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FLAT CONNECTIONS, GEOMETRIC INVARIANTS AND THE SYMPLECTIC NATURE OF THE FUNDAMENTAL GROUP OF SURFACES

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In this paper we associate a new geometric invariant to the space of flat connections on a G $(= \mathrm{SU}(2))$-bundle on a compact Riemann surface M and relate it to the symplectic structure on the space $\mathrm{Hom}(\pi_1(M), G)/G$ consisting of representations of the fundamental group $\pi_1(M)$ of M into G modulo the conjugate action of G on representations.

Introduction. Our setup is as follows. Let $G = \mathrm{SU}(2)$ and M be a compact Riemann surface and $E \to M$ be the trivial G-bundle. (Any $\mathrm{SU}(2)$-bundle over M is topologically trivial.) Let $\mathscr{C}$ (resp. $\mathscr{C}^*$) be the space of all (resp. irreducible) connections and $\mathscr{F}$ (resp. $\mathscr{F}^*$) the subspace of all (resp. irreducible) flat connections on this G-bundle. We put the Fréchet topology on $\mathscr{C}$ and the subspace topology on $\mathscr{F}$.

Given a loop $\sigma: S^1 \to \mathscr{F}$, we can extend σ to the closed unit disc $\tilde{\sigma}: D^2 \to \mathscr{C}$, since $\mathscr{C}$ is contractible. On the trivial G-bundle $E \times D^2 \to M \times D^2$ we define a "tautological" connection form ϑ_σ as follows.

$$\vartheta_\sigma|_{(e,t)} = \tilde{\sigma}(t) \quad \forall\, (e, t) \in E \times D^2.$$

Clearly restriction of ϑ_σ to the bundle $E \times \{t\} \to M \times \{t\}$ is $\tilde{\sigma}(t)$ $\forall\, t \in D^2$. Let $K(\theta_\sigma)$ be the curvature form of ϑ_σ. Evaluation of the second Chern polynomial on this curvature form $K(\vartheta_\sigma)$ gives a closed 4-form on $M \times D^2$, which when integrated along D^2 yields a 2-form on M. This 2-form is closed since $\dim M = 2$ and thus defines an element in $H^2(M, \mathbb{R}) \approx \mathbb{R}$. It is seen that this class is independent of the extension of σ. We thus have a map

$$\chi: \Omega(\mathscr{F}) \to H^2(M, \mathbb{R}) \approx \mathbb{R}$$

where $\Omega(\mathscr{F})$ is the loop space of $\mathscr{F}$.

It is seen that χ induces a map

$$\overline{\chi}: \Omega(\mathscