a Banach space when $0 < p < 1$.

The dual space of $H^p(B_n)$ for $0 < p < 1$ has been studied by several authors. In the one dimensional case Romberg [8] described the dual of H^p for all $p \in (0, 1)$ except $p = 1/(k + 1)$ (where k is any positive integer). Duren, Romberg, and Shields [5] completed the characterization of the dual of H^p for all $0 < p < 1$ in the case $n = 1$. They described the dual of H^p of the unit disk in terms of the Lipschitz spaces and the Zygmund class. These results were then generalized to the polydisk by Frazier [6] and to the unit ball by Hahn and Mitchell [7]. The basic ideas and constructions in these papers are somewhat similar. See [4] for more information on H^p of the disk with $0 < p < 1$.

Shapiro [10] gave a different and quite general approach to the duality problem of Bergman and Hardy spaces with $0 < p < 1$. The results in [10] were stated and proved for the disk, but it is clear that some techniques and ideas there work in more general situations. Shapiro's approach, based on the notion of Mackey topologies, is more geometric.

Coifman and Rochberg [2] also considered the duality problem for weighted Bergman spaces with small exponents. Their arguments were based on the theory of atomic decomposition for certain classes of holomorphic functions.

Our formulation of the duality and the method of approach are different from the ones mentioned above, the main difference being in the duality pairing. Let $\mathscr{B}(B_n)$ denote the Bloch space of B_n consisting of holomorphic functions f in B_n with

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup\{(1 - |z|^2)|\nabla f(z)| : z \in B_n\} < +\infty,$$

where $\nabla f(z) = (\partial f(z)/\partial z_1, \ldots, \partial f(z)/\partial z_n)$ is the complex gradient of f at z, and

$$|\nabla f(z)| = \left[\sum_{k=1}^{n} \left|\frac{\partial f}{\partial z_k}(z)\right|^2\right]^{1/2}.$$

It is well known that $\mathscr{B}(B_n)$ is a Banach space with the above norm. We describe the dual spaces of $L_a^p(B_n, dv_\alpha)$ and $H^p(B_n)$ for $0 < p < 1$ in terms of the Bloch functions. Our main results are Theorems 5 and 6, stated below as Theorems A and B, respectively.

THEOREM A. *Suppose* $0 < p \leq 1$, $\alpha > -1$, *and* $\beta = (n+1+\alpha)/p - (n+1)$. *Then the dual space of* $L_a^p(B_n, dv_\alpha)$ *is isomorphic to* $\mathscr{B}(B_n)$ *(with equivalent norms) under the duality pairing*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{B_n} f(rz)\overline{g(z)}(1 - |z|^2)^\beta \, dv(z),$$

$$f \in L_a^p(B_n, dv_\alpha), \quad g \in \mathscr{B}(B_n).$$

THEOREM B. *Suppose* $0 < p < 1$ *and* $\beta = (n/p) - (n+1)$. *Then the dual of* $H^p(B_n)$ *is isomorphic to* $\mathscr{B}(B_n)$ *(with equivalent norms) under the duality pairing*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{B_N} f(rz)\overline{g(z)}(1 - |z|^2)^\beta \, dv(z),$$

$$f \in H^p(B_n), \quad g \in \mathscr{B}(B_n).$$

It is likely that experts in this field could deduce the above results from those obtained in [5], [6], [7], [10]. Nevertheless, we think a direct proof here is more natural and desirable.

The author thanks P. Duren, B. Korenblum, S. Krantz, D. Luecking, R. Rochberg, and J. Shapiro for their interest, criticism, and pointing out additional references after the first version of the paper was circulated.

2. A class of fractional derivatives and integrals. In this section we introduce a certain type of radial fractional derivatives and integrals. Recall that $H(B_n)$ is the space of all holomorphic functions in B_n. We equip $H(B_n)$ with the topology of "uniform convergence on compact sets". Thus a linear operator T on $H(B_n)$ is continuous if and only if $Tf_k \to Tf$ uniformly on compact sets whenever $f_k \to f$ uniformly on compact sets. For $f \in H(B_n)$ and $0 < r < 1$ we define f_r in $H(B_n)$ by $f_r(z) = f(rz)$.

