PACIFIC JOURNAL OF MATHEMATICS
Vol. 162, No. 1, 1994

VERTEX OPERATOR CONSTRUCTION OF STANDARD MODULES FOR $A_n^{(1)}$

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We generalize the vertex operator formula for the affine Lie algebra $A_n^{(1)}$ in the "homogeneous picture" and by using it we construct a basis of any given standard $A_n^{(1)}$-module parametrized by coloured partitions. We also obtain a similar explicit construction of vacuum spaces of standard $A_2^{(1)}$-modules.

1. Introduction. In this paper we give an explicit construction of standard (i.e. integrable highest weight) representations of affine Lie algebra $\tilde{\mathfrak{g}}$ of the type $A_n^{(1)}$.

As usual, for $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbf{C})$ we fix a Cartan subalgebra $\mathfrak{h}$ and root vectors x_α, and we identify $\mathfrak{h} \cong \mathfrak{h}^*$ via bilinear form $\langle x, y\rangle = \operatorname{tr} xy$. We denote by c the canonical central element of the affine Lie algebra $\tilde{\mathfrak{g}}$ and we write $x(i) = x \otimes t^i$ for $x \in \mathfrak{g}$ and $i \in \mathbf{Z}$. As usual we use triangular decompositions

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+, \qquad \tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+ .$$

Let $\mathfrak{n}_0 \subset \mathfrak{n}_+$ be the nilpotent radical of a maximal parabolic subalgebra of $\mathfrak{g}$ such that its Levi factor is (isomorphic to) $\mathfrak{gl}(n, \mathbf{C})$. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be the set of weights of $\mathfrak{n}_0$ (see §2). Then

$$\{x_\beta(j)\,;\ \beta \in \Gamma,\ \ j \in \mathbf{Z}\}$$

is a commutative family in $\tilde{\mathfrak{g}}$.

Let $L(\Lambda)$ be a standard $\tilde{\mathfrak{g}}$-module with a highest weight vector v_Λ. On $L(\Lambda)$ we have a projective representation $\beta \mapsto e_\beta$ of the root lattice Q of $\mathfrak{g}$ (see §5). Let

$$x_\alpha(\zeta) = \sum_{j \in \mathbf{Z}} x_\alpha(j)\zeta^j .$$

By using the formal Laurent series technique we extend the vertex operator formula for level 1 $A_n^{(1)}$-modules and for level $k \geq 1$ $A_1^{(1)}$-modules to all standard $A_n^{(1)}$-modules, based on a simple observation that the vertex operator formula for level 1 representation can be written as an equality of products of exponentials:

$$\exp\left(\sum_{\beta\in\Gamma} x_\beta(\zeta)\right)$$

$$= \exp\left(\sum_{i<0} -\varphi(i)\zeta^i/i\right) \exp\left(\sum_{\gamma\in\Gamma'} \varepsilon(\gamma,\varphi)x_\gamma(\zeta)\right)$$

$$\times \exp\left(\sum_{i>0} -\varphi(i)\zeta^i/i\right) e_\varphi \zeta^{-c-\varphi},$$

where $\varphi \in \Gamma$, $\Gamma' = s_\varphi\Gamma$ (s_φ being a reflection corresponding to the root φ), and $\varepsilon(\gamma,\varphi) \in \{\pm 1\}$. Written in this way the vertex operator formula holds for every standard module (see §6, Theorem 6.4). The above formula is to be understood as the equality of coefficients in two formal Laurent series. For example, the coefficient of ζ^m of the left-hand side has the unique summand $x_\beta(m)$ of weight β, and hence $x_\beta(m)$ can be expressed in terms of elements e_φ, $\varphi(i)$'s and $x_\gamma(i)$'s. Another consequence of the vertex operator formula is:

$$\sum_{j_1+\cdots+j_{k+1}=m} x_{\beta_1}(j_1)\cdots x_{\beta_{k+1}}(j_{k+1}) = 0, \tag{1.1}$$

where $m \in \mathbf{Z}$, $\beta_1, \dots, \beta_{k+1} \in \Gamma$, $k = \Lambda(c)$.

Set $t_0 = \prod_{\beta\in\Gamma} e_\beta$. Since $L(\Lambda) = U(\tilde{\mathfrak{n}}_-)v_\Lambda$, by using the vertex operator formula (as mentioned above) we see that a set of vectors of the form

$$t_0^p x_{\beta_1}(j_1)\cdots x_{\beta_s}(j_s)v_\Lambda, \tag{1.2}$$

where $p \in \mathbf{Z}$, $s \geq 0$, $\beta_1, \dots, \beta_s \in \Gamma$ and $j_1 \leq \cdots \leq j_s < 0$, is a spanning set of $L(\Lambda)$ (see §8, Theorem 8.2). This set of vectors is not a basis of $L(\Lambda)$—we reduce it further by expressing one monomial

$$x(\mu) = x_{\beta_1}(j_1)\cdots x_{\beta_{k+1}}(j_{k+1})$$

appearing in (1.1) in terms of the rest of them. The final result is a spanning set of vectors of the form (1.2) satisfying certain combinatorial conditions, which, in fact, is a basis of $L(\Lambda)$ (Lemma 9.4 and Remark 9.5).

Monomials of the form

$$x(\nu) = x_{\beta_1}(j_1)\cdots x_{\beta_s}(j_s),$$

where $s \geq 0$, $\beta_1, \dots, \beta_s \in \Gamma$ and $j_1 \leq \cdots \leq j_s < 0$, we call coloured partitions. When we reduce a spanning set (1.2) to a basis of $L(\Lambda)$

we use induction, and for this reason we introduce an order on the set of coloured partitions (§3) with three basic properties: it allows arguments by induction (Lemma 3.2), it respects the semigroup structure of coloured partitions (Lemma 3.3) and the set of monomials appearing in (1.1) has the smallest element (Lemma 3.4).

We may call the smallest element appearing in (1.1) the leading term of (1.1). Denote by $D(\Lambda)$ the set of all leading terms for all $m < 0$ and $\beta_1, \dots, \beta_{k+1} \in \Gamma$. By induction we see that vectors of the form (1.2) which contain $x(\mu) \in D(\Lambda)$ as a factor may be erased from the spanning set. We also identify a certain set $I(\Lambda)$ of monomials $x(\mu)$ such that $x(\mu)v_\Lambda = 0$ (Lemma 9.2). In §4 we study a set of all monomials (i.e. coloured partitions) which do not contain as a factor any $x(\nu)$ in $D(\Lambda) \cup I(\Lambda)$. For such coloured partitions we say that they satisfy difference and initial conditions.

Roughly speaking, the main theorem (Theorem 9.1) states that the set of vectors

$$t_0^p x(\nu) v_\Lambda, \tag{1.3}$$

where $p \leq 0$ and $x(\nu)$ satisfy the difference and initial conditions, is a basis of $L(\Lambda)$.

In order to prove the linear independence of such a set of vectors, we first study a particular basis of level 1 standard $\tilde{\mathfrak{g}}$-module in which vectors of the form $x(\nu)v_\Lambda$ have a simple expansion (Lemma 7.2(i)). The construction relies on the observation that if the Fock space for the homogeneous Heisenberg subalgebra of $\mathfrak{sl}(2, \mathbf{C})^\sim$ is identified with the algebra of symmetric functions, then the exponential

$$\exp\left(\sum_{i<0} -\alpha(i)\zeta^i/i\right)$$

is to be identified with the generating function for complete symmetric functions. However, the basis $\{K(\nu)(1 \otimes e^\lambda)\}$ corresponding to Schur functions is better suited for our purposes (see §7).

The second step uses Frenkel's observation that a standard module of level $k \geq 2$ may be viewed as a subspace of level 1 standard module by the use of a full subalgebra. The main point is the expansion of (to be basis) elements of the form (1.3) in terms of Schur functions basis (Lemma 9.7):

$$x(\nu)v_\Lambda \simeq aK(\nu^0)(1 \otimes e^\lambda) + \sum_{\kappa > \nu^0} b_\kappa K(\kappa)(1 \otimes e^\lambda). \tag{1.4}$$

In this formula a combinatorial argument is used to show $a \neq 0$. Another combinatorial argument shows that a map $\nu \mapsto \nu^0$ is (roughly speaking) injective (Lemma 4.6). In this way the linear independence follows.

This construction does not describe the vacuum space (for the homogeneous Heisenberg subalgebra) of a standard module, but still by its main ideas and techniques may be regarded as a part of a general approach proposed by Lepowsky and Wilson.

In §10 we extend a construction of the vacuum spaces (for the homogeneous Heisenberg subalgebra) of standard $A_1^{(1)}$-modules (Theorem 10.2) to standard $A_2^{(1)}$-modules (Theorem 10.3). In this case even a spanning result requires a delicate study of (vertex operator formula) relations (Lemma 10.9). In the proof of linear independence the analog of expansion (1.4) (Lemma 10.12) is used. This example suggests the combinatorial difficulties one may expect in the case of $A_n^{(1)}$, $n > 2$, but we fail to understand them.

Theorems 6.4, 8.2 and 9.1 are formulated in **[P]**.

Finally let us make a few remarks:

It should be noticed that the coefficient of ζ^m in the vertex operator formula is an operator of degree m on $L(\Lambda)$ (with respect to the usual homogeneous grading). For this reason we prefer to use the formal indeterminate ζ. However, from the point of view of vertex operator algebra theory and conformal field theory it is far more natural to express the level 1 setup using $z = \zeta^{-1}$ instead of ζ, and the level k setup in terms of $z = \zeta^{-k}$.

Although the starting point of our construction is the vertex operator construction of level 1 modules given by Frenkel and Kac, we obtain a different basis. Some combinatorial evidence (see Remark 9.10) suggest that there might be some connection between the basis of the form (1.3) (or the corresponding Schur functions) and the construction of level 1 standard modules in terms of Maya diagrams and paths given by Date, Jimbo, Kuniba, Miwa and Okado.

From $A_1^{(1)}$ case it seemed that one should use the vertex operator formula to obtain (and reduce further) a spanning set of $L(\Lambda)$ of the form $e_\beta x(\nu) v_\Lambda$, where $\beta \in Q$ and $x(\nu)$ satisfy the difference conditions. The construction of standard $\tilde{\mathfrak{g}}$-modules in terms of Maya diagrams and paths suggested to use a spanning set (1.3) instead. I thank E. Date, M. Jimbo and T. Miwa for stimulating conversations which inspired us to formulate the correct initial conditions. It turned out that all other ideas necessary to construct a basis came through the work with J. Lepowsky and A. Meurman, to whom I express my gratitude.

2. Affine Lie algebra $A_n^{(1)}$. Let $\mathfrak{g} = \mathfrak{sl}(n+1, C)$, $n \geq 1$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, R the corresponding root system, Q the root lattice of R. Fix a basis $\{\alpha_1, \dots, \alpha_n\}$ of R. Let $\langle x, y\rangle = \operatorname{tr} xy$ for $x, y \in \mathfrak{g}$ and identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via $\langle\ ,\ \rangle$.

Fix a bilinear map $\varepsilon: Q \times Q \to \{\pm 1\}$, i.e.

$$\varepsilon(\alpha+\beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma),$$
$$\varepsilon(\alpha, \beta+\gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma),$$

such that

$$\varepsilon(\alpha, \alpha) = -1 \quad \text{for } \alpha \in R,$$
$$\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{\langle\alpha, \beta\rangle} \quad \text{for } \alpha, \beta \in Q.$$

Then there exist root vectors $x_\alpha \in \mathfrak{g}$, $\alpha \in R$, such that (cf. **[FK]**, **[F]**, see also **[LP1]**)

$$[x_\alpha, x_\beta] = \begin{cases} \varepsilon(\alpha, \beta)x_{\alpha+\beta} & \text{if } \alpha+\beta \in R, \\ -\alpha & \text{if } \alpha+\beta = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}] + \mathbf{C}c + \mathbf{C}d$ be the affine Lie algebra associated with $\mathfrak{g}$—a Kac-Moody Lie algebra of the type $A_n^{(1)}$ (cf. **[K]**). As usual set $x(j) = x \otimes t^j$ for $x \in \mathfrak{g}$ and $j \in \mathbf{Z}$. Then commutation relations in $\tilde{\mathfrak{g}}$ are given by

$$[c, \tilde{\mathfrak{g}}] = 0,$$
$$[d, x(j)] = jx(j),$$
$$[x(i), y(j)] = [x, y](i+j) + i\langle x, y\rangle\delta_{i+j,0}c.$$

We identify $\mathfrak{g}$ with $\mathfrak{g} \otimes t^0 \subset \tilde{\mathfrak{g}}$.

Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$. Set $\tilde{\mathfrak{h}} = \mathfrak{h} + \mathbf{C}c + \mathbf{C}d$, $\tilde{\mathfrak{n}}_\pm = \mathfrak{g} \otimes t^{\pm 1}\mathbf{C}[t^{\pm 1}] + \mathfrak{n}_\pm$. Then we have a triangular decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+$.

Define $\delta \in \tilde{\mathfrak{h}}^*$ by $\delta(d) = 1$, $\delta|\mathfrak{h} + \mathbf{C}c = 0$, and $\alpha_0 = \delta - \theta$, where $\theta \in R$ is the maximal root. Set $\alpha_0^\vee = c - \theta$, $\alpha_i^\vee = \alpha_i$ for $i = 1, \dots, n$, and define fundamental weights $\Lambda_i \in \tilde{\mathfrak{h}}^*$, $i = 0, \dots, n$, by $\Lambda_i(\alpha_j^\vee) = \delta_{ij}$, $\Lambda_i(d) = 0$.

Let $e_1, \dots, e_{n+1}$ be the canonical basis in $\mathbf{R}^{n+1}$, and $R = \{e_i - e_j;\ i \neq j\}$, $\alpha_1 = e_1 - e_2, \dots, \alpha_n = e_n - e_{n+1}$. For $i \in \{1, \dots, n+1\}$ set

$\Gamma_i = \{e_i - e_j\,;\ j \neq i\}$. Notice that $R = \Gamma_1 \cup \cdots \cup \Gamma_{n+1}$ and that each Γ_i is a basis of Q.

Set $\gamma_j = e_1 - e_{n+2-j}$, $j = 1, \ldots, n$, and $\gamma_1 > \gamma_2 > \cdots > \gamma_n$. Set

$$\Gamma = \Gamma_1 = \{\gamma_1, \ldots, \gamma_n\}, \qquad \widetilde{\Gamma}_- = \{x_\gamma(j)\,;\ \gamma \in \Gamma,\ j < 0\}$$

and define an order on $\widetilde{\Gamma}_-$ by $x_\beta(i) < x_\gamma(j)$ if $i < j$ or $i = j$, $\beta < \gamma$.

Set $\tilde{\mathfrak{n}}_0 = \operatorname{span}_{\mathbf{C}} \widetilde{\Gamma}_-$. Notice that $\tilde{\mathfrak{n}}_0$ is a commutative subalgebra of $\tilde{\mathfrak{n}}_-$.

Denote by

$$\mathfrak{s} = \sum_{j \in \mathbf{Z}\setminus\{0\}} \mathfrak{h} \otimes t^j + \mathbf{C}c$$

the infinite dimensional (graded) Heisenberg subalgebra and by $\mathfrak{s}_- = \mathfrak{s} \cap \tilde{\mathfrak{n}}_-$.

For integral dominant

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_n\Lambda_n, \qquad k_i \in \mathbf{Z}_+,$$

(where $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$) set

$$k = \Lambda(c) = k_0 + k_1 + \cdots + k_n,$$
$$g_j = \Lambda(\gamma_j) = k_1 + \cdots + k_{n+1-j}, \qquad j = 1, \ldots, n.$$

Then $k \geq g_1 \geq g_2 \geq \cdots \geq g_n \geq 0$ determines Λ, and we shall also write

$$\Lambda = [k\,;\ g_1, g_2, \ldots, g_n].$$

3. Coloured partitions. Let S be a set. Denote by $\mathscr{P}(S)$ the set of all functions $\mu\colon S \to \mathbf{Z}_+$ with finite support $\operatorname{supp}(\mu) = \{a \in S\,;\ \mu(a) \neq 0\}$. We will call such functions a partition with $\mu(a)$ parts a. Clearly $\mathscr{P}(S)$ is a semigroup with pointwise addition $\mu + \nu$. Define the length of μ by

$$l(\mu) = \sum_{a \in S} \mu(a)$$

and set

$$\mathscr{P}_m(S) = \{\mu\,;\ l(\mu) = m\}.$$

Then we have

$$\mathscr{P}(S) = \sum_{m \geq 0} \mathscr{P}_m(S), \qquad \mathscr{P}_m(S) + \mathscr{P}_n(S) \subset \mathscr{P}_{n+m}(S).$$

Let $\delta_1, \delta_2, \ldots : \mathscr{P}(S) \to \mathbf{Z}$ be a sequence (or well ordered set) of additive functionals, and set $\mu > \nu$ if there exists s such that

$$\delta_s(\mu) > \delta_s(\nu) \quad \text{and} \quad \delta_r(\mu) = \delta_r(\nu) \quad \text{for all } r < s.$$

Clearly we have:

LEMMA 3.1. *Let* $\mu, \nu, \kappa \in \mathscr{P}(S)$.