THEOREM 1. *For each* $\alpha > -1$ *there exists a unique linear operator* D^α *on* $H(B_n)$ *with the following properties*:

(1) D^α *is continuous on* $H(B_n)$.

(2) $D^\alpha f_r(z) = D^\alpha f(rz)$ *for all* $f \in H(B_n)$, $0 < r < 1$, *and* $z \in B_n$.

(3) $D_z^\alpha[(1 - \langle z, w \rangle)^{-(n+1)}] = C_\alpha(1 - \langle z, w \rangle)^{-(n+1+\alpha)}$ *for each* $w \in B_n$.

Proof. We first prove uniqueness. Given f in $H(B_n)$ and $0 < r < 1$ we can write

$$f_r(z) = \int_{B_n} \frac{f_r(w)\, dv(w)}{(1 - \langle z, w \rangle)^{n+1}}, \qquad z \in B_n.$$

The above integral converges uniformly for z in any compact subset of B_n. Thus properties (1) and (3) imply that

$$D^\alpha f_r(z) = \int_{B_n} D_z^\alpha[(1 - \langle z, w \rangle)^{-(n+1)}] f_r(w)\, dv(w)$$

$$= C_\alpha \int_{B_n} \frac{f_r(w)\, dv(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.$$

This proves the uniqueness of D^α.

To prove existence we first define $D^\alpha \colon H^\infty(B_n) \to H(B_n)$ by the formula

$$D^\alpha f(z) = C_\alpha \int_{B_n} \frac{f(w)\, dv(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \qquad z \in B_n.$$

This along with the reproducing property of the Bergman kernel shows that property (3) holds. Also using homogeneous expansions we easily obtain $D^\alpha f_r(z) = D^\alpha f(rz)$ for each $f \in H^\infty(B_n)$. Now for a general f in $H(B_n)$ we define $D^\alpha f$ as follows. For each $z \in B_n$ there exists $r \in (0, 1)$ and $\tilde{z} \in B_n$ such that $z = r\tilde{z}$. We define $D^\alpha f(z) = D^\alpha f_r(\tilde{z})$. (Note that $f_r \in H^\infty(B_n)$ and hence $D^\alpha f_r$ is defined.) To see that $D^\alpha f$ is well-defined, suppose $z = r_1 \tilde{z}_1 = r_2 \tilde{z}_2$ with $r_1 < r_2$, say $r_1 = r r_2$ for some $r \in (0, 1)$. Since f_{r_2} is in $H^\infty(B_n)$ we have

$$D^\alpha f_{r_1}(\tilde{z}_1) = D^\alpha f_{r r_2}(\tilde{z}_1) = D^\alpha f_{r_2}(r\tilde{z}_1) = D^\alpha f_{r_2}(\tilde{z}_2).$$

Thus $D^\alpha f$ is well-defined. Furthermore, $D^\alpha f$ is holomorphic since for each $r \in (0, 1)$ we have

$$D^\alpha f(rz) = D^\alpha f_r(z) = C_\alpha \int_{B_n} \frac{f_r(w)\, dv(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \qquad z \in B_n.$$

Thus D^α as defined above maps $H(B_n)$ to $H(B_n)$. The linearity of D^α is obvious. Property (2) follows easily from the definition of D^α. It remains to show that D^α is continuous on $H(B_n)$.

Let $f_k \to f$ in $H(B_n)$. It is clear that for each $r \in (0, 1)$ we have $f_k(rz) \to f(rz)$ uniformly for $z \in B_n$ as $k \to +\infty$. Thus for each $r \in (0, 1)$ and $k \to +\infty$ we have

$$D^\alpha f_k(rz) = C_\alpha \int_{B_n} \frac{f_k(\sqrt{r}w)\,dv(w)}{(1 - \langle \sqrt{r}z, w \rangle)^{n+1+\alpha}}$$

$$\to C_\alpha \int_{B_n} \frac{f(\sqrt{r}w)\,dv(w)}{(1 - \langle \sqrt{r}z, w \rangle)^{n+1+\alpha}}$$

$$= D^\alpha f(rz)$$

uniformly for $z \in B_n$. Since r is arbitrary, we see that $D^\alpha f_k \to D^\alpha f$ uniformly on each compact subset of B_n. This completes the proof of Theorem 1.