(i) *If* $\mu \geq \nu$ *and* $\nu \geq \kappa$, *then* $\mu \geq \kappa$.

(ii) *If* $\mu \geq \nu$, *then* $\mu + \kappa \geq \nu + \kappa$.

(iii) *If* $\delta_1, \delta_2, \ldots$ *is such that* $\delta_i(\mu) = \delta_i(\nu)$ *for all* $i = 1, 2, \ldots$ *implies* $\mu = \nu$, *then* $\geq$ *is a linear order on* $\mathscr{P}(S)$.

Now take $S = \widetilde{\Gamma}_- = \{x_\beta(j);\ \beta \in \Gamma, j < 0\}$. We will call $\mu \in \mathscr{P}(\widetilde{\Gamma}_-)$ a coloured partition with $\mu(x_\beta(j))$ parts $x_\beta(j)$ of degree j and colour (weight) β. Recall that we have defined the order on $\widetilde{\Gamma}_-$ by $x_\beta(i) < x_\gamma(j)$ if $i < j$ or $i = j$, $\beta < \gamma$. Then a coloured partition μ may be written as a sequence

$$x_{\beta_1}(j_1) \leq x_{\beta_2}(j_2) \leq \cdots \leq x_{\beta_s}(j_s),$$

where the element $x_\beta(j)$ appears in this sequence $\mu(x_\beta(j))$ times. We may visualize μ by its "Young diagram" representing a part $x_\beta(j)$ with $-j$ boxes of colour β. For example,

$$x_{\gamma_3}(-4), \quad x_{\gamma_5}(-3), \quad x_{\gamma_2}(-3), \quad x_{\gamma_2}(-1)$$

is represented by the Young diagram on the left-hand side of Figure 1, where 3 stands for γ_3, etc. (Sometimes we shall also write $\beta(i)$ instead of $x_\beta(i)$.)

For coloured partitions we write $\nu \cup \kappa$ and $\nu = \varnothing$ instead of $\nu + \kappa$ and $\nu = 0$, and $\nu \subset \kappa$ if $\nu(a) \leq \kappa(a)$ for all $a \in \widetilde{\Gamma}_-$. If $\nu \subset \kappa$, we say that κ contains ν.

Define the length $l(\mu)$, degree $|\mu|$ and weight $w(\mu)$ of μ by

$$l(\mu) = s = \sum_{a \in \widetilde{\Gamma}_-} \mu(a),$$

$$|\mu| = j_1 + \cdots + j_s = \sum_{a \in \widetilde{\Gamma}_-} \mu(a) \deg(a),$$

$$w(\mu) = \beta_1 + \cdots + \beta_s = \sum_{a \in \widetilde{\Gamma}_-} \mu(a) w(a),$$

where $\deg x_\beta(j) = j$ and $w(x_\beta(j)) = \beta$.

If μ and ν are given by

$$\mu\colon a_1 \leq a_2 \leq \cdots \leq a_s, \qquad \nu\colon b_1 \leq b_2 \leq \cdots \leq b_t,$$

then we shall write

$$\mu < \nu$$

if $\mu \neq \nu$ and one of the following statements are true

(i) $l(\mu) > l(\nu)$,

(ii) $l(\mu) = l(\nu)$, $|\mu| < |\nu|$,

(iii) $l(\mu) = l(\nu)$, $|\mu| = |\nu|$ and

$$\deg a_s = \deg b_s, \ldots, \deg a_{i+1} = \deg b_{i+1}, \quad \deg a_i < \deg b_i$$

for some $s \geq i \geq 1$,

(iv) $l(\mu) = l(\nu)$, $|\mu| = |\nu|$, $\deg a_i = \deg b_i$ for $i = 1, \ldots, s$ and

$$w(a_s) = w(b_s), \ldots, w(a_{i+1}) = w(b_{i+1}), w(a_i) < w(b_i)$$

for some $s \geq i \geq 1$.

For example,

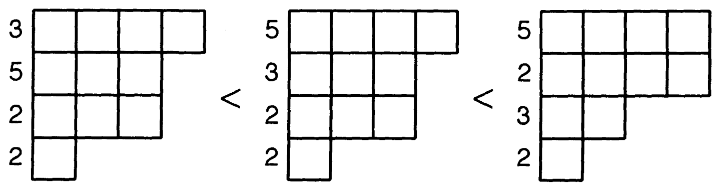

FIGURE 1

Obviously, we have:

LEMMA 3.2. *The relation* $\leq$ *is a (reverse) well order on* $\mathscr{P}(\widetilde{\Gamma}_-)$. *The element* 0 *is the largest element in* $\mathscr{P}(\widetilde{\Gamma}_-)$.

Notice that the relation $\leq$ may be defined by a sequence of functionals

$$-l, \quad |\ |, \quad \delta_1, \delta_2, \ldots, \delta_{1,1}, \ldots, \delta_{1,n}, \quad \delta_{2,1}, \ldots, \delta_{2,n}, \ldots$$

where

$$\delta_j(\mu) = \sum_{\beta \in \Gamma} \mu(x_\beta(-j)), \qquad \delta_{j,r}(\mu) = \mu(x_{\gamma_r}(-j)).$$

Hence we have:

LEMMA 3.3. *Let* $\mu, \nu, \kappa \in \mathscr{P}(S)$. *If* $\mu \geq \nu$, *then* $\mu + \kappa \geq \nu + \kappa$.

This is a crucial property of the order $\geq$ and our construction may be regarded as a "commutative version" of the construction in [**LW**, Proposition 6.2].

Later on we shall need the following:

LEMMA 3.4. *Let* $m \in -\mathbf{Z}_+$, $\varphi = q_1\gamma_1 + \cdots + q_n\gamma_n$, $q_1, \dots, q_n \in \mathbf{Z}_+$, *and* $q_1 + \cdots + q_n = k + 1 \geq 2$. *Let*

$$A = \{\mu \in \mathscr{P}(\widetilde{\Gamma}_-)\,;\, l(\mu) = k+1,\ |\mu| = m,\ w(\mu) = \varphi\}.$$

(i) *The set* $A \neq \varnothing$ *has the smallest element.*

(ii) *Let* ν *be a coloured partition* $x_{\beta_1}(j_1) \leq \cdots \leq x_{\beta_{k+1}}(j_{k+1})$, $\nu \in A$. *Then* ν *is the smallest element of* A *if and only if*

$$j_1 = j_{k+1} \quad \textit{or} \quad j_1 = -1 + j_{k+1}, \qquad \beta_1 \geq \beta_{k+1}.$$

Proof. Since A is finite, (i) is clear. It is also clear that for the smallest element

$$\mu\colon x_{\beta_1}(j_1) \leq \cdots \leq x_{\beta_{k+1}}(j_{k+1})$$

must be either $j_1 = j_{k+1}$ of $j_1 = -1 + j_{k+1}$.

In the case

$$j_1 = \cdots = j_t = -1 + j_{t+1} = \cdots = -1 + j_{k+1}$$

write a sequence

$$\psi_1 \geq \cdots \geq \psi_t \geq \psi_{t+1} \geq \cdots \geq \psi_{k+1},$$

where γ_1 appears q_1 times, $\dots$, γ_n appears q_n times. Then for

$$\beta_1 = \psi_t, \dots, \beta_t = \psi_1,$$
$$\beta_{t+1} = \psi_{k+1}, \dots, \beta_{k+1} = \psi_{t+1}$$

μ is the smallest element in A. Hence $\beta_1 \geq \beta_{k+1}$. Conversely, $\beta_t \geq \cdots \geq \beta_1 \geq \beta_{k+1} \geq \cdots \geq \beta_{t+1}$ determines the sequence (ψ_i). □

REMARK 3.5. If $\nu \in A$ is not the smallest element, then $j_1 \leq -1 + j_{k+1}$, and if $j_1 = -1 + j_{k+1}$, then $\beta_1 < \beta_{k+1}$. (Cf. a definition of difference conditions in the following section.)

4. Difference and initial conditions. In this section we fix a natural number k and a sequence of nonnegative integers $k \geq g_1 \geq g_2 \geq \cdots \geq g_n \geq 0$. We shall write

$$\Lambda = [k\,;\, g_1, g_2, \dots, g_n], \quad g_i = \Lambda(\gamma_i), \qquad i = 1, \dots, n.$$

Let us denote by $D(\Lambda)$ a set of coloured partitions ν of the form

$$x_{\beta_1}(j_1) \leq \cdots \leq x_{\beta_{k+1}}(j_{k+1})$$

such that

$$j_1 = j_{k+1} \quad \text{or} \quad j_1 = -1 + j_{k+1}, \quad \beta_1 \geq \beta_{k+1}.$$

We shall say that a coloured partition μ satisfies the difference conditions $D(\Lambda)$ if μ *does not* contain any $\nu \in D(\Lambda)$. Equivalently, a coloured partition μ of the form

$$x_{\beta_1}(j_1) \leq \cdots \leq x_{\beta_r}(j_r)$$

satisfies the difference conditions $D(\Lambda)$ if

$$j_s \leq -2 + j_{s+k} \quad \text{or} \quad j_s = -1 + j_{s+k}\,, \qquad \beta_s < \beta_{s+k}$$

for $s = 1, \ldots, r-k$.

Let us denote by $I(\Lambda)$ a set of coloured partitions of the form

$$x_\beta(-1) \leq x_{\beta_1}(-1) \leq \cdots \leq x_{\beta_s}(-1), \qquad s = k - \Lambda(\beta).$$

We shall say that a coloured partition μ satisfies the initial conditions $I(\Lambda)$ if μ *does not* contain any $\nu \in I(\Lambda)$. Equivalently, a coloured partition μ satisfies the initial conditions $I(\Lambda)$ if μ has at most $k-g_i$ parts of degree -1 and colours $\geq \gamma_i$, $i = 1, \ldots, n$.

Consider a set of points $A \subset \mathbf{Z}^2$

$$\begin{array}{ll} (n,0),\ (n,-1), \ldots, (n, -g_1+1); & \\ (n-1,0),\ (n-1,-1), \ldots, (n-1, -g_2+1); & \\ \qquad (n-1,1), \ldots, (n-1, k-g_1), & \\ (n-2,0),\ (n-2,-1), \ldots, (n-2, -g_3+1); & \\ \qquad (n-2,1), \ldots, (n-2, k-g_2), & \\ & \\ (1,0),\ (1,-1), \ldots, (1, -g_n+1); & \\ \qquad (1,1), \ldots, (1, k-g_{n-1}),\ (0,1), \ldots, (0, k-g_n). & \end{array}$$

Notice that in the first row we listed g_1 points, in second row $g_2 + (k-g_1)$ points, and so on.

Let

$$B = \{(p + (n+1)r, q - kr) \in \mathbf{Z}^2 ; (p, q) \in A,\ r \in \mathbf{Z}\}.$$

LEMMA 4.1. (i) $\#A = k \cdot n$.

(ii) *For each horizontal line* $l = \{(a, b); a \in \mathbf{Z}\}$ *we have* $\#(B \cap l) = n$ *and* $B \cap l$ *is an interval.*

Proof. The first statement is clear. To prove the second statement, let $0 \geq b > -k$. If for some r we have

$$-g_{r+1} + 1 > b, \quad -g_r + 1 \leq b,$$

then for $n \geq t \geq r+1 > r \geq s \geq 1$ we have $g_t \leq g_{r+1} \leq g_r \leq g_s$ and

$$-g_s + 1 \leq b, \quad -k + (k - g_t) \geq b.$$

Hence

$$B \cap l = \{(n+1-r, b), \dots, (n+1-1, b), \\ (0+n+1, b), \dots, (n-r-1+n+1, b)\}.$$

Since B is periodic, the statement follows. □

Label each point of the interval $B \cap l$ by colours γ_1 to γ_n (from left to right). In particular, to each point of $A \subset B$ we associate a colour. Define a coloured partition ν_Λ by associating to each point $(p, q) \in A$ of colour β a part $x_\beta(-p-1)$.

For example, if $\Lambda = [2; 2, 1, 1]$, then we have (writing $s(j)$ instead of $x_{\gamma_s}(j)$)

$$\nu_\Lambda = (3(-4), 1(-4), 2(-3), 3(-2), 1(-2), 2(-1)).$$

From the above construction it is easy to see that we can construct ν_Λ in another way: write

$$\Lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_k}$$

as the sum of fundamental weights

$$\Lambda_0 = [1; 0, \dots, 0],$$
$$\Lambda_i = [1; 1, \dots, 1, 0, \dots, 0], \qquad 1 \leq i \leq n,$$

where zero appears $i-1$ times. Then

$$\nu_\Lambda = \nu_{\Lambda_{i_1}} \cup \cdots \cup \nu_{\Lambda_{i_k}},$$

where

$\nu_{\Lambda_0} : \gamma_n(-n), \dots, \gamma_2(-2), \gamma_1(-1)$,
$\nu_{\Lambda_n} : \gamma_1(-n-1);\ \gamma_n(-n+1), \dots, \gamma_2(-1)$,
$\nu_{\Lambda_i} : \gamma_{n-i+1}(-n-1), \dots, \gamma_1(-i-1);\ \gamma_n(-i+1), \dots, \gamma_{n-i+2}(-1)$,
$\nu_{\Lambda_1} : \gamma_n(-n-1), \dots, \gamma_2(-3), \gamma_1(-2)$.

LEMMA 4.2. (i) $l(\nu_\Lambda) = k \cdot n$.
(ii) $-|\nu_\Lambda| = g_1 + \cdots + g_n + kn(n+1)/2$.
(iii) $w(\nu_\Lambda) = k(\gamma_1 + \cdots + \gamma_n)$.

Proof. (i) follows from Lemma 4.1(i). From the construction it is clear that each colour in A appears k times, and hence (iii) holds. To prove (ii), notice that there is $k-g_n$ parts of degree -1, $(k-g_{n-1})+g_n$ parts of degree $-2, \dots, g_1$ parts of degree $-n-1$. □

Proposition 4.3. *A coloured partition ν_Λ satisfies the difference conditions $D(\Lambda)$ and initial conditions $I(\Lambda)$.*

Proof. For each horizontal line $l_b = \{(a, b); a \in \mathbf{Z}\}$ we have that $B \cap l_b$ is an interval with n points. From construction of A we see that for $0 \geq b > -k+1$ the beginning of the segment $B \cap l_{b-1}$ is to the right from the beginning of the segment $B \cap l_b$. Since B is periodic, this is true in general. By the way B is coloured, it is clear that on each vertical line $\{(a, b); b \in \mathbf{Z}\} \cap B$ the colours are descending (while going up): $\beta_{(a,b)} \leq \beta_{(a,b-1)}$. (Recall that $\gamma_1 > \cdots > \gamma_n$.) This means that colours of the parts of ν_Λ of degree $-a-1$ are arranged in the Young diagram of ν_Λ in the same way as the colours on the vertical line $A \cap \{(a, b); b \in \mathbf{Z}\}$. Now to check that the difference conditions hold for ν_Λ is the same as to check whether for adjacent points (a, b), $(a+1, b) \in A$ (on horizontal line) their colours satisfy relation $\beta_{(a+1,b)} < \beta_{(a,b)}$. But this is true by construction.

By inspecting the construction of ν_Λ, we see that on the first vertical line in A colours $\beta \geq \gamma_1$ appear $k-g_1$ times, colours $\beta \geq \gamma_2$ appear $k-g_2$ times, ..., and hence ν_Λ satisfy initial conditions as well. □

For a coloured partition μ and $j \geq 0$ denote by μ_j a coloured partition defined by

$$\mu_j(x_\beta(q-j)) = \mu(x_\beta(q)),\ \mu_j(x_\beta(r)) = 0 \quad \text{for } r \geq -j.$$

Clearly the Young diagram of μ_j is obtained by adding to each part of μ additional j boxes.

For a coloured partition μ and $q \geq 1$ set

$$\mu_{q,\Lambda} = \mu_{q(n+1)} + (\nu_\Lambda)_{(q-1)(n+1)} + \cdots + (\nu_\Lambda)_{n+1} + \nu_\Lambda.$$

For example, if μ is given by

$$3(-2),\ 1(-2),\ 2(-1)$$

and Λ being as in the previous example, then $\mu_{1,\Lambda}$ is given by

$$3(-6),\ 1(-6),\ 2(-5),\ 3(-4),\ 1(-4),\ 2(-3),\ 3(-2),\ 1(-2),\ 2(-1).$$

Proposition 4.4. *Let $\mu \in \mathscr{P}(\tilde{\Gamma}_-)$. Then it is equivalent*

(i) *μ satisfies the difference conditions $D(\Lambda)$ and initial conditions $I(\Lambda)$.*

(ii) *$\mu_{q,\Lambda}$ satisfies the difference conditions $D(\Lambda)$ for $q \geq 1$.*

Proof. It is enough to consider the case when $q = 1$. Clearly one has to compare the parts of $\mu_{1\cdot(n+1)}$ of degree $-1-(n+1)$ with parts of ν_Λ of degree $-(n+1)$, let us denote them by

$$a_s \le \cdots \le a_1 < b_{g_1} \le \cdots \le b_1 .$$

In ν_Λ there is g_1 parts of degree $-(n+1)$. If $g_1 = 0$, i.e. there are no parts of degree $-(n+1)$, then the difference conditions are clearly satisfied. But then $g_1 = \cdots = g_n = 0$, so μ satisfies the initial conditions if it satisfies the difference conditions. Hence in this case the proposition holds.

Now assume $g_1 \ge 1$, and let (i) hold. We have to compare colours of parts b_j and a_{k-g_1+j}. In b_i's colour γ_1 appears $g_1 - g_2$ times, γ_2 appears $g_2 - g_3$ times, ..., γ_n appears g_n times, i.e.

$$\begin{aligned} &w(b_1) = \cdots = w(b_{g_1-g_2}) = \gamma_1 , \\ &w(b_{g_1-g_2+1}) = \cdots = w(b_{g_1-g_3}) = \gamma_2 , \\ &\cdots \\ &w(b_{g_1-g_n+1}) = \cdots = w(b_{g_1}) = \gamma_n . \end{aligned}$$

Since μ satisfies the initial conditions, we have

$$\begin{aligned} &w(a_{k-g_1+1}) < \gamma_1 , \\ &w(a_{k-g_2+1}) < \gamma_2 , \\ &\cdots \\ &w(a_{k-g_n+1}) < \gamma_n , \text{ i.e. } s \le k - g_n . \end{aligned}$$

Hence $w(b_1) = \gamma_1 > w(a_{k-g_1+1}), \ldots$, and the difference conditions hold for $\mu_{1,\Lambda}$.

The other implication is proved similarly. □

Later on we shall need the following construction (recall that $k \ge 1$ is fixed): For a coloured partition

$$\nu = (x_{\beta_1}(j_1), \ldots, x_{\beta_s}(j_s)), \qquad x_{\beta_1}(j_1) \le \cdots \le x_{\beta_s}(j_s),$$

set

$$\nu' = (x_{\beta_1}(kj_1), \ldots, x_{\beta_s}(kj_s))$$

and

$$\begin{aligned} \nu^0 &= (x_{\beta_1}(kj_1 + s - 1), \ldots, x_{\beta_s}(kj_s)) \\ &= (x_{\beta_1}(p_1), \ldots, x_{\beta_s}(p_s)) . \end{aligned}$$

Clearly, we have a map $\nu \to \nu' \to \nu^0$ from $\mathscr{P}(\widetilde{\Gamma}_-)$ into "coloured sequences" which may be visualized as multiplying the number of boxes in the Young diagram of ν by k, and then going upwards erasing $0, 1, 2, \dots$ boxes. For example ($k = 2$, $\nu = \nu_\Lambda$ as before):

$$\nu = (3(-4), 1(-4), 2(-3), 3(-2), 1(-2), 2(-1)),$$
$$\nu' = (3(-8), 1(-8), 2(-6), 3(-4), 1(-4), 2(-2)),$$
$$\nu^0 = (3(-3), 1(-4), 2(-3), 3(-2), 1(-3), 2(-2)).$$

First we list some obvious properties of ν^0:

LEMMA 4.5. *Let ν satisfy the difference conditions $D(\Lambda)$. Then*

(i) $p_r \le p_{r+k}$,
(ii) $p_r = p_{r+k}$ *implies* $\beta_r < \beta_{r+k}$,
(iii) $p_{s-j} \equiv j \mod k$,
(iv) $p_r \le -1$,
(v) *for* $r \ne q$ *we have* $x_{\beta_r}(p_r) \ne x_{\beta_q}(p_q)$.

Let $j > 1$. Call the sequence of all parts of ν of degree $-j$ a j-block of ν (if nonempty), denote it by $B(\nu, j)$. Clearly, $\#B(\nu, j) \le k$.

(vi) $\{p_r;\ x_{\beta_r}(j_r) \in B(\nu, j)\}$ *is an interval in* $\mathbf{Z}$, *denote it by* $[a_j, b_j]$.
(vii) *If* $i > j$, *then* $b_i \le b_j$.

We may think of ν^0 as a coloured partition, i.e. $\nu^0 \in \mathscr{P}(\widetilde{\Gamma}_-)$.

LEMMA 4.6. *Fix $\varphi = a_1\gamma_1 + \cdots + a_n\gamma_n$, $a_1, \dots, a_n \in \mathbf{Z}_+$. The map $\nu \to \nu^0$ from the set*

$$\{\nu \in \mathscr{P}(\widetilde{\Gamma}_-)\,;\ \nu \text{ satisfies difference condition } D(\Lambda) \text{ and } w(\nu) = \varphi\}$$

into $\mathscr{P}(\widetilde{\Gamma}_-)$ is an injection.

Proof. Let μ and ν be coloured partitions which satisfy the difference conditions and $w(\nu) = w(\mu) = \varphi$. Then $l(\nu) = l(\mu) = a_1 + \cdots + a_n = s$. Let

$$\nu = (x_{\beta_1}(j_1), \dots, x_{\beta_s}(j_s)), \qquad x_{\beta_1}(j_1) \le \cdots \le x_{\beta_s}(j_s),$$
$$\mu = (x_{\varphi_1}(i_1), \dots, x_{\varphi_s}(i_s)), \qquad x_{\varphi_1}(i_1) \le \cdots \le x_{\varphi_s}(i_s),$$

and

$$\nu^0 = (x_{\beta_1}(p_1), \dots, x_{\beta_s}(p_s)), \qquad \mu^0 = (x_{\varphi_1}(q_1), \dots, x_{\varphi_s}(q_s)).$$

We need to prove that $\mu \ne \nu$ implies $\mu^0 \ne \nu^0$.

For this purpose define a relation $<$ by conditions (i), (ii), (iii) in §3, i.e. by a sequence of functionals $-l$, $|\ |$, $\delta_1, \delta_2, \dots$. Let $\mu \ne \nu$.

(a) If $\nu < \mu$, then $\nu^0 < \mu^0$.

The case $|\nu| < |\mu|$ is obvious, so assume $|\nu| = |\mu|$. Let

$$j_s = i_s, \dots, j_{r+1} = i_{r+1}, \qquad j_r < i_r.$$

Then

$$p_s = q_s, \dots, p_{r+1} = q_{r+1}, \qquad p_r < q_r.$$

By Lemma 4.5 we have

$$p_r \le -k + q_r < q_r.$$

Moreover

$$\begin{aligned} &p_{r-1} \le -k + q_r + 1 < q_r, \\ &p_{r-2} \le -k + q_r + 2 < q_r, \\ &\dots \\ &p_{r-k+1} \le -k + q_r + k - 1 < q_r, \\ &p_{r-k} \le p_r < q_r, \\ &p_{r-k-1} \le p_{r-1} < q_r. \end{aligned}$$

Hence $p_{r-j} < q_r$ for $j \ge 0$, and $p_j = q_j$ for $s \ge j \ge r+1$. If we arrange the parts of ν^0 and μ^0 by degrees, we see that in degree q_r μ^0 has (at least) one part more than ν^0, and that $\nu^0 < \mu^0$.

(b) Let $j_s = i_s, \dots, j_1 = i_1$. Then ν and μ differ in "colouring". To prove $\nu^0 \ne \mu^0$ it is enough to show that the colouring of ν is determined by the colouring of ν^0.

Consider

$$\begin{aligned} \nu' &= (x_{\beta_1}(kj_1), \dots, x_{\beta_s}(kj_s)), \quad x_{\beta_1}(kj_1) \le \dots \le x_{\beta_s}(kj_s), \\ \nu^0 &= (x_{\beta_1}(kj_1 + s - 1), \dots, x_{\beta_{s-1}}(kj_{s-1} + 1), x_{\beta_s}(kj_s)) \\ &= (x_{\beta_1}(p_1), \dots, x_{\beta_s}(p_s)). \end{aligned}$$

In the sequence $p_1, \dots, p_s$ consider all elements equal -1, say $p_{t_1}, \dots, p_{t_r}$. By Lemma 4.5(ii) we have $\beta_{t_1} < \beta_{t_2} < \dots < \beta_{t_r}$. Hence ν^0 (starting from the right) looks like

$$\dots < x_\psi(-2) < x_{\beta_{t_1}}(-1) < \dots < x_{\beta_{t_r}}(-1).$$

The point is, if we know ν^0 and $j_1, \dots, j_s$, then we know $kj_s + s - 1, \dots, kj_{s-1}+1, kj_s$, and we know the places for colours $\beta_{t_1}, \dots, \beta_{t_r}$.

Next we consider all elements equal -2 in the sequence $p_1, \dots, p_s$ and, arguing as above, we reconstruct positions of colours in another part of ν. Hence in finite number of steps we determine ν completely. □

5. Standard representations. A highest weight $\tilde{\mathfrak{g}}$-module V is generated by a highest weight vector v_Λ such that

$$\begin{aligned} h \cdot v_\Lambda &= \Lambda(h) v_\Lambda \quad \text{for all } h \in \tilde{\mathfrak{h}}, \\ x \cdot v_\Lambda &= 0 \quad \text{for all } x \in \tilde{\mathfrak{n}}_+, \end{aligned}$$

where $\Lambda \in \tilde{\mathfrak{h}}^*$ is the highest weight of V. A highest weight $\tilde{\mathfrak{g}}$-module V is a direct sum of weight subspaces $V_\mu = \{v \in V;\ h \cdot v = \mu(h) v$ for all $h \in \tilde{\mathfrak{h}}\}$, $\mu \in \tilde{\mathfrak{h}}^*$.

Standard $\tilde{\mathfrak{g}}$-module (i.e. integrable highest weight $\tilde{\mathfrak{g}}$-module) we may define (cf. **[K]**) as an irreducible highest weight module with highest weight

$$\Lambda = k_0 \Lambda_0 + k_1 \Lambda_1 + \cdots + k_n \Lambda_n,$$

where $k_i \in \mathbf{Z}_+$, for $i = 0, \ldots, n$, and we denote it by $L(\Lambda)$. The central element c acts on $L(\Lambda)$ as a scalar

$$k = \Lambda(c) = k_0 + k_1 + \cdots + k_n$$

called a level of $L(\Lambda)$.

On each standard module $L(\Lambda)$ we define operators

$$\begin{aligned} s_\alpha &= \exp x_\alpha(0) \exp x_{-\alpha}(0) \exp x_\alpha(0), \\ s_{\delta-\alpha} &= \exp x_{-\alpha}(1) \exp x_\alpha(-1) \exp x_{-\alpha}(1), \\ e_\alpha &= s_{\delta-\alpha} s_\alpha, \end{aligned}$$

for each $\alpha \in R$. Then a map $\alpha \to e_\alpha$ extends to a projective representation of Q on $L(\Lambda)$ such that

$$e_\alpha e_\beta = \varepsilon(\alpha, \beta)^k e_{\alpha+\beta} \quad \text{for all } \alpha, \beta \in Q,$$

where k is level of $L(\Lambda)$.

On a standard $\tilde{\mathfrak{g}}$-module we have

$$\begin{aligned} e_\alpha d e_\alpha^{-1} &= d + \alpha - \frac{1}{2}\langle \alpha, \alpha \rangle c, \\ e_\alpha \beta e_\alpha^{-1} &= \beta - \langle \alpha, \beta \rangle c, \\ e_\alpha \beta(j) e_\alpha^{-1} &= \beta(j) \quad \text{for } j \neq 0, \\ e_\alpha x_\gamma(j) e_\alpha^{-1} &= (-1)^{\langle \alpha, \gamma \rangle} x_\gamma(j - \langle \alpha, \gamma \rangle), \end{aligned}$$

where $\alpha, \beta \in Q$, $j \in \mathbf{Z}$, $\gamma \in R$. (Cf. **[FK]** or the next section.)

Denote by T the group generated by e_α, $\alpha \in R$. We may identify T with $\{\pm e_\varphi;\ \varphi \in Q\}$.

6. Vertex operator formula. For a given standard module $L(\Lambda)$ of level k and $\alpha \in R$ denote by $x_\alpha(\zeta)$ a formal Laurent series

$$x_\alpha(\zeta) = \sum_{j\in\mathbf{Z}} x_\alpha(j)\zeta^j$$

in ζ with coefficients in $\operatorname{End}(L(\Lambda))$, where $x_\alpha(j)$ are fixed as in §2.

For $\alpha \in Q$ define a formal Laurent series $\zeta^{-c-\alpha}$ with coefficients in $\operatorname{End}(L(\Lambda))$ by

$$\zeta^{-c-\alpha}v_\mu = v_\mu\zeta^{-k-\langle\alpha,\mu\rangle}$$

whenever $v_\mu \in L(\Lambda)$ is such that $h\cdot v_\mu = \mu(h)\nu_\mu$ for all $h \in \mathfrak{h}$.

For $\alpha \in \mathfrak{h}$ define a formal Laurent series

$$E^\pm(\alpha, \zeta) = \exp\left(\sum_{i>0} \alpha(\pm i)\zeta^{\pm i}/(\pm i)\right).$$

Then the vertex operator formula due to I. Frenkel and V. G. Kac [**FK**, Theorem 1] (in our notation) states:

THEOREM 6.1. *Let Λ be a fundamental weight. Then on $L(\Lambda)$*

$$x_\alpha(\zeta) = E^-(-\alpha, \zeta)E^+(-\alpha, \zeta)e_\alpha\zeta^{-1-\alpha}$$

for $\alpha \in R$.

It will be convenient to recall the Frenkel-Kac vertex operator construction of a fundamental $\tilde{\mathfrak{g}}$-module (our notation is as in [**LP1**]):

Recall that we denote by

$$\mathfrak{s} = \sum_{j\in\mathbf{Z}\setminus\{0\}} \mathfrak{h}\otimes t^j + \mathbf{C}c$$

the infinite dimensional (graded) Heisenberg subalgebra of $\tilde{\mathfrak{g}}$ and by $\mathfrak{s}_- = \mathfrak{s}\cap\tilde{\mathfrak{n}}_-$. On the symmetric algebra $S(\mathfrak{s}_-)$ we define a representation of $\mathfrak{s}$ so that for $h \in \mathfrak{h}$ and $i \in \mathbf{Z}$ the elements $h(i)$ act as multiplication operators $h(i)$ if $i < 0$ and as derivations $i\langle h, h\rangle\partial/\partial h(-i)$ if $i > 0$, and set $c = 1$. Grading on $\mathfrak{s}_-$ induces the grading on $S(\mathfrak{s}_-)$ and we denote by d the degree operator. Define a formal Laurent series $E^\pm(\alpha, \zeta)$ with coefficients in $\operatorname{End}(S(\mathfrak{s}_-))$ as before. Then we have [**LP1**, Lemma 3.2]:

LEMMA 6.2. *Let $\varphi, \psi \in \mathfrak{h}$. Then*

$$E^+(\varphi, \zeta_1)E^-(\psi, \zeta_2) = (1-\zeta_1/\zeta_2)^{\langle\varphi,\psi\rangle}E^-(\psi, \zeta_2)E^+(\varphi, \zeta_1).$$

(Here ζ_1 *and* ζ_2 *are commuting indeterminates and the expression* $(1-\zeta_1/\zeta_2)^a$ *is understood to be the formal power series in* ζ_1/ζ_2 *obtained by means of the binomial expansion.)*

Let Q be the root lattice of R and P the weight lattice. Then $Q \subset P$. Let $\lambda_1, \dots, \lambda_n$ be the fundamental weights. Set $\lambda_0 = 0$. Denote by $\mathbf{C}[Q]$ and $\mathbf{C}[P]$ the group algebras of Q and P with basis elements of the form e^μ and multiplication $e^\mu e^\nu = e^{\mu+\nu}$. For $\varphi \in Q$ and fixed $i \in \{0, \dots, n\}$ define a linear map

$$e_\varphi : e^{\lambda_i}\mathbf{C}[Q] \to e^{\lambda_i}\mathbf{C}[Q],$$
$$e_\varphi : e^{\lambda_i+\mu} \to \varepsilon(\varphi, \mu)e^{\lambda_i+\mu+\varphi}, \quad \mu \in Q.$$

Hence we have a projective representation $\varphi \to e_\varphi$ of Q such that $e_\varphi e_\psi = \varepsilon(\varphi, \psi)e_{\varphi+\psi}$.