REMARK. The proof of the above theorem shows that for each $\alpha > -1$ the operator D^α is given by

$$D^\alpha f(z) = C_\alpha \lim_{r \to 1^-} \int_{B_n} \frac{f(rw)\,dv(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \qquad f \in H(B_n), \ z \in B_n.$$

If f in $H(B_n)$ is integrable with respect to volume measure, then it is clear that

$$D^\alpha f(z) = C_\alpha \int_{B_n} \frac{f(w)\,dv(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \qquad z \in B_n.$$

THEOREM 2. *For each $\alpha > -1$ the operator D^α is invertible on $H(B_n)$.*

Proof. For each $\alpha > -1$ define an operator D_α on $H(B_n)$ as follows.

$$D_\alpha f(z) = \lim_{r \to 1^-} \int_{B_n} \frac{(1 - |w|^2)^\alpha f(rw)\,dv(w)}{(1 - \langle z, w \rangle)^{n+1}}, \qquad f \in H(B_n), \ z \in B_n.$$

Using arguments similar to those in the proof of Theorem 1 we can show that each D_α is a well-defined continuous linear operator on $H(B_n)$. Moreover, by Fubini's theorem and Proposition 7.1.2 of [9] we easily check that $D^\alpha D_\alpha f = D_\alpha D^\alpha f$ for all $f \in H^\infty(B_n)$. By the continuity of D^α and D_α and the density of $H^\infty(B_n)$ in $H(B_n)$ we conclude that D_α is the inverse of D^α on $H(B_n)$ for each $\alpha > -1$.

REMARK. If $\alpha > -1$ and f is in $L_a^1(B_n, dv_\alpha)$, then we have

$$D_\alpha f(z) = \int_{B_n} \frac{(1 - |w|^2)^\alpha f(w)\, dv(w)}{(1 - \langle z, w \rangle)^{n+1}}, \qquad z \in B_n.$$

We call D^α a radial fractional differential operator of order α and D_α a radial fractional integral operator of order α. Note that when $-1 < \alpha < 0$ "integration" here is really "differentiation" and vice versa.

3. Weighted Bergman spaces.

In this section we characterize the dual of $L_a^p(B_n, dv_\alpha)$ for each $\alpha > -1$ and $0 < p \leq 1$. Recall that

$$\|f\|_{\alpha, p} = \left[\int_{B_n} |f(z)|^p\, dv_\alpha(z) \right]^{1/p}, \qquad f \in L_a^p(B_n, dv_\alpha).$$

Also recall that for $\alpha > -1$ the operator P_α is given by

$$P_\alpha f(z) = \int_{B_n} \frac{f(w)\, dv_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \qquad z \in B_n.$$

The proof of the next lemma is due to D. Luecking.

LEMMA 3. *For each* $0 < p \leq 1$ *and* $\alpha > -1$ *there exists a constant* $C > 0$ *such that*

$$\int_{B_n} |f(z)|(1 - |z|^2)^{(n+1+\alpha)/p - (n-1)}\, dv(z) \leq C \|f\|_{\alpha, p}$$

for all f *in* $L_a^p(B_n, dv_\alpha)$.

Proof. Fix z in B_n and let $D(z)$ be the open Euclidean ball in $\mathbf{C}^n$ with center z and radius $(1 - |z|)/2$. By the subharmonicity of $|f|^p$ we have

$$|f(z)|^p \leq \frac{1}{v(D(z))} \int_{D(z)} |f(w)|^p\, dv(w).$$

Since $1 - |w| \sim 1 - |z|$ for $w \in D(z)$ and $v(D(z)) \sim (1 - |z|^2)^{n+1}$, it is easy to find a constant $C_1 > 0$ such that

$$|f(z)| \leq C_1 (1 - |z|^2)^{-(n+1+\alpha)/p} \|f\|_{\alpha, p}$$

for all f in $L_a^p(B_n, dv_\alpha)$ and $z \in B_n$. For $0 < p \leq 1$ we can write $|f(z)| = |f(z)|^p |f(z)|^{1-p}$ and estimate the second factor using the above inequality. This easily produces the desired result.

LEMMA 4. *Suppose $\alpha > -1$ and f is holomorphic in B_n. If either f or the function $(1-|z|^2)^{-\alpha}f(z)$ is bounded on B_n, then the function $(1-|z|^2)^{\alpha}D^{\alpha}f(z)$ is in $L^1(B_n, dv)$ and*

$$\int_{B_n} f(z)\overline{g(z)}\, dv(z) = \int_{B_n} D^{\alpha}f(z)\overline{g(z)}(1-|z|^2)^{\alpha}\, dv(z)$$

for all g in $H^{\infty}(B_n)$.

Proof. That the function $(1-|z|^2)^{\alpha}D^{\alpha}f(z)$ is in $L^1(B_n, dv)$ follows from the definition of D^{α} and 1.4.10 of [9]. The desired result then follows from Fubini's theorem and the fact that the operator P_{α} reproduces bounded holomorphic functions (see 7.1.2 of [9]). (Note that the applicability of Fubini's theorem here also follows from 1.4.10 of [9] and the assumption about f.)

THEOREM 5. *Suppose $0 < p \le 1$, $\alpha > -1$, and $\beta = (n+1+\alpha)/p - (n+1)$. Then we have $L_a^p(B_n, dv_{\alpha})^* \simeq \mathscr{B}(B_n)$ with*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{B_n} f(rz)\overline{g(z)}(1-|z|^2)^{\beta}\, dv(z),$$

$$f \in L_a^p(B_n, dv_{\alpha}), \quad g \in \mathscr{B}(B_n).$$

Proof. Fix $0 < p \le 1$, $\alpha > -1$, and $F \in L_a^p(B_n, dv_{\alpha})^*$. Since $\|f - f_r\|_{\alpha, p} \to 0$ as $r \to 1^-$, we have $F(f) = \lim_{r \to 1^-} F(f_r)$ for all $f \in L_a^p(B_n, dv_{\alpha})$. Write

$$f_r(z) = \int_{B_n} \frac{f_r(w)\, dv(w)}{(1 - \langle z, w \rangle)^{n+1}}, \quad z \in B_n.$$

Since the above integral converges in $L_a^p(B_n, dv_{\alpha})$, the continuity of F implies that

$$F(f_r) = \int_{B_n} f_r(w) F_z\left[\frac{1}{(1 - \langle z, w \rangle)^{n+1}}\right] dv(w).$$

Let

$$h(w) = \overline{F_z\left[\frac{1}{(1 - \langle z, w \rangle)^{n+1}}\right]}, \quad w \in B_n.$$

Then h is holomorphic in B_n and

$$F(f_r) = \int_{B_n} f_r(w)\overline{h(w)}\, dv(w).$$

It follows from the boundedness of F on $L_a^p(B_n, dv_\alpha)$ and 1.4.10 of [9] that Lemma 4 can be applied to obtain (The case $\beta = 0$ does not require the use of Lemma 4.)

$$F(f_r) = \int_{B_n} f_r(w)\overline{D^\beta h(w)}(1 - |w|^2)^\beta \, dv(w) \, ,$$

where

$$\beta = \frac{n + 1 + \alpha}{p} - (n + 1) \, .$$

By Property (3) of Theorem 1 we have

$$D^\beta h(w) = C_\beta F_z \overline{\left[\frac{1}{(1 - \langle z, w \rangle)^{(n+1+\alpha)/p}} \right]} \, .$$

Let $g = D^\beta h$. Then

$$F(f_r) = \int_{B_n} f_r(w)\overline{g(w)}(1 - |w|^2)^\beta \, dv(w)$$

and

$$\frac{\partial g}{\partial w_k}(w) = \frac{(n + 1 + \alpha)C_\beta}{p} F_z \overline{\left[\frac{z_k}{(1 - \langle z, w \rangle)^{(n+1+\alpha)/p+1}} \right]} \, , \qquad w \in B_n \, .$$

Using 1.4.10 of [9] and the boundedness of F on $L_a^p(B_n, dv_\alpha)$ we easily see that g is in the Bloch space $\mathscr{B}(B_n)$.