Define a grading on $\mathbf{C}[P]$ by $de^\mu = -\frac{1}{2}\langle\mu, \mu\rangle e^\mu$.

Define the action of $\mathfrak{h}$ on $\mathbf{C}[P]$ by $he^\mu = \langle\mu, h\rangle e^\mu$. As before we define a formal Laurent series ζ^α for $\alpha \in \mathfrak{h}$.

For $i \in \{0, \dots, n\}$ set

$$V_i = S(\mathfrak{s}_-) \otimes e^{\lambda_i}\mathbf{C}[Q].$$

Then on V_i we have the action of the Lie algebra $\mathfrak{s}$ (acting on the first tensorand), the action of $\mathfrak{h}$ and Q (acting on the second tensorand) and the grading defined by $d = d \otimes 1 + 1 \otimes d$. Clearly V_i is irreducible for action of these operators.

By using Lemma 6.2 it is easy to see that coefficients of the formal Laurent series

$$E^-(-\alpha, \zeta)E^+(-\alpha, \zeta) \otimes e_\alpha \zeta^{-1-\alpha}$$

satisfy the same commutation relations as Lie algebra elements $x_\alpha(j)$, so by the vertex operator formula V_i is a $\tilde{\mathfrak{g}}$-module equivalent to $L(\Lambda_i)$. (To be precise, the action of $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ is equivalent, and the grading is shifted by $-\frac{1}{2}\langle\lambda_i, \lambda_i\rangle$.) Moreover, operator e_α is equal to $s_{\delta-\alpha}s_\alpha$ (introduced in §5).

If $\alpha, \beta \in R$ and $\langle\alpha, \beta\rangle \geq 1$, then the family $\{x_\alpha(j), x_\beta(j);\ j \in \mathbf{Z}\}$ is commutative and the formal Laurent series $x_\alpha(\zeta)x_\beta(\zeta)$ is well defined. As a consequence of the vertex operator construction and Lemma 6.2 we have:

PROPOSITION 6.3. *Let* Λ *be a fundamental weight and* $\alpha, \beta \in R$, $\langle\alpha, \beta\rangle \geq 1$. *Then on* $L(\Lambda)$

$$x_\alpha(\zeta)x_\beta(\zeta) = 0.$$

Similarly, for $\beta_1, \dots, \beta_s \in \Gamma_i$ the coefficients of $x_{\beta_1}(\zeta), \dots, x_{\beta_s}(\zeta)$ commute and the formal Laurent series $x_{\beta_1}(\zeta)\cdots x_{\beta_s}(\zeta)$ is well defined. Since by the complete reducibility theorem [**K**, Theorem 10.7] a standard module $L(\Lambda)$ of level k is a submodule of the tensor product of k fundamental modules, Proposition 6.3 implies that for $\beta_1, \dots, \beta_{k+1} \in \Gamma_i$

$$x_{\beta_1}(\zeta)\cdots x_{\beta_{k+1}}(\zeta) = 0$$

on $L(\Lambda)$. Hence the formal Laurent series

$$\exp\left(\sum_{\beta\in\Gamma_i} x_\beta(\zeta)\right)$$

is well defined on $L(\Lambda)$.

Now we can state a generalization of the vertex operator formula:

THEOREM 6.4. *Let $i, j \in \{1, \dots, n+1\}$, $i \neq j$, and set $\varphi = e_i - e_j \in \Gamma_i \cap (-\Gamma_j)$. Then on $L(\Lambda)$*

(6.1)
$$\exp\left(\sum_{\beta\in\Gamma_i} x_\beta(\zeta)\right) = E^-(-\varphi, \zeta)\exp\left(\sum_{\gamma\varepsilon\Gamma_j} \varepsilon(\gamma, \varphi)x_\gamma(\zeta)\right)E^+(-\varphi, \zeta)e_\varphi\zeta^{-c-\varphi}.$$

Proof. In the level 1 case these are vertex operator formulas. Since the relation (6.1) holds for fundamental modules, it holds on tensor products of fundamental modules, and hence on every standard module (cf. [**LP2**, Theorem 5.6]). □

Formula (6.1) can be written by components:

$$(6.2)\quad \varepsilon(\psi - k\varphi, \varphi)(q_1!\cdots q_n!)/(p_1!\cdots p_n!)x_{\beta_1}(\zeta)^{p_1}\cdots x_{\beta_n}(\zeta)^{p_n} = E^-(-\varphi, \zeta)x_{\gamma_1}(\zeta_1)^{q_1}\cdots x_{\gamma_n}(\zeta)^{q_n}E^+(-\varphi, \zeta)e_\varphi\zeta^{-k-\varphi},$$

$$(6.3)\quad x_{\beta_1}(\zeta)^{r_1}\cdots x_{\beta_n}(\zeta)^{r_n} = 0,$$

where for fixed $i \neq j$ we take $\Gamma_i = \{\beta_1, \dots, \beta_n\}$, $\Gamma_j = \{\gamma_1, \dots, \gamma_n\}$, $\varphi \in \Gamma_i \cap (-\Gamma_j)$, $p_1\beta_1 + \cdots + p_n\beta_n = \psi = q_1\gamma_1 + \cdots + q_n\gamma_n + k\varphi$, $r_1 + \cdots + r_n = k+1$, $p_s, q_s, r_s \geq 0$.

7. Schur functions. In this section we set $V = S(\mathfrak{s}_-) \otimes \mathbf{C}[P] = V_0 + \cdots + V_n$ and we consider formal Laurent series in commuting indeterminates $\zeta_1, \ldots, \zeta_m$, $m \geq 1$, with coefficients in $\operatorname{End}(V)$.

Denote by S_m the symmetric group and by $\varepsilon(w)$ a sign of permutation $w \in S_m$. The symmetric group S_m acts on $\mathbf{Z}^m$ by permuting the coordinates. Set $\delta_m = (m-1, \ldots, 1, 0) \in \mathbf{Z}^m$. For $\mu = (j_1, \ldots, j_m) \in \mathbf{Z}^m$ write $\zeta^\mu = \zeta_1^{j_1} \cdots \zeta_m^{j_m}$. Then we have

$$\prod_{1 \leq i < j \leq m} (\zeta_i - \zeta_j) = \sum_{w \in S_m} \varepsilon(w) \zeta^{w\delta_m}. \tag{7.1}$$

Notice that for $\beta_1, \ldots, \beta_m \in \Gamma = \Gamma_1$ formal Laurent series $x_{\beta_1}(\zeta_1), \ldots, x_{\beta_m}(\zeta_m)$ commute.

For $\beta_1, \ldots, \beta_m \in \Gamma$ set

$$\begin{aligned} K(\beta_1(\zeta_1), \ldots, \beta_m(\zeta_m)) &= e_{\beta_1} \cdots e_{\beta_m} \prod_{i<j,\, \beta_i = \beta_j} (\zeta_i^{-1} - \zeta_j^{-1}) \\ &\quad \times \prod_{i=1}^m E^-(-\beta_i, \zeta_i) \prod_{i=1}^m E^+(-\beta_i, \zeta_i) \prod_{i=1}^m \zeta_i^{-1-\beta_i}, \end{aligned} \tag{7.2}$$

$$K(\beta_1(\zeta_1), \ldots, \beta_m(\zeta_m)) = \sum_{j_1, \ldots, j_m \in \mathbf{Z}} K(\beta_1(j_1), \ldots, \beta_m(j_m)) \zeta_1^{j_1} \cdots \zeta_m^{j_m}.$$

We shall also write

$$K(\beta_1(j_1), \ldots, \beta_m(j_m)) = K(\beta_1, \ldots, \beta_m; j_1, \ldots, j_m).$$

By using the vertex operator formula and Lemma 6.2 we get:

LEMMA 7.1.

(i) $$K(\beta_1(\zeta_1), \ldots, \beta_m(\zeta_m)) = \prod_{1 \leq i < j \leq m} \zeta_i (1 - \zeta_i/\zeta_j)^{-1} x_{\beta_1}(\zeta_1) \cdots x_{\beta_m}(\zeta_m).$$

(ii) $$x_{\beta_1}(\zeta_1) \cdots x_{\beta_m}(\zeta_m) = \prod_{1 \leq i < j \leq m} (\zeta_i^{-1} - \zeta_j^{-1}) K(\beta_1(\zeta_1), \ldots, \beta_m(\zeta_m)).$$

(iii) $$K(\beta_{w(1)}(\zeta_{w(1)}), \ldots, \beta_{w(m)}(\zeta_{w(m)})) = \varepsilon(w) K(\beta_1(\zeta_1), \ldots, \beta_m(\zeta_m)).$$

In particular, Lemma 7.1 and (7.1) imply:

LEMMA 7.2. *For* $\beta_1, \dots, \beta_m \in \Gamma$, $j_1, \dots, j_m \in \mathbf{Z}$ *and* $w \in S_m$ *we have*:

(i) $$x_{\beta_1}(j_1)\cdots x_{\beta_m}(j_m) = \sum_{w\in S_m} \varepsilon(w)K(\beta_1(j_1+w(m)-1), \dots, \beta_m(j_m+w(1)-1)).$$

(ii) $$K(\beta_{w(1)}(j_{w(1)}), \dots, \beta_{w(m)}(j_{w(m)})) = \varepsilon(w)K(\beta_1(j_1), \dots, \beta_m(j_m)).$$

Let $\lambda \in P$ be dominant. Elements in V of the form

$$K(\beta_1(j_1), \dots, \beta_m(j_m))(1\otimes e^\lambda)$$

we will call Schur functions.

For $\beta = \gamma_i \in \Gamma$ denote by $\mathscr{P}^{(i)} \subset \mathscr{P}(\widetilde{\Gamma}_-)$ the set of all coloured partitions of colour $\beta = \gamma_i$:

$$x_\beta(j_1) \le \cdots \le x_\beta(j_m).$$

For $\beta = \gamma_i \in \Gamma$ define elements $s_\nu^{(i)} \in S(\mathfrak{s}_-)$ by

$$\prod_{1\le i<j\le m} (\zeta_i^{-1} - \zeta_j^{-1})E^-(-\beta, \zeta_1)\cdots E^-(-\beta, \zeta_m) = \sum_\mu s_{-\mu}^{(i)}\zeta^{-(\mu+\delta_m)} = \sum_\kappa s_{\kappa+\delta_m}^{(i)}\zeta^\kappa$$

summed over all $\mu \in \mathbf{Z}^m$ (or $\kappa \in \mathbf{Z}^m$).

By **[LP2**, Proposition 7.3] the family

$$\{s_{\tau^{(1)}}^{(1)}\cdots s_{\tau^{(n)}}^{(n)};\ \tau^{(i)} \in \mathscr{P}^{(i)} \text{ for } i = 1, \dots, n\}$$

is a basis of $S(\mathfrak{s}_-)$. (Here we identify $(\beta = \gamma_i,\ j_1 \le \cdots \le j_m < 0)$ with $\gamma_i(j_1) \le \cdots \le \gamma_i(j_m)$.) Also notice that for $j_1 \le \cdots \le j_m \le 0$ we have

$$s_{(j_1, \dots, j_m)}^{(i)} = s_{(j_1, \dots, j_m, 0)}^{(i)}.$$

It is clear that a coloured partition $\nu \in \mathscr{P}(\widetilde{\Gamma}_-)$ can be written in the unique way as

$$\nu = \nu^{(1)} + \cdots + \nu^{(n)},$$

where $\nu^{(i)} \in \mathscr{P}^{(i)}$.

For a sequence

$$\nu = (x_{\beta_1}(j_1), \dots, x_{\beta_m}(j_m))$$

we set

$$K(\nu) = K(\beta_1(j_1), \ldots, \beta_m(j_m)).$$

Let $\nu \in \mathscr{P}(\widetilde{\Gamma}_-)$ with weight $w(\nu) = \varphi = m_1\gamma_1 + \cdots + m_n\gamma_n$. Set $g_i = \langle \gamma_i, \lambda \rangle$. Then it is clear from the definition that

$$K(\nu)(1 \otimes e^\lambda) = \prod_{i=1}^{n} s^{(i)}_{\kappa^{(i)}+\delta_{m_i}} \otimes \varepsilon e^{\lambda+\varphi}, \tag{7.3}$$

where $\varepsilon \in \{\pm 1\}$ and

$$\kappa^{(i)} + (-1 - g_i, \ldots, -1 - g_i) = \nu^{(i)}, \qquad i = 1, \ldots, n.$$

LEMMA 7.3. *The following two statements are equivalent:*

(i) $K(\beta_1(j_1), \ldots, \beta_m(j_m))(1 \otimes e^\lambda) \neq 0$.

(ii) $j_r \leq -1 - \langle \lambda, \beta_r \rangle$ *for all* $r = 1, \ldots, m$ *and all parts* $\beta_r(j_r)$ *are mutually different, i.e.* $\beta_r = \beta_s$ *implies* $j_r \neq j_s$.

Proof. Let $K(\nu)(1 \otimes e^\lambda) \neq 0$. By Lemma 7.2(ii) all parts of ν must be different. It follows from definition (7.2) that

$$K(\beta_1(\zeta_1), \ldots, \beta_m(\zeta_m))(1 \otimes e^\lambda)$$

may have a nontrivial coefficient of $\zeta_1^{j_1} \cdots \zeta_m^{j_m}$ only if $j_r \leq -1 - \langle \lambda, \beta_r \rangle$.

Conversely, if (ii) holds, then clearly $\kappa^{(i)} \in -\mathbf{Z}_+^{m_i}$. Moreover, all parts of partition $\nu^{(i)}$ being different, we have $\kappa^{(i)} + \delta_{m_i} \in -\mathbf{Z}_+^{m_i}$. Hence by [LP2, Proposition 7.3(b)] $s_{\kappa^{(i)}+\delta_{m_i}} \neq 0$ and the lemma follows. □

LEMMA 7.4. *Fix* $\varphi = m_1\gamma_1 + \cdots + m_n\gamma_n$, $m_1, \ldots, m_n \in \mathbf{Z}_+$. *Set*

$$A = \{K(\nu)(1 \otimes e^\lambda);\ \nu \in \mathscr{P}(\widetilde{\Gamma}_-),\ w(\nu) = \varphi\}.$$

Then $A \backslash \{0\}$ *is linearly independent.*

Proof. Since φ is fixed, the length $l(\nu^{(i)}) = m_i$ is fixed, and hence $A\backslash\{0\}$ is a subset of the basis of V of the form

$$s^{(1)}_{\nu^{(1)}} \cdots s^{(n)}_{\nu^{(n)}} \otimes \varepsilon e^{\lambda+\varphi}$$

and the lemma follows.

REMARK 7.5. If $\beta \in \Gamma$, then Lemma 7.1(i) implies

$$\begin{aligned} &x_\beta(\xi) K(\beta_1(\zeta_1), \ldots, \beta_m(\zeta_m)) \\ &\quad = \prod_{1 \leq j \leq m} \xi^{-1}(1 - \xi/\zeta_j) K(\beta(\xi), \beta_1(\zeta_1), \ldots, \beta_m(\zeta_m)), \end{aligned}$$

and the action of $x_\beta(j)$ on Schur functions can be given a simple combinatorial interpretation in terms of Young diagrams. In order to obtain a similar description for the action $x_{-\beta}(j)$ for $\beta \in \Gamma$ one may use the fact that elements defined by

$$\prod_{1\le i<j\le m} (\zeta_i^{-1} - \zeta_j^{-1}) \prod_{1\le i\le m} E^-(-\beta, \zeta_i) = \sum s_{-\lambda}\zeta^{-(\lambda+\delta_m)},$$

$$\prod_{1\le i<j\le m} (\zeta_i^{-1} - \zeta_j^{-1}) \prod_{1\le i\le m} E^-(\beta, -\zeta_i) = \sum \bar{s}_{-\mu}\zeta^{-(\mu+\delta_m)}$$

satisfy

$$\bar{s}_{-\mu} = s_{-\mu'}$$

for a partition μ, where μ' denotes the transposed partition (cf. [**M**, §I.3]).

8. A spanning set of $L(\Lambda)$. Recall that $T = \langle e_\alpha;\ \alpha \in R\rangle$. Set

$$t_0 = e_{\gamma_1} \cdots e_{\gamma_n}$$

and denote by $T_0 \subset T$ a subgroup generated by t_0.