On the other hand, if g is in $\mathscr{B}(B_n)$, we shall show that the formula

$$F(f) = \lim_{r \to 1^-} \int_{B_n} f_r(z)\overline{g(z)}(1 - |z|^2)^\beta \, dv(z) \, , \qquad f \in L_a^p(B_n, dv_\alpha)$$

well defines a bounded linear functional on $L_a^p(B_n, dv_\alpha)$, where $\beta = (n + 1 + \alpha)/p - (n + 1)$. By [1] there exists a function $\varphi \in L^\infty(B_n)$ such that

$$g(z) = P_\beta \varphi(z) = \int_{B_n} \frac{\varphi(w) \, dv_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}} \, , \qquad z \in B_n \, .$$

Using Fubini's theorem and the reproducing property of P_β we easily obtain

$$\int_{B_n} f_r(z)\overline{g(z)}(1 - |z|^2)^\beta \, dv(z) = \int_{B_n} f_r(w)\overline{\varphi(w)}(1 - |w|^2)^\beta \, dv(w) \, .$$

By Lemma 3 we have

$$F(f) = \lim_{r \to 1^-} \int_{B_n} f_r(z)\overline{g(z)}(1 - |z|^2)^\beta \, dv(z)$$

$$= \int_{B_n} f(z)\overline{\varphi(z)}(1 - |z|^2)^\beta \, dv(z)$$

with $|F(f)| \le C\|\varphi\|_\infty\|f\|_{\alpha,p}$ for all f in $L_a^p(B_n, dv_\alpha)$.

4. Hardy spaces $H^p(B_n)$ with $0 < p < 1$. In this section we describe the bounded linear functionals on the Hardy space $H^p(B_n)$ with $0 < p < 1$. The result and method are similar to those in the previous section.

THEOREM 6. *Suppose* $0 < p < 1$ *and* $\alpha = n/p - (n+1)$. *Then* $H^p(B_n)^* \simeq \mathscr{B}(B_n)$ *with*

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{B_n} f(rz)\overline{g(z)}(1 - |z|^2)^\alpha \, dv(z),$$

$$f \in H^p(B_n), \quad g \in \mathscr{B}(B_n).$$

Proof. For $0 < p < 1$ it is clear that $\alpha = n/p - (n+1) > -1$. If g is in $\mathscr{B}(B_n)$ then by [1] there exists $\varphi \in L^\infty(B_n)$ such that $g = P_\alpha\varphi$. Writing g as an integral, applying Fubini's theorem, and using the reproducing property of P_α, we see that

$$\int_{B_n} f_r(z)\overline{g(z)}(1 - |z|^2)^\alpha \, dv(z) = \int_{B_n} f_r(z)\overline{\varphi(z)}(1 - |z|^2)^\alpha \, dv(z)$$

for each $f \in H^p(B_n)$ and $r \in (0, 1)$. By Theorem 11 of [7] there exists a constant $C > 0$ such that

$$\int_{B_n} |f(z)|(1 - |z|^2)^\alpha \, dv(z) \le C\|f\|_{H^p}$$

for all f in $H^p(B_n)$. This clearly shows that

$$F(f) = \lim_{r \to 1^-} \int_{B_n} f(rz)\overline{g(z)}(1 - |z|^2)^\alpha \, dv(z)$$

$$= \int_{B_n} f(z)\overline{\varphi(z)}(1 - |z|^2)^\alpha \, dv(z), \qquad f \in H^p(B_n),$$

defines a bounded linear functional on $H^p(B_n)$.

Conversely, if F is a bounded linear functional on $H^p(B_n)$, then by the first few lines of the proof of Theorem 5 we have $F(f) = \lim_{r \to 1^-} F(f_r)$ and

$$F(f_r) = \int_{B_n} f_r(w)\overline{h(w)} \, dv(w),$$

where

$$h(w) = \overline{F_z\left[\frac{1}{(1 - \langle z, w \rangle)^{n+1}}\right]}, \qquad w \in B_n.$$