Recall that

$$\tilde{\mathfrak{n}}_0 = \sum_{\alpha\in\Gamma,\ j<0} \mathbf{C}x_\alpha(j)$$

is a commutative subalgebra of $\tilde{\mathfrak{n}}_-$.

Let $L(\Lambda)$ be a standard $\tilde{\mathfrak{g}}$-module of level k and let v_Λ be a highest weight vector.

LEMMA 8.1. $L(\Lambda) = TU(\tilde{\mathfrak{n}}_0)v_\Lambda$.

Proof. Set

$$\tilde{\mathfrak{n}} = \sum_{\alpha\in\Gamma,\ j\in\mathbf{Z}} \mathbf{C}x_\alpha(j).$$

Then $\tilde{\mathfrak{n}} \subset \tilde{\mathfrak{g}}$ is a commutative subalgebra. Notice that $\tilde{\mathfrak{n}}$ is invariant for the adjoint action of the Heisenberg subalgebra $\mathfrak{s}$ and that $\tilde{\mathfrak{n}}$ and $\mathfrak{s}$ are invariant for the adjoint action of the group T. Since $R = \Gamma_1 \cup \cdots \cup \Gamma_{n+1}$, by using a relation (see (6.2))

$$x_{\beta_1}(\zeta) \in \mathbf{C}E^-(-\varphi, \zeta)x_{\gamma_1}(\zeta)^{q_1} \cdots x_{\gamma_n}(\zeta)^{q_n}E^+(-\varphi, \zeta)e_\varphi\zeta^{-k-\varphi}$$

we see that

$$L(\Lambda) = TU(\tilde{\mathfrak{n}})U(\mathfrak{s})v_\Lambda.$$

From the relation (see (6.2))

$$E^-(-\varphi, \zeta)E^+(-\varphi, \zeta) \in \mathbf{C}x_\beta(\zeta)^k\zeta^{k+\varphi}e_\varphi^{-1}$$

we see that $U(\mathfrak{s})v_\Lambda \subset TU(\tilde{\mathfrak{n}})v_\Lambda$ (cf. [LP2, Propositions 7.1 and 7.2]), so

$$L(\Lambda) = TU(\tilde{\mathfrak{n}})v_\Lambda$$

and

$$L(\Lambda) = TU(\tilde{\mathfrak{n}}_0)v_\Lambda . \qquad \square$$

THEOREM 8.2. *For a given weight subspace* $L(\Lambda)_\mu$ *there exists* $p_0 \geq 0$ *such that*

$$L(\Lambda)_\mu \subset t_0^{-p}U(\tilde{\mathfrak{n}}_0)v_\Lambda$$

whenever $p \geq p_0$.

In particular

$$L(\Lambda) = T_0U(\tilde{\mathfrak{n}}_0)v_\Lambda .$$

Proof. Since $\dim L(\Lambda)_\mu < \infty$, by Lemma 8.1 there exists a spanning set S of finitely many vectors of the form

$$e_{\gamma_1}^{q_1}\cdots e_{\gamma_n}^{q_n} x_{\beta_1}(j_1)\cdots x_{\beta_s}(j_s)v_\Lambda ,$$

where $q_1, \dots, q_n \in \mathbf{Z}$, $\beta_1, \dots, \beta_s \in \Gamma$ and $j_1, \dots, j_s < 0$. Let $p_0 \geq 0$ be such that $p_0 \geq -q_1, \dots, -q_n$ for each vector in S, and let $p \geq p_0$. Then $t_0^p = \pm e_{\gamma_1}^p \cdots e_{\gamma_n}^p$, and vectors $t_0^p S$ are of the form

$$\begin{aligned}\pm\, & e_{\gamma_1}^{q_1+p}\cdots e_{\gamma_n}^{q_n+p} x_{\beta_1}(j_1)\cdots x_{\beta_s}(j_s)v_\Lambda \\ &= \pm x_{\beta_1}(j_1')\cdots x_{\beta_s}(j_s')e_{\gamma_1}^{q_1+p}\cdots e_{\gamma_n}^{q_n+p}v_\Lambda \in U(\tilde{\mathfrak{n}}_0)v_\Lambda ,\end{aligned}$$

the last statement being a consequence of relations (see (6.2))

$$x_\gamma(\zeta)^k = k!\,E^-(-\gamma, \zeta)E^+(-\gamma, \zeta)e_\gamma\zeta^{-k-\gamma}$$

$(\gamma \in \Gamma)$, applied to v_Λ and reading off the coefficient of $\zeta^{-k-\Lambda(\gamma)}$. $\square$

9. A basis of $L(\Lambda)$. In this section we construct a basis of a standard module of level $k \geq 1$.

To each coloured partition ν we associate $x(\nu) \in U(\tilde{\mathfrak{n}}_0)$ by

$$x(\nu) = x_{\beta_1}(j_1)\cdots x_{\beta_s}(j_s) = \prod_{a\in\tilde{\Gamma}_-} a^{\nu(a)} .$$

For a coloured partition ν define $\varepsilon(\nu) \in \{\pm 1\}$ by

$$t_0x(\nu)t_0^{-1} = \varepsilon(\nu)x(\nu_{n+1}) .$$

Then $\varepsilon(\nu \cup \mu) = \varepsilon(\nu)\varepsilon(\mu)$, $\varepsilon(\nu_j) = \varepsilon(\nu)$ and $\varepsilon(\nu_\Lambda) = 1$.

By Lemma 9.8 (in this section) $x(\nu_\Lambda)v_\Lambda \neq 0$. Since $x(\nu_\Lambda)v_\Lambda \in L(\Lambda)_{t_0\Lambda}$ and $\dim L(\Lambda)_{t_0\Lambda} = \dim L(\Lambda)_\Lambda = 1$, we may define $a_\Lambda \neq 0$ by

$$t_0v_\Lambda = a_\Lambda x(\nu_\Lambda)v_\Lambda .$$

THEOREM 9.1. *Let* $L(\Lambda)$ *be a standard level* k $\tilde{\mathfrak{g}}$*-module,* $\mu \in \mathfrak{h}^*$ *and* $m \in \mathbf{Z}$. *Let* $p \geq 0$ *be such that* $L(\Lambda)_{\mu - m\delta} \subset t_0^{-p} U(\tilde{n}_0) v_\Lambda$. *Then the set of vectors*

$$\varepsilon(\nu)^p a_\Lambda^p t_0^{-p} x(\nu) v_\Lambda, \qquad \nu \in \mathscr{P}(\tilde{\Gamma}_-)$$

such that

(1) ν *satisfies the difference conditions* $D(\Lambda)$ *and the initial conditions* $I(\Lambda)$,

(2) $w(\nu) = \mu - \Lambda|\mathfrak{h} + kp(\gamma_1 + \cdots + \gamma_n)$,

(3) $|\nu| = -m - n(n+1)kp^2/2 - p\langle \gamma_1 + \cdots + \gamma_n, \mu \rangle$, *is a basis of* $L(\Lambda)_{\mu - m\delta}$.

Moreover, this basis does not depend on a choice of p.

Notice that

$$\begin{aligned} \varepsilon(\nu)^p a_\Lambda^p t_0^{-p} x(\nu) v_\Lambda &= \varepsilon(\nu)^p a_\Lambda^p t_0^{-p-1} t_0 x(\nu) t_0^{-1} t_0 v_\Lambda \\ &= \varepsilon(\nu)^{p+1} a_\Lambda^{p+1} t_0^{-(p+1)} x(\nu_{n+1}) x(\nu_\Lambda) v_\Lambda \\ &= \varepsilon(\nu_{1,\Lambda})^{p+1} a_\Lambda^{p+1} t_0^{-(p+1)} x(\nu_{1,\Lambda}) v_\Lambda . \end{aligned}$$

Since under our assumptions $\nu \to \nu_{1,\Lambda}$ is injective, it is clear that a basis does not depend on a choice of p.

The rest of this section is devoted to the proof of Theorem 9.1.

Let $\tilde{\mathfrak{g}}_{(k)} \subset \tilde{\mathfrak{g}}$ be the full subalgebra of $\tilde{\mathfrak{g}}$ of depth $k \geq 1$ defined as

$$\tilde{\mathfrak{g}}_{(k)} = \mathfrak{g} \otimes \mathbf{C}[t^k, t^{-k}] + \mathbf{C}c + \mathbf{C}d .$$

Then $\tilde{\mathfrak{g}}_{(k)} \cong \tilde{\mathfrak{g}}$ via the isomorphism

$$\tau_k : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_{(k)}$$

given by

$$\begin{aligned} \tau_k(x(j)) &= x(kj), \qquad x \in \mathfrak{g},\ j \in \mathbf{Z}, \\ \tau_k(c) &= kc, \\ \tau_k(d) &= d/k . \end{aligned}$$

If $\pi : \tilde{\mathfrak{g}} \to \operatorname{End} V$ defines a $\tilde{\mathfrak{g}}$-module structure on a vector space V, then the restriction of π to the full subalgebra $\tilde{\mathfrak{g}}_{(k)}$ defines the representation

$$\pi_{(k)} = \pi \circ \tau_k$$

of $\tilde{\mathfrak{g}}$, and we denote this $\tilde{\mathfrak{g}}$-module by $V_{(k)}$. If V is a standard $\tilde{\mathfrak{g}}$-module of level 1, then $V_{(k)}$ is a direct sum of standard $\tilde{\mathfrak{g}}$-modules of level k. Moreover, if we take

$$V = S(\mathfrak{s}_-) \otimes \mathbf{C}[P],$$

then all standard modules of level k appear in $V_{(k)}$. Let

$$\begin{aligned} \Lambda &= k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_n\Lambda_n, \qquad k_i \in \mathbf{Z}_+, \\ k &= k_0 + k_1 + \cdots + k_n, \end{aligned}$$

and set $\lambda = \Lambda|\mathfrak{h}$. Then the $\tilde{\mathfrak{g}}$-submodule of $V_{(k)}$ generated by the vector $1 \otimes e^\lambda$ is equivalent to $L(\Lambda)$ and $1 \otimes e^\lambda$ is a highest weight vector, i.e.

$$\begin{aligned} L(\Lambda) &\cong U(\tilde{\mathfrak{g}}_{(k)})(1 \otimes e^\lambda) \subset S(\mathfrak{s}_-) \otimes \mathbf{C}[P], \\ v_\Lambda &\cong 1 \otimes e^\lambda, \\ x_{\beta_1}(j_1) \cdots x_{\beta_s}(j_s) v_\Lambda &\cong x_{\beta_1}(kj_1) \cdots x_{\beta_s}(kj_s)(1 \otimes e^\lambda). \end{aligned}$$

Fix

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_n\Lambda_n = [k; g_1, \ldots, g_n],$$

where $g_i = \Lambda(\gamma_i)$, $i = 1, \ldots, n$, $k = k_0 + \cdots + k_n$.

LEMMA 9.2. *If* $\nu \in I(\Lambda)$, *then* $x(\nu)(v_\Lambda) = 0$.

Proof. By using the full subalgebra, we have

$$\begin{aligned} x(\nu)v_\Lambda &= x_{\beta_1}(j_1) \cdots x_{\beta_s}(j_s) v_\Lambda \\ &\cong x_{\beta_1}(kj_1) \cdots x_{\beta_s}(kj_s)(1 \otimes e^\lambda) \\ &= x(\nu')(1 \otimes e^\lambda). \end{aligned}$$

Now let $\nu \in I(\Lambda)$, i.e.

$$\nu' = (x_{\beta_1}(-k), \ldots, x_{\beta_s}(-k)),$$

where $\beta_1 \le \cdots \le \beta_s$, $s - 1 = k - \Lambda(\beta_1)$. By Lemma 7.2(i)

$$\begin{aligned} &x(\nu')(1 \otimes e^\lambda) \\ &\quad = \sum_{w \in S_s} \varepsilon(w) K(\beta_1, \ldots, \beta_s; (-k, \ldots, -k) + w\delta_s)(1 \otimes e^\lambda), \end{aligned}$$

and by Lemma 7.3 each of the summands on the right-hand side equals zero since for some r

$$j_r = -k + s - 1 = -\Lambda(\beta_1) \ge -\Lambda(\beta_r) > -1 - \Lambda(\beta_r). \qquad \square$$

LEMMA 9.3. *Let* $p \in \mathbf{Z}$ *and* $\nu \in \mathscr{P}(\widetilde{\Gamma}_-)$. *Then* $t_0^{-p}x(\nu)v_\Lambda \in L(\Lambda)_{\mu-m\delta}$, *where* $\mu \in \mathfrak{h}^*$ *and* $m \in \mathbf{Z}$ *are given by*

$$\mu = w(\nu) + \Lambda|\mathfrak{h} - kp(\gamma_1 + \cdots + \gamma_n),$$
$$-m = |\nu| + \Lambda(d) + n(n+1)kp^2/2 + p\langle\gamma_1 + \cdots + \gamma_n|\mu\rangle.$$

Proof. The statement follows by using the formulas for adjoint action of e_φ listed in §5. □

LEMMA 9.4. *Let* $\varphi = a_1\gamma_1 + \cdots + a_n\gamma_n$, $a_1, \ldots, a_n \in \mathbf{Z}_+$, *and* $r \in \mathbf{Z}_+$. *Then*

$$\begin{aligned}\operatorname{span}\{x(\nu)v_\Lambda;\ \nu \in \mathscr{P}(\widetilde{\Gamma}_-),\ w(\nu) = \varphi,\ |\nu| = -r\}\\ = \operatorname{span}\{x(\nu)v_\Lambda;\ \nu \in \mathscr{P}(\widetilde{\Gamma}_-),\ w(\nu) = \varphi,\ |\nu| = -r \text{ and}\\ \nu \text{ satisfies conditions } D(\Lambda) \text{ and } I(\Lambda)\}.\end{aligned}$$

Proof. If ν contains a partition $\tau \in I(\Lambda)$, then by Lemma 9.2 $x(\tau)v_\Lambda = 0$, and hence $x(\nu)v_\Lambda = 0$.

If ν contains a partition $\tau \in D(\Lambda)$, relation (6.3) and Lemma 3.4 implies that

$$x(\tau)v_\Lambda = \sum_{\kappa>\tau} a_\kappa x(\kappa)v_\Lambda$$

for some $a_\kappa \in \mathbf{C}$, and hence by Lemma 3.3

$$x(\nu)v_\Lambda = \sum_{\mu>\nu} b_\mu x(\mu)v_\Lambda$$

for some $b_\mu \in \mathbf{C}$. Now the lemma follows by induction. □

REMARK 9.5. Lemmas 9.3 and 9.4 imply that vectors described in Theorem 9.1 form a spanning set of $L(\Lambda)_{\mu-m\delta}$. What remains to prove is the linear independence of this set.

LEMMA 9.6. *Let* $\nu \in \mathscr{P}(\widetilde{\Gamma}_-)$ *satisfy the difference conditions* $D(\Lambda)$ *and initial conditions* $I(\Lambda)$. *Then*

$$K(\nu^0)(1 \otimes e^\lambda) \neq 0.$$

Proof. Let ν be given by

$$x_{\beta_1}(j_1) \leq \cdots \leq x_{\beta_r}(j_r) < x_{\beta_{r+1}}(-1) \leq \cdots \leq x_{\beta_s}(-1),$$

where $j_r \leq -2$. Since difference conditions are satisfied, we have $s - r \leq k$.

Moreover, since initial conditions are satisfied, we have for $r+1 \leq i \leq s$

$$s - i < k - \Lambda(\beta_i).$$

Now the corresponding ν^0 is given by

$$x_{\beta_1}(kj_1 + s - 1), \dots, x_{\beta_r}(kj_r + s - r),$$
$$x_{\beta_{r+1}}(-k + s - r - 1), \dots, x_{\beta_s}(-k),$$

and for $r+1 \leq i \leq s$

$$p_i = -k + s - i \leq -1 - \Lambda(\beta_i).$$

Now let $x_{\beta_t}(p_t)$ be a part of ν^0 such that $p_t \geq -k$. We want to show that

$$p_t \leq -1 - \Lambda(\beta_t).$$

For $t = r+1, \dots, s$ this is true. Let $t < r+1$. By Lemma 4.5(vii) $p_t \leq p_{r+1}$. Moreover, by Lemma 4.5(i), (ii), (iii) we have

$$p_s = -k \leq p_t \leq p_{t+k} \leq p_{t+2k} \leq \cdots \leq p_{r+1};$$

hence

$$p_t = p_{t+k} = p_{t+2k} = \cdots = p_i, \qquad \beta_t < \beta_i$$

for some $i \in \{r+1, \dots, s\}$. But then

$$p_t = p_i \leq -1 - \Lambda(\beta_i) < -1 - \Lambda(\beta_t).$$

Since by Lemma 4.5(v) all parts of ν^0 are mutually different, Lemma 7.3 implies that $K(\nu^0)(1 \otimes e^\lambda) \neq 0$. □

LEMMA 9.7. *Let $\varphi = a_1\gamma_1 + \cdots + a_n\gamma_n$, $a_1, \dots, a_n \in \mathbf{Z}_+$. Let $\nu \in \mathscr{P}(\widetilde{\Gamma}_-)$ satisfy the difference conditions $D(\Lambda)$ and let $w(\nu) = \varphi$. Consider ν^0 as coloured partition. Then there exists an integer $a \neq 0$ such that*

$$x(\nu)v_\Lambda \cong aK(\nu^0)(1 \otimes e^\lambda) + \sum b_\kappa K(\kappa)(1 \otimes e^\lambda),$$

summed over $\kappa \in \mathscr{P}(\widetilde{\Gamma}_-)$, $w(\kappa) = \varphi$, $\kappa > \nu^0$.