If $\alpha \neq 0$, then 1.4.10 of [9] and the boundedness of F on $H^p(B_n)$ imply that either h is bounded in B_n or there exists a constant $C > 0$ such that $|h(w)| \leq C\|F\|(1 - |w|^2)^\alpha$ for all $w \in B_n$. Thus we can apply Lemma 4 to obtain

$$F(f_r) = \int_{B_n} f_r(w)\overline{g(w)}(1 - |w|^2)^\alpha\, dv(w),$$

where

$$g(w) = D^\alpha h(w) = C_\alpha \overline{F_z \left[\frac{1}{(1 - \langle z, w \rangle)^{n/p}} \right]}, \qquad w \in B_n.$$

This is obviously true if $\alpha = 0$. For each $1 \leq k \leq n$ we have

$$\frac{\partial g}{\partial w_k}(w) = \frac{nC_\alpha}{p} \overline{F_z \left[\frac{z_k}{(1 - \langle z, w \rangle)^{n/p+1}} \right]}, \qquad w \in B_n.$$

It follows easily from 1.4.10 of [9] and the boundedness of F on $H^p(B_n)$ that there exists a constant $C > 0$ such that

$$(1 - |w|^2) \left| \frac{\partial g}{\partial w_k}(w) \right| \leq C\|F\|$$

for all w in B_n and $1 \leq k \leq n$. Thus g is in $\mathscr{B}(B_n)$ and the proof of Theorem 6 is completed.

5. Further remarks. In order to generalize our results to bounded symmetric domains we need to answer the following questions:

(1) Does each weighted Bergman projection P_α map $L^\infty(\Omega)$ onto the same space?

(2) If the answer to the above question is affirmative, then how to describe the space $P_\alpha L^\infty(\Omega)$ in terms of partial derivatives?

In the case of the polydisk we can settle the above questions satisfactorily. It is proved in [11] that each P_α maps L^∞ of the polydisk onto the same space X, which consists of holomorphic functions f on the polydisk such that

$$\left(\prod_{k=1}^{n} (1 - |z_k|^2) \right) \frac{\partial^m f}{\partial z_{i_1} \cdots \partial z_{i_m}}(z)$$

is bounded on the polydisk for each $0 \leq m \leq n$ and $1 \leq i_1 < \cdots \prec i_m \leq n$. Using this result it is then shown in [11] that the dual space of each weighted Bergman space and Hardy space with $0 < p < 1$ on the polydisk is isomorphic to the above space X (with equivalent norms) under certain integral pairing over the polydisk. Note that the

duality problem for both the weighted Bergman space and the Hardy space of the polydisk with $0 < p < 1$ were also studied in [3] and [6].

REFERENCES

[1]   B. R. Choe, *Projections, the weighted Bergman spaces, and the Bloch space*, Proc. Amer. Math. Soc., **108** (1990), 127–136.

[2]   R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p*, Astérisque, **77** (1980), 11–66.

[3]   A. E. Djrbashian and F. A. Shamoian, *Topics in the Theory of A_α^p Spaces*, B. G. Teubner Verlagsgesellschaft, Leipzig, 1988.

[4]   P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[5]   P. L. Duren, B. W. Romberg and A. L. Shields, *Linear functionals on H^p spaces with $0 < p < 1$*, J. Reine Angew. Math., **238** (1969), 32–60.

[6]   A. P. Frazier, *The dual space of H^p of the polydisk for $0 < p < 1$*, Duke Math. J., **39** (1972), 369–379.

[7]   K. T. Hahn and J. Mitchell, *Representation of linear functionals in H^p spaces over bounded symmetric domains*, J. Math. Anal. Appl., **56** (1976), 379–396.

[8]   B. W. Romberg, *The H^p spaces with $0 < p < 1$*, Doctoral dissertation, University of Rochester, 1960.

[9]   W. Rudin, *Function Theory in the Unit Ball of $\mathbf{C}^n$*, Springer, New York, 1980.

[10] J. Shapiro, *Mackey topologies, reproducing kernels, and diagonal maps on the Hardy and Bergman spaces*, Duke Math. J., **43** (1976), 187–202.

[11] K. Zhu, *Weighted Bergman projections on the polydisk*, to appear in Houston J. Math.

STATE UNIVERSITY OF NEW YORK
ALBANY, NY 12222
E-mail address: kz258@albnyvms.bitnet