Proof. Let

$$\nu = (x_{\beta_1}(j_1), \dots, x_{\beta_m}(j_m)),$$
$$b = (\beta_1, \dots, \beta_m),$$
$$\mu = (kj_1, \dots, kj_m),$$
$$\mu^0 = (kj_1 + m - 1, \dots, kj_m) = (p_1, \dots, p_m).$$

By using the full subalgebra and Lemma 7.2(i) we have

$$\begin{aligned} x(\nu)v_\Lambda &\cong x(\nu')(1\otimes e^\lambda) \\ &= \sum_{w\in S_m} \varepsilon(w)K(b\,;\mu+w\delta_m)(1\otimes e^\lambda). \end{aligned}$$

Recall that $\delta_m = (m-1, \dots, 0)$, so the parts of $w\delta_m$ are 0, 1, 2, ... Write $\tau = (b\,;\mu + w\delta_m) = (\beta_1, \beta_2, \dots; t_1, t_2, \dots) = (x_{\beta_1}(t_1), x_{\beta_2}(t_2), \dots)$ and consider it as a coloured partition. We want to see that either $K(\tau)(1\otimes e^\lambda) = 0$ or $\tau \geq \nu^0$. Let $j = -j^{(1)} = j_m$ and consider $j^{(1)}$-block of ν (see Lemma 4.5)

$$x_{\beta_r}(-j) \leq \cdots \leq x_{\beta_m}(-j)$$

and the corresponding sequence in ν^0

$$\begin{gathered} x_{\beta_r}(p_r), \dots, x_{\beta_m}(p_m), \\ p_r = -kj + m - r, \dots, p_m = -kj. \end{gathered}$$

By Lemma 4.5(vii) $p_r \geq p_i$ for $i = 1, \dots, m$. If $w\delta_m$ is not of the form

$$(\dots, i_r, \dots, i_m), \quad \{i_r, \dots, i_m\} = \{0, 1, \dots, m-r\}, \tag{9.1}$$

then τ has a part which is strictly greater than p_r, and hence $\tau > \nu^0$.

Hence consider τ such that $w\delta_m$ has the form (9.1). Now consider next $j^{(2)}$-block of ν, where $-j^{(2)} = j_{r-1}$. By the same argument we see that if $w\delta_m$ is not of the form

$$\begin{gathered} (\dots, i_q, \dots, i_{r-1}; i_r, \dots, i_m), \\ \{i_r, \dots, i_m\} = [0, -1 + \#B(\nu, j^{(1)})], \\ \{i_q, \dots, i_{r-1}\} = [\#B(\nu, j^{(1)}), -1 + \#B(\nu, j^{(1)}) + \#B(\nu, j^{(2)})], \end{gathered}$$

then $\tau > \nu^0$. By proceeding in this manner we see that it is enough to consider τ of the form

$$\tau = (\beta_1, \dots, \beta_m; t_1, \dots, t_m),$$

where

$$(t_1, \dots, t_m) = \mu + w\delta_m, \tag{9.2}$$

and the permutation w leaves each interval $[c_s, c_{s+1} - 1]$ invariant, where

$$c_s = \#B(\nu, j^{(1)}) + \cdots + \#B(\nu, j^{(s)}).$$

In particular, for coloured partitions ν^0 and τ the condition (iv) in the definition of order $\leq$ (see §3) should be checked. So consider colours of ν^0 and τ, first for the first block $B(\nu, j^{(1)})$:

$$x_{\beta_r}(p_r), \dots, x_{\beta_m}(p_m),$$
$$x_{\beta_r}(t_r), \dots, x_{\beta_m}(t_m),$$
$$\{p_r, \dots, p_m\} = \{t_r, \dots, t_m\},$$
$$\beta_r \leq \cdots \leq \beta_m.$$

Let $\beta_r = \cdots = \beta_i < \beta_{i+1}$, where $r \leq i \leq m$. By Lemma 4.5 we have $p_t \leq p_r$, and $p_t = p_r$ implies $\beta_t < \beta_r$. Hence the largest part in ν^0 is $\beta_r(p_r)$. If $p_r = t_j$ for $j > i$, then $x_{\beta_r}(p_r) < x_{\beta_j}(t_j)$ and hence $\nu^0 < \tau$. So let τ be such that $p_r = t_j$ for $r \leq j \leq i$, i.e. $x_{\beta_r}(p_r) = x_{\beta_j}(t_j)$. If another part of degree $p_t = p_r$ appears in ν^0, it must be with colour $\beta_t < \beta_r$, so (cf. Lemma 4.5) consider the next block $B(\nu, j^{(2)})$. Then ν^0 looks like $\cdots < x_{\beta_t}(p_t) < x_{\beta_r}(p_r)$, ($\beta_t$ being the smallest colour in the second block). As above, we conclude that it is enough to consider τ such that $x_{\beta_t}(p_t) = x_{\beta_j}(t_j)$ for some

$$m - \#B(\nu, j^{(1)}) - \#B(\nu, j^{(2)}) + 1 \leq j \leq m - \#B(\nu, j^{(1)}).$$

After considering parts of ν^0 and τ of degree p_r, we consider parts of degree $p_{r+1} = -1 + p_r$, etc. In finite number of steps we see that either $\nu^0 < \tau$, or $\nu^0 = \tau$. Moreover, if $\nu^0 = \tau$, then τ is of the form (9.2), where for each $i \in [c_s, c_{s+1} - 1]$ the permutation w leaves the interval

$$\{j \in [c_s, c_{s+1} - 1];\ \varphi_j = \varphi_i\}$$

invariant.

Now let $\tau = \nu^0$. Then the above property of τ implies

$$\varepsilon(w)K(\beta;\ \mu + w\delta_m) = \varepsilon(w)\varepsilon(w)K(\beta;\ \mu + \delta_m).$$

Hence the term $K(\nu^0)(1 \otimes e^\lambda)$ appears with a non-zero coefficient. □

LEMMA 9.8. *Let $\varphi = a_1\gamma_1 + \cdots + a_n\gamma_n$, $a_1, \dots, a_n \in \mathbf{Z}$. Then the set of vectors $x(\nu)v_\Lambda$ such that $\nu \in \mathscr{P}(\tilde{\Gamma}_-)$ satisfies the difference conditions $D(\Lambda)$ and initial conditions $I(\Lambda)$ and $w(\nu) = \varphi$ is linearly independent.*

Proof. By Lemmas 9.7 and 9.6 we have

$$x(\nu)v_\Lambda \cong aK(\nu^0)(1 \otimes e^\lambda) + \sum_{w(\kappa)=\varphi,\ \kappa > \nu^0} b_\kappa K(\kappa)(1 \otimes e^\lambda),$$

where $aK(\nu^0)(1\otimes e^\lambda)\neq 0$. Moreover, in the notation of Lemma 7.4 $K(\nu^0)(1\otimes e^\lambda)$, $K(\kappa)(1\otimes e^\lambda)\in A$. Since by Lemma 4.6 $\nu\to\nu^0$ is an injection, the lemma follows by induction on order $>$. □

REMARK 9.9. Lemma 9.8 completes the proof of Theorem 9.1.

REMARK 9.10. Let $L(\Lambda_0)=S(\mathfrak{s}_-)\otimes\mathbf{C}[Q]$ be the basic $A_n^{(1)}$ module, $n\geq 2$, and consider its restriction to the subalgebra $\tilde{\mathfrak{g}}_1\subset\tilde{\mathfrak{g}}$ of the type $A_{n-1}^{(1)}$, where

$$\tilde{\mathfrak{g}}_1=\sum_{\alpha\in R_1,\ j\in\mathbf{Z}}\mathbf{C}x_\alpha(j)+\mathbf{C}c+\mathbf{C}d+\operatorname{span}_{\mathbf{C}}R_1,$$

$$R_1=\{e_i-e_j\,;\,i\neq j,\ i,j=2,\ldots,n+1\}.$$

Let $Q_1=\mathbf{Z}\alpha_2+\cdots+\mathbf{Z}\alpha_n$ be a root lattice of R_1 and $\mathbf{C}[Q_1]$ its group algebra viewed as a subalgebra of $\mathbf{C}[Q]$. Set

$$W_i=S(\mathfrak{s}_-)\otimes e^{\gamma_1+\cdots+\gamma_i}\mathbf{C}[Q_1]$$

for $i=1,\ldots,n$. Then W_i is a $\tilde{\mathfrak{g}}_1$-module. If we set $h=\gamma_1+\cdots+\gamma_n$, then $h\perp R_1$ and for $i=1,\ldots,n$

$$W_i\cong L(\Lambda'_{n-i})\otimes\mathbf{C}[h(-1),h(-2),\ldots],\tag{9.3}$$

where $L(\Lambda'_j)$ is the fundamental $\tilde{\mathfrak{g}}_1$-module for a fundamental weight Λ'_j, $j=0,\ldots,n-1$.

Notice that

$$v_i=x_{\gamma_i}(-i)\cdots x_{\gamma_1}(-1)(1\otimes e^0)\in\mathbf{C}\otimes e^{\gamma_1+\cdots+\gamma_i}$$

for $i=1,\ldots,n$.

By Theorem 9.1 elements of the form

$$v(\nu)=\varepsilon(\nu)^p a_{\Lambda_0}^{p_-}t_0^{-p}x(\nu)v_{\Lambda_0}\in W_i\tag{9.4}$$

such that ν satisfies difference conditions and that p is large enough so that $\nu\supset\nu_{\Lambda_0}$ (see Proposition 4.4) is a basis of W_i. Since

$$v(\nu)=v(\nu_{n+1}\cup\nu_{\Lambda_0})=\cdots=v(\nu_{q,\Lambda_0})=\cdots,$$

we may identify $v(\nu)$ with an infinite sequence $(\nu_{q,\Lambda_0};\ q\geq 0)$. For such a sequence (or "long enough" coloured partition) consider a corresponding sequence of "colours"

$$\beta_1,\beta_2,\beta_3,\ldots,\beta_j\in\Gamma,$$

β_j being the weight of jth part of ν_{q,Λ_0}. Clearly, for some j_0 a sequence $(\beta_j)_{j\geq j_0}$ is periodic with period n:

$$\cdots\gamma_n,\ldots,\gamma_1,\gamma_n,\ldots,\gamma_1,\ldots$$

and (9.4) holds if and only if the length $l = l(\nu) \equiv i \bmod n$ and $p = (l-i)/n$. We will call such a sequence (β_j) a path (corresponding to ν).

For each path (β_j) there are many coloured partitions satisfying difference conditions and having (β_j) as a corresponding path. Denote by $\nu((\beta_j))$ the largest one (with respect to order $<$). For a path

$$\beta_1, \ldots, \beta_{(p-1)n+i}, \beta_{(p-1)n+i+1}, \ldots, \beta_{pn+i}$$

$\nu((\beta_j))$ has parts

$$x_{\beta_1}(j_1), \ldots, x_{\beta_{(p-1)n+i}}(j_{(p-1)n+i}), x_{\gamma_n}(-n), \ldots, \gamma_1(-1),$$

and if we set

$$H(\beta_r, \beta_{r+1}) = \begin{cases} 1, & \beta_r \geq \beta_{r+1}, \\ 0, & \beta_r < \beta_{r+1}, \end{cases}$$

then

$$j_r = j_{r+1} - 1 - H(\beta_r, \beta_{r+1}). \tag{9.5}$$

Denote by $(\overline{\beta}_j)$ a path (of length $l = np + i$)

$$\gamma_i, \ldots, \gamma_1, \gamma_n, \ldots, \gamma_1, \ldots, \gamma_n, \ldots, \gamma_1.$$

Then $v_i \in \mathbf{C}v(\nu((\overline{\beta}_i)))$.

Denote by $\mathfrak{h}_1 = \operatorname{span}_{\mathbf{C}} R_1$. Then $\mathfrak{h}_1$-weight of the vector $v(\nu((\beta_j))) \in W_i$ equals

$$w(\nu((\beta_j)))|\mathfrak{h}_1 = \left(\sum_{j\geq 1} \beta_j - \overline{\beta}_j\right)|\mathfrak{h}_1. \tag{9.6}$$

We also see that the degree of $v(\nu((\beta_j))) \in W_i$ equals

$$\deg v(\nu((\beta_j))) = |\nu((\beta_j))| - |\nu((\overline{\beta}_j))|.$$

By the way $\nu((\beta_j))$ is constructed (9.5), we see that

$$\deg v(\nu((\beta_j))) = -\sum_{r\geq 1} r(H(\beta_r, \beta_{r+1}) - H(\overline{\beta}_r, \overline{\beta}_{r+1})). \tag{9.7}$$

Formulas (9.6) and (9.7) are used in **[DJKMO 1]** to define the weight and degree of path (β_j).

Finally notice that for a given path (β_j) the set of all coloured partitions satisfying difference conditions and having (β_j) as a corresponding path may be obtained from $\nu((\beta_j))$ and partitions

$$n_1 \geq n_2 \geq \cdots \geq 0, \quad n_1 + n_2 + \cdots < \infty$$

by adding n_1 boxes to the first part of $\nu((\beta_j))$, n_2 boxes to the second part of $\nu((\beta_j))$, ... of the Young diagram of $\nu((\beta_j))$. Since by Theorem 9.1 $v(\nu) \in W_i$ form a basis, the above argument and (9.3) imply that $\dim L(\Lambda'_{n-i})_\mu$ equals to the number of paths (β_j) such that the weight of (β_j) given by formulas (9.6) and (9.7) equals μ. This was proved in **[DJKMO 1 and 2]**.

10. Basis of vacuum spaces of standard modules for $A_1^{(1)}$ and $A_2^{(1)}$. For the homogeneous Heisenberg subalgebra $\mathfrak{s}$ set $\mathfrak{s}_+ = \mathfrak{s} \cap \tilde{\mathfrak{n}}_+$, and denote by $\Omega(\Lambda)$ the vacuum space of a standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$:

$$\Omega(\Lambda) = \{v \in L(\Lambda)\,;\, \mathfrak{s}_+ v = (0)\}.$$

Then we have the following linear isomorphism due to Lepowsky and Wilson (cf. **[LW]**, **[LP1]**)

$$U(\mathfrak{s}_-) \otimes \Omega(\Lambda) \to L(\Lambda), \qquad u \otimes v \to u \cdot v.$$

In this section we construct a basis of $\Omega(\Lambda)$ for $A_2^{(1)}$ standard modules. This is a generalization of the construction given in **[LP2]** for $A_1^{(1)}$ standard modules. We include the $A_1^{(1)}$ case as well: although the proofs are (almost) the same to the original ones, they illuminate similarities and differences of results in §§9 and 10.

Let $L(\Lambda)$ be a standard $\tilde{\mathfrak{g}}$-module of level k.

For $\beta_1, \ldots, \beta_m \in \Gamma$ set

$$(10.1)\quad Z(\beta_1, \ldots, \beta_m; \zeta_1, \ldots, \zeta_m) = \prod_{j=1}^{m} E^-(\beta_j/k, \zeta_j) x_{\beta_1}(\zeta_1) \cdots x_{\beta_m}(\zeta_m) \prod_{j=1}^{m} E^+(\beta_j/k, \zeta_j),$$

$$Z(\beta_1, \ldots, \beta_m; \zeta_1, \ldots, \zeta_m) = \sum Z(\beta_1, \ldots, \beta_m; j_1, \ldots, j_m) \zeta_1^{j_1} \cdots \zeta_m^{j_m},$$

summed over all $j_1, \ldots, j_m \in \mathbf{Z}$. Clearly, for every permutation σ we have

$$Z(\beta_{\sigma(1)}, \ldots, \beta_{\sigma(m)}; j_{\sigma(1)}, \ldots, j_{\sigma(m)}) = Z(\beta_1, \ldots, \beta_m; j_1, \ldots, j_m).$$

For a coloured partition

$$\nu = (\beta_1(j_1), \ldots, \beta_m(j_m))$$

set

$$Z(\nu) = Z(\beta_1, \ldots, \beta_m; j_1, \ldots, j_m).$$

It is clear from Lemma 8.1 and (10.1) that

$$L(\Lambda) = TU(\mathfrak{s}_-)\,\mathrm{span}\{Z(\mu)v_\Lambda\,;\, \mu \in \mathscr{P}(\Gamma_-)\}\,.$$

It is easy to see (cf. **[LP1**, Proposition 2.7]) that the action of the Heisenberg subalgebra $\mathfrak{s}$ commutes with each $Z(\nu)$. In particular, each $Z(\nu)$ preserves $\Omega(\Lambda)$. Hence we have:

LEMMA 10.1.

$$\Omega(\Lambda) = T\,\mathrm{span}\{Z(\mu)v_\Lambda\,;\, \mu \in \mathscr{P}(\Gamma_-)\}\,.$$

In this section we prove the following two theorems:

THEOREM 10.2. *Let* $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$ *and* $\Gamma = \{\alpha\}$. *The set of vectors*

$$e_{n\alpha}Z(\mu)v_\Lambda\,,$$

where $n \in \mathbf{Z}$, *and (coloured) partition* μ *does not contain any partition of the form*

$$I(\Lambda) : (\alpha(-1), \dots, \alpha(-1)) \text{ of length } k - \Lambda(\alpha) + 1\,,$$
$$D'(\Lambda) : \alpha(j_1) \leq \cdots \leq \alpha(j_k)\,, \qquad |j_1 - j_k| \leq 1\,,$$

is a basis of the vacuum space $\Omega(\Lambda)$ *of the standard* $\tilde{\mathfrak{g}}$*-module* $L(\Lambda)$.

THEOREM 10.3. *Let* $\mathfrak{g} = \mathfrak{sl}(3, \mathbf{C})$ *and* $\Gamma = \{\beta\,, \alpha\}$, $\beta < \alpha$. *The set of vectors*

$$e_\varphi Z(\mu)v_\Lambda\,,$$

where $\varphi \in Q$ *and coloured partition* μ *does not contain any partition of the form*

$$I(\Lambda) : \beta_1(-1) \leq \cdots \leq \beta_s(-1)\,, \qquad \beta_i \in \Gamma,\ s = k - \Lambda(\beta_1) + 1\,,$$
$$D(\Lambda) : \beta_1(j_1) \leq \cdots \leq \beta_{k+1}(j_{k+1})\,, \qquad \beta_i \in \Gamma\,,$$
$$j_1 = j_{k+1} \quad \text{or} \quad j_1 = -1 + j_{k+1}\,, \qquad \beta_1 \geq \beta_{k+1}\,,$$
$$D'(\Lambda) : \text{(a) } \gamma(j_1) \leq \cdots \leq \gamma(j_k)\,,\ |j_1 - j_k| \leq 1\,, \qquad \gamma \in \Gamma\,,$$
$$\text{(b) } \beta(j-1)^a\alpha(j-1)^b\beta(j)^c\alpha(j)^d\beta(j+1)^e\,,$$
$$j \leq -2\,,\ a\,, b\,, d\,, e \geq 1\,,\ c \geq 0\,,$$
$$a + b + c = k\,,\ c + d + e = k\,,\ b + c + d \leq k\,,$$
$$\text{(c) } \alpha(j-1)^a\beta(j)^b\alpha(j)^c\beta(j+1)^d\alpha(j+1)^e\,,$$
$$j \leq -2\,,\ a\,, b\,, d\,, e \geq 1\,,\ c \geq 0\,,$$
$$a + b + c = k\,,\ c + d + e = k\,,\ b + c + d \leq k\,,$$

is a basis of the vacuum space $\Omega(\Lambda)$ *of the standard* $\tilde{\mathfrak{g}}$*-module* $L(\Lambda)$. *(Here* $\gamma(i)^a$ *denotes that the part* $\gamma(i)$ *appears* a *times.)*

REMARK 10.4. Notice that coloured partitions listed in $D'(\Lambda)$ satisfy the difference conditions $D(\Lambda)$.

First we prove a spanning:

Clearly the definition (10.1) and Lemma 9.2 imply (cf. [**LP2**, Proposition 6.4]):

LEMMA 10.5. *If a coloured partition* μ *contains* $\nu \in I(\Lambda)$, *then*

$$Z(\mu)v_\Lambda = 0.$$

Together with Lemma 6.2 we now recall [**LP1**, Lemma 3.1] (notice a difference in the definition of $E^{\pm}(\varphi, \zeta)$):

LEMMA 10.6. *On a level* $k \geq 1$ *module* $L(\Lambda)$ *we have*

(i) $E^+(\varphi, \zeta_1)E^-(\psi, \zeta_2) = (1 - \zeta_1/\zeta_2)^{\langle \varphi, \psi \rangle k} E^-(\psi, \zeta_2)E^+(\varphi, \zeta_1)$,

(ii) $E^+(\varphi, \zeta_1)x_\psi(\zeta_2) = (1 - \zeta_1/\zeta_2)^{-\langle \varphi, \psi \rangle} x_\psi(\zeta_2)E^+(\varphi, \zeta_1)$,

(iii) $x_\varphi(\zeta_1)E^-(\psi, \zeta_2) = (1 - \zeta_1/\zeta_2)^{-\langle \varphi, \psi \rangle} E^-(\psi, \zeta_2)x_\varphi(\zeta_1)$.

By applying Lemma 10.6, the definition (10.1) and the relations

$$x_\gamma(\zeta)^k = k!\, E^-(-\gamma, \zeta)E^+(-\gamma, \zeta)e_\gamma\zeta^{-k-\gamma},$$
$$x_{\beta_1}(\zeta)^{r_1} \cdots x_{\beta_n}(\zeta)^{r_n} = 0,$$

for $\beta_1, \ldots, \beta_n \in \Gamma$, $r_1 + \cdots + r_n = k + 1$, (see (6.2) and (6.3)), we get (cf. [**LP2**, Theorem 5.8]):

LEMMA 10.7. (i) *For* $\beta_1, \ldots, \beta_m \in \Gamma$, $m \geq k$, $1 \leq s \leq m - k + 1$, $\beta_s = \cdots = \beta_{s+k-1} = \gamma$, *we have*

$$\begin{aligned}&\lim_{\zeta_s, \ldots, \zeta_{s+k-1} \to \zeta} Z(\beta_1, \ldots, \beta_m; \zeta_1, \ldots, \zeta_m)\\ &\quad= k!\, e_\gamma \zeta^{-k-\gamma} \prod_{i=1}^{s-1} (1 - \zeta/\zeta_i)^{\langle \gamma, \beta_i \rangle} \prod_{i=s+k}^{m} (1 - \zeta/\zeta_i)^{\langle \gamma, \beta_i \rangle}\\ &\quad\cdot Z(\beta_1, \ldots, \beta_{s-1}, \beta_{s+k}, \ldots, \beta_m; \zeta_1, \ldots, \zeta_{s-1}, \zeta_{s+k}, \ldots, \zeta_m).\end{aligned}$$

(ii) *For* $\beta_1, \ldots, \beta_m \in \Gamma$, $m \geq k + 1$, $1 \leq s \leq m - k$, *we have*

$$\lim_{\zeta_s, \ldots, \zeta_{s+k} \to \zeta} Z(\beta_1, \ldots, \beta_m; \zeta_1, \ldots, \zeta_m) = 0.$$

REMARK 10.8. Let a coloured partition μ contain a partition of the form

$$\gamma(j_1) \leq \cdots \leq \gamma(j_k), \qquad |j_1 - j_k| \leq 1, \ \gamma \in \Gamma.$$

Then Lemma 10.7 implies that

$$Z(\mu)v_\Lambda = \sum_{\nu > \mu} a_\nu Z(\nu) v_\Lambda$$

for some $a_\nu \in \mathbf{C}[Q]$. In the case of $A_1^{(1)}$ standard modules this relation together with Lemma 10.5 implies that the set of vectors defined in Theorem 10.2 is a spanning set of $\Omega(\Lambda)$ (cf. [**LP2**, Theorem 6.5]).

LEMMA 10.9. *Let* $\mathfrak{g} = \mathfrak{sl}(3, \mathbf{C})$. *Let a coloured partition* μ *contain a partition of the form* (a), (b) *or* (c) *listed in Theorem* 10.3. *Then*

$$Z(\mu)v_\Lambda = \sum_{\nu > \mu} a_\nu Z(\nu) v_\Lambda \tag{10.2}$$

for some $a_\nu \in \mathbf{C}[Q]$.

Proof. In the case when μ contains a partition of the form (a) the statement follows from Lemma 10.7(i).

Now let μ contain a partition τ of the form (b): let $\mu = \mu' \cup \tau$,

$$\tau = \beta(j-1)^a \alpha(j-1)^b \beta(j)^c \alpha(j)^d \beta(j+1)^e,$$
$$j \leq -2, \ a, b, d, e \geq 1, \ c \geq 0,$$
$$a + b + c = k, \ c + d + e = k, \ b + c + d \leq k.$$

First notice that our assumptions imply that $d \leq a$ and that the number of parts of τ of colour β is $\geq k$, say $a + c + e = k + r$. Clearly, $0 \leq r = a - d < a$. Let s be the number of parts τ of colour α, i.e. $s = b + d$. Notice that $r + s = a + b$.

Define a sequence of coloured partitions

$$\tau_r < \tau_{r+1} < \cdots < \tau_a = \tau$$

by

$$\tau_r = \beta(j-1)^r \alpha(j-1)^s \beta(j)^{c+d} \beta(j+1)^e,$$
$$\tau_{r+1} = \beta(j-1)^{r+1} \alpha(j-1)^{s-1} \beta(j)^{c+d-1} \alpha(j) \beta(j+1)^e,$$
$$\cdots$$
$$\tau_{r+i} = \beta(j-1)^{r+i} \alpha(j-1)^{s-i} \beta(j)^{c+d-i} \alpha(j)^i \beta(j+1)^e,$$
$$\cdots$$
$$\tau_a = \beta(j-1)^a \alpha(j-1)^b \beta(j)^c \alpha(j)^d \beta(j+1)^e.$$

Notice that we keep constant the number of parts of a given colour, as well as the number of parts of a given degree. Define an "upper triangular matrix" of partitions of length $k+1$

$$\tau_{pp} < \tau_{pp+1} < \cdots < \tau_{pq} < \dots, \quad p = r, \dots, a, \quad p \le q \le a,$$

by

$$\begin{aligned} \tau_{pp} &= \beta(j-1)^p \alpha(j-1)^{a+b-p} \beta(j)^{c+1}, \qquad p = r, \dots, a-1, \\ \tau_{aa} &= \beta(j-1)^a \alpha(j-1)^b \beta(j)^c \alpha(j), \\ \tau_{p,p+i} &= \beta(j-1)^{p+i} \alpha(j-1)^{a+b-p-i} \beta(j)^{c+1-i} \alpha(j)^i, \end{aligned}$$

for $p = r, \dots, a-1$ and for $i = 0, \dots, a-p$ such that $c+1-i \ge 0$. Notice that, whenever τ_{pq} $(p \le q)$ is defined, we have for $p = r, \dots, a-1$ and $q = p, \dots, a$

$$\begin{aligned} \tau_q &= \beta(j-1)^q \alpha(j-1)^{a+b-q} \beta(j)^{c+a-q} \alpha(j)^{d-a+q} \beta(j)^e \\ &= \tau_{pq} \cup \beta(j)^{a-p-1} \alpha(j)^{d-a+p} \beta(j)^e. \end{aligned}$$

Let

$$\mu_q = \mu' \cup \tau_q, \qquad q = r, \dots, a.$$

Then

$$\mu_r < \mu_{r+1} < \cdots < \mu_a = \mu$$

and for $p = r, \dots, a-1$, $p \le q \le a$, we have

$$\mu_q = \tau_{pq} \cup \mu' \cup \beta(j)^{a-p-1} \alpha(j)^{d-a+p} \beta(j)^e.$$

Notice that $\tau_{pp} \in D(\Lambda)$ for $p \in \{r, \dots, a-1\}$ (and $\tau_{aa} \notin D(\Lambda)$). Hence by applying Lemma 10.7(ii) we have for $p = r, \dots, a-1$

$$\sum_{i=0}^{a-p} c_{p,p+i} Z(\tau_{p,p+i}) + \sum_{\kappa > \tau_{pa}} c_\kappa Z(\kappa) = 0,$$

and

$$\sum_{i=0}^{a-p} c_{pp+i} Z(\mu_{p+i}) + \sum_{\nu > \mu_a} c_\nu Z(\nu) = 0 \tag{10.3}$$

for some c_κ, $c_\nu \in \mathbf{C}$, and

$$\begin{aligned} c_{pp+i} &= \binom{c+1+p}{p+i} \binom{a+b-p}{i} \\ &= \frac{(c+1+p)!(a+b-p)!}{(p+i)!(a+b-p-i)!(c+1)!} \binom{c+1}{i}, \\ c_{pq} &= 0 \quad \text{for } p > q. \end{aligned}$$

Notice that calculating the determinant of a submatrix of (c_{pq}) reduces to calculating the determinant of the corresponding submatrix of the matrix $(\binom{c+1}{q-p})$. Hence one can easily see (by induction on c and d) that

$$\det\left(\binom{c+1}{q-p}\right)_{p=r,\dots,a-1;q=r+1,\dots,a} > 0,$$

and

$$\det(c_{pq})_{p=r,\dots,a-1;q=r+1,\dots,a} \neq 0. \tag{10.4}$$

By using Gauss elimination procedure for the set of relations (10.3) we get

$$c_{rr}Z(\mu_r) + cZ(\mu_a) = \sum_{\nu>\mu_\alpha} d_\nu Z(\nu)$$

for some $c, d_\nu \in \mathbf{C}$. Now (10.4) implies $c \neq 0$. Since $\mu_r \supset \tau_r \supset \beta(j)^{c+d}\beta(j+1)^e$, by using Lemma 10.7(i) we get (10.2).

In the case when μ contains a partition of the form (c) the proof is similar. □

REMARK 10.10. In the case of $A_2^{(1)}$ standard modules Lemmas 10.5, 10.7(ii) and 10.9 imply (by induction) that the set of vectors defined in Theorem 10.3 is a spanning set of $\Omega(\Lambda)$.

In the rest of this section we prove the linear independence: Assume that $k \geq 2$. Set (cf. **[LP2**, §7])

$$A = \operatorname{span}\{\gamma(j);\ \gamma \in \Gamma,\ j < 0,\ j \equiv 0 \bmod k\},$$
$$\overline{A} = \operatorname{span}\{\gamma(j);\ \gamma \in \Gamma,\ j < 0,\ j \not\equiv 0 \bmod k\}.$$

Then

$$S(\mathfrak{s}_-) \cong U(\mathfrak{s}_-) \cong S(A) \otimes S(\overline{A}), \qquad S(\overline{A}) \subset S(\mathfrak{s}_-).$$

Define an algebra homomorphism

$$U(\mathfrak{s}_-) \to S(\overline{A}),$$
$$v \mapsto \overline{v}$$

by mapping

$$\gamma(j) \mapsto 0 \qquad \text{for } j \equiv 0 \bmod k, \quad j < 0,\ \gamma \in \Gamma,$$
$$\gamma(j) \mapsto \gamma(j) \quad \text{for } j \not\equiv 0 \bmod k, \quad j < 0,\ \gamma \in \Gamma.$$

Extend this map to

$$S(\mathfrak{s}_-) \otimes \mathbf{C}[P] \to S(\overline{A}) \otimes \mathbf{C}[P],$$
$$v \otimes e^\beta \mapsto \overline{v} \otimes e^\beta.$$

In particular, for Schur functions we have

$$\overline{K}(\mu) = \overline{K(\mu)} \in S(\overline{A}) \otimes \mathbf{C}[P].$$

Now recall that by using the full subalgebra of level k we have (see §9)

$$L(\Lambda) \cong U(\tilde{\mathfrak{g}}_{(k)})(1 \otimes e^\lambda) \subset S(\mathfrak{s}_-) \otimes \mathbf{C}[P],$$
$$v_\Lambda \cong 1 \otimes e^\lambda,$$
$$x_{\beta_1}(j_1) \cdots x_{\beta_m}(j_m) v_\Lambda \cong x_{\beta_1}(kj_1) \cdots x_{\beta_m}(kj_m)(1 \otimes e^\lambda).$$

Moreover, we have:

LEMMA 10.11.

$$Z(\nu) v_\Lambda \cong \sum_{w \in S_m} \varepsilon(w) \overline{K}(b\,;\, \mu + w\delta_m)(1 \otimes e^\lambda),$$

where

$$\nu = (\beta_1(j_1), \ldots, \beta_m(j_m)),$$
$$b = (\beta_1, \ldots, \beta_m),$$
$$\mu = (kj_1, \ldots, kj_m).$$

Proof. Clearly (7.2) implies

(10.5)

$$K(\beta_1(\zeta_1), \ldots, \beta_m(\zeta_m)) = \prod_{i=1}^m E_{(k)}^-(-\beta_i, \zeta_i^k) \overline{K}(\beta_1(\zeta_1), \ldots, \beta_m(\zeta_m)) \prod_{i=1}^m E_{(k)}^+(-\beta_i, \zeta_i^k),$$

where

$$E_{(k)}^\pm(\beta, \zeta^k) = \exp\left(\sum_{j>0} \beta(\pm kj)\zeta^{\pm kj}/(\pm kj)\right).$$

For a Laurent series

$$A(\zeta_1, \ldots, \zeta_m) = \sum_{j_1, \ldots, j_m \in \mathbf{Z}} a_{j_1 \cdots j_m} \zeta_1^{j_1} \cdots \zeta_m^{j_m}$$

write

$$P_k(A(\zeta_1, \ldots, \zeta_m)) = \sum_{j_1, \ldots, j_m \equiv 0 \bmod k} a_{j_1 \cdots j_m} \zeta_1^{j_1} \cdots \zeta_m^{j_m}.$$

Then we have

$$
\begin{aligned}
&Z(\beta_1, \dots, \beta_m, \zeta_1^k, \dots, \zeta_m^k) \\
&\quad = \prod_{i=1}^m E^-(\beta_i/k, \zeta_i^k) x_{\beta_1}(\zeta_1^k) \cdots x_{\beta_m}(\zeta_m^k) \prod_{i=1}^m E^+(\beta_i/k, \zeta_i^k) \\
&\quad \cong \prod_{i=1}^m E^-_{(k)}(\beta_i, \zeta_i^k) P_k(x_{\beta_1}(\zeta_1), \dots, x_{\beta_m}(\zeta_m)) \prod_{i=1}^m E^+_{(k)}(\beta_i, \zeta_i^k) \\
&\quad = P_k\left(\prod_{i=1}^m E^-_{(k)}(\beta_i, \zeta_i^k) x_{\beta_1}(\zeta_1) \cdots x_{\beta_m}(\zeta_m) \prod_{i=1}^m E^+_{(k)}(\beta_i, \zeta_i^k)\right) \\
&\quad = P_k\left(\prod_{1 \le i < j \le m} (\zeta_i^{-1} - \zeta_j^{-1}) \overline{K}(\beta_1(\zeta_1), \dots, \beta_m(\zeta_m))\right).
\end{aligned}
$$

(The last equality follows from (10.5) and Lemma 7.1.) By comparing the coefficients on both sides and using (7.1) the lemma follows. □

The proof of Lemma 9.7 together with Lemma 10.11 imply:

LEMMA 10.12. *Let ν satisfy the difference conditions $D(\Lambda)$. Consider ν^0 as a coloured partition. Then there exists an integer $a \neq 0$ such that*

$$Z(\nu) v_\Lambda \cong a \overline{K}(\nu^0)(1 \otimes e^\lambda) + \sum_{\kappa > \nu^0} b_\kappa \overline{K}(\kappa)(1 \otimes e^\lambda)$$

for some $b_\kappa \in \mathbf{C}$.

Denote by $\mathscr{C}_k(\Lambda)$ the set of all coloured partitions $\nu = (\beta_1(j_1), \dots, \beta_m(j_m))$ such that

(i) $j_r \le -1 - \langle \lambda, \beta_r \rangle$ for all $r = 1, \dots, m$ and all parts of ν are mutually different.

(ii) ν does not contain any partition of length k of the form

$$\gamma(j-k+1), \gamma(j-k+2), \dots, \gamma(j-1), \gamma(j)$$

for $j \le -1$, $\gamma \in \Gamma$.

PROPOSITION 10.13. (i) *The family*

$$\{e_\psi \overline{K}(\nu)(1 \otimes e^\lambda); \psi \in kQ, \ \nu \in \mathscr{C}_k(\Lambda)\}$$

is a basis of $S(\overline{A}) \otimes e^\lambda \mathbf{C}[Q]$.

(ii) *If* ν *is a coloured partition,* $\nu \notin \mathscr{C}_k(\Lambda)$, *then*

$$\overline{K}(\nu)(1\otimes e^{\lambda}) = \sum_{\nu\in\mathscr{C}(\Lambda),\ \mu>\nu} a_\mu \overline{K}(\mu)(1\otimes e^{\lambda})$$

for some $a_\mu \in \mathbf{C}[kQ]$.

Proof. (i) For $\beta = \gamma_i \in \Gamma$ define elements $s_\nu^{(i)} \in S(\mathfrak{s}_-)$ by

$$(10.6)\qquad \prod_{1\le r<s\le m} (\zeta_r^{-1} - \zeta_s^{-1}) E^-(-\beta, \zeta_1)\cdots E^-(-\beta, \zeta_m) = \sum_\mu s_{-\mu}^{(i)} \zeta^{-(\mu+\delta_m)}$$

summed over all $\mu \in \mathbf{Z}^m$. Also denote by $\mathscr{B}_k^{(i)}$ the set of all partitions $\tau^{(i)}$

$$\beta(j_1), \dots, \beta(j_s), j_1 \le \cdots \le j_s < 0$$

such that every part of $\tau^{(i)}$ occurs in $\tau^{(i)}$ at most $k-1$ times. Then by [**LP2**, Proposition 7.6] the family

$$\{\overline{s}_{\tau(1)}^{(1)} \cdots \overline{s}_{\tau(n)}^{(n)}\,;\ \tau^{(i)} \in \mathscr{B}_k^{(i)} \text{ for } i = 1, \dots, n\}$$

is a basis of $S(\overline{A})$.

Now fix $\mu \in Q$. Then the set of vectors

$$(10.7)\qquad \overline{s}_{\tau(1)}^{(1)} \cdots \overline{s}_{\tau(n)}^{(n)} \otimes e^{\lambda+\mu},$$

where $\tau^{(i)} \in \mathscr{B}_k^{(i)}$ for $i = 1, \dots, n$, is a basis of $S(\overline{A}) \otimes e^{\lambda+\mu}$. Let $\mu = r_1\gamma_1 + \cdots + r_n\gamma_n$. For fixed $\tau^{(1)}, \dots, \tau^{(n)}$ let integers $p_1, \dots, p_n$ be such that

$$l(\tau^{(i)}) + p_i \equiv r_i \mod k, \qquad 0 \le p_i < k.$$

Notice that for $\tau^{(i)} = (\beta(j_1), \dots, \beta(j_s))$ we have

$$\overline{s}_{\tau(i)}^{(i)} = \overline{s}_{(j_1,\dots,j_s)}^{(i)} = \overline{s}_{(j_1,\dots,j_s,0,\dots,0)}^{(i)},$$

where p_i zeros are added. Write

$$(10.8)\qquad \begin{aligned}\sigma^{(i)} &= (\beta(j_1), \dots, \beta(j_s), \beta(0), \dots, \beta(0)),\\ \kappa^{(i)} &= \sigma^{(i)} - \delta_{s+p_i},\\ \nu^{(i)} &= \kappa^{(i)} + (-1-g_i, \dots, -1-g_i),\end{aligned}$$

where $g_i = \langle \gamma_i, \lambda\rangle$. Set $\nu = \nu^{(1)} \cup \cdots \cup \nu^{(n)}$. Then (cf. (7.3))

$$(10.9)\qquad \overline{K}(\nu)(1\otimes e^{\lambda}) = \prod_{i=1}^n \overline{s}_{\sigma(i)}^{(i)} \otimes \varepsilon e^{\lambda+\varphi},$$

where $\varepsilon \in \{\pm 1\}$ and $\psi = \mu - \varphi \in kQ$. Obviously $\nu \in \mathscr{C}_k(\Lambda)$.

Conversely, for $\nu \in \mathscr{C}_k(\Lambda)$, $\nu = \nu^{(1)} \cup \cdots \cup \nu^{(n)}$, the partition $\sigma^{(i)}$ defined by (10.8) has at most $k-1$ parts equal $\beta(0)$, and hence ν and ψ uniquely determine a basis element of the form (10.7).

(ii) For $\beta = \gamma_i \in \Gamma$ define elements $h_\nu^{(i)} \in S(\mathfrak{s}_-)$ by

$$\prod_{j=1}^{m} E^-(-\beta, \zeta_j) = \sum h_{-\nu}^{(i)} \zeta^{-\nu} \tag{10.10}$$

summed over all $\nu \in \mathbf{Z}^m$.

Then (10.6) and (10.10) imply

$$\sum s_\sigma^{(i)} \zeta^\sigma = \prod_{1 \le r < s \le m} (1 - \zeta_r/\zeta_s) \sum h_\sigma^{(i)} \zeta^\sigma,$$

$$\sum h_\sigma^{(i)} \zeta^\sigma = \prod_{1 \le r < s \le m} (1 - \zeta_r/\zeta_s)^{-1} \sum s_\sigma^{(i)} \zeta^\sigma,$$

summed over all $\sigma \in \mathbf{Z}^m$. Hence for a partition σ

$$s_\sigma^{(i)} = h_\sigma^{(i)} + \sum_{\mu > \sigma} a_\mu h_\mu^{(i)}, \qquad h_\sigma^{(i)} = s_\sigma^{(i)} + \sum_{\mu > \sigma} b_\mu s_\mu^{(i)},$$

for some $a_\mu b_\mu \in \mathbf{C}$. The proof of **[LP2**, Proposition 7.5**]** shows (notice that our order $<$ is slightly different) that for $\sigma \notin \mathscr{B}_k^{(i)}$

$$\overline{h}_\sigma^{(i)} = \sum_{\mu > \sigma} a_\mu \overline{h}_\mu^{(i)},$$

for some $a_\mu \in \mathbf{C}$, and hence for $\sigma \notin \mathscr{B}_k^{(i)}$

$$\overline{s}_\sigma^{(i)} = \sum_{\mu > \sigma} a_\mu \overline{s}_\mu^{(i)}, \tag{10.11}$$

for some $a_\mu \in \mathbf{C}$.

Now assume $\nu = \nu^{(1)} \cup \cdots \cup \nu^{(n)} \notin \mathscr{C}_k(\Lambda)$. Define $\sigma^{(i)}$ by (10.8). Then for some $i \in \{1, \ldots, n\}$ $\sigma = \sigma^{(i)}$ either has $p_i < k$ parts of the form $\beta(0)$ and $(\beta(j_1), \ldots, \beta(j_s)) \notin \mathscr{B}_k$, or it has $p_i \ge k$ parts of the form $\beta(0)$.

In the first case by using (10.11), (10.9) and Lemma 3.3 we see that the statement (ii) holds. In the second case we may erase in σ k parts of the form $\beta(0)$ and

$$\overline{K}(\nu)(1 \otimes e^\lambda) = \varepsilon e_{k\beta} \overline{K}(\mu)(1 \otimes e^\lambda),$$

where $\varepsilon \in \{\pm 1\}$, $l(\mu) = l(\nu) - k$, and the statement (ii) follows as well. □

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$. Notice that if ν does not contain any partition of the form $D'(\Lambda)$, then ν satisfies the difference conditions $D(\Lambda)$.

LEMMA 10.14. *Let* $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$. *Suppose that* ν *does not contain any partition in* $I(\Lambda) \cup D'(\Lambda)$. *Then* $\nu^0 \in \mathscr{C}_k(\Lambda)$.

Proof. Let

$$\nu = (\alpha(j_1), \dots, \alpha(j_s)), \qquad j_1 \leq \cdots \leq j_s,$$
$$\nu^0 = (\alpha(p_1), \dots, \alpha(p_s)).$$

Now recall Lemma 4.5: Since ν does not contain partitions of the form $D'(\Lambda)$, for a j-block $B(\nu, j)$ of ν we have $\#B(\nu, j) \leq k-1$. By (vi) we have an interval

$$\{p_r; \beta(j_r) \in B(\nu, j)\} = [a_j, b_j],$$

and by (vii) $b_i \leq b_j$ for $i > j$. Since we have only one colour α, and all parts of ν^0 are mutually different, we have that $b_i < a_j$ for $i > j$. Now consider two adjacent intervals

$$[a_i, b_i] \cup [a_j, b_j].$$

Assumption $b_i = a_j - 1$ implies (Lemma 4.5(iii)) $i = j+1$ and $\#([a_i, b_i] \cup [a_j, b_j]) = k$, which is impossible since ν does not contain any partition of the form $D'(\Lambda)$.

Hence ν^0 does not contain any interval of k elements, and $\nu^0 \in \mathscr{C}_k(\Lambda)$ (cf. Lemma 9.6). □

LEMMA 10.15. *Let* $\mathfrak{g} = \mathfrak{sl}(3, \mathbf{C})$. *Suppose that* ν *does not contain any partition in* $I(\Lambda) \cup D(\Lambda) \cup D'(\Lambda)$. *Then* $\nu^0 \in \mathscr{C}_k(\Lambda)$.

Proof. Let

$$\nu = (\beta_1(j_1), \dots, \beta_s(j_s)),$$
$$\nu^0 = (\beta_1(p_1), \dots, \beta_s(p_s)),$$

and let ν^0 contain parts

$$\beta(i), \ i \in [r, t], \ \#[r, t] \geq k.$$

By Lemma 4.5 there exists a j-block $B(\nu, j)$

$$\{p_i; \beta(j_i) \in B(\nu, j)\} = [r_1, t], \qquad r \leq r_1.$$

If $\#[r_1, t] = k$, then ν contains a partition $\beta(-j)^k$, which is of the form (a).

Let $r < r_1$. Then the part $\beta(r_1 - 1) = \beta_q(p_q)$ appears in ν^0. Let $\beta(r_1) = \beta_m(p_m)$. First notice that $q < m$: otherwise we would have a part $\beta_{m+1}(p_{m+1})$ of ν^0, where $\beta_{m+1}(p_{m+1}) = \alpha(r_1 - 1)$ or

$p_{m+1} > r-1$, both of these impossible because of Lemma 4.5(i), (ii). Hence $q < m$.

Because of Lemma 4.5(vii) and (ii) we have that either ν contains two blocks $B(\nu, j+1)$, $B(\nu, j+2)$ and $q = m+1-2k$, or ν contains a block $B(\nu, j+1)$ and $q = m+1-k$. In the first case all colours of parts in $B(\nu, j+1)$ are α, and

$$\#[r_1, t] + \#B(\nu, j+1) = k, \qquad \#B(\nu, j+2) = k.$$

Now it is clear that these three blocks form a partition which is either listed in (b), or it contains a partition listed in (a).

In the second case $(j+1)$-block $B(\nu, j+1)$ contains at least one part $\beta(-j-1)$, and again

$$\#B(\nu, j) + \#B(\nu, j+1) = k.$$

If all the parts of $B(\nu, j+1)$ have colour β, then clearly ν contains a partition listed in (a). If there is at least one part in $B(\nu, j+1)$ with colour α, we repeat the above argument for $B(\nu, j+1)$ instead of $B(\nu, j)$, and see that either ν contains a partition of the form (a) or (b), or it contains a third block $B(\nu, j+2)$ which contains at least one part $\beta(-j-2)$, and

$$\#[r_2, r_1-1] + \#B(\nu, j+2) = k,$$

where $[r_2, r_1-1] = \{p_i; \beta(j_i) \in B(\nu, j+1)\}$.

Now it is clear that these three blocks contain a partition which is either in (a), $D(\Lambda)$ or in (b).

The proof that ν contains τ of the form (a), $D(\Lambda)$ or (c) in the case when ν^0 contains parts

$$\alpha(i), \ i \in [r, t], \#[r, t] \geq k$$

is similar. □

The Proof of Theorems 10.2 *and* 10.3. The set of vectors defined in Theorems 10.2 and 10.3 are spanning sets (Remarks 10.8 and 10.10). The linear independence follows from Lemmas 4.6, 10.12, 10.14 and 10.15 and Proposition 10.13. □

Acknowledgments. Partial support is gratefully acknowledged from Ministarstvo znanosti, tehnologije i informatike Republike Hrvatske.

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Received August 10, 1990 and in revised form July 1, 1991.

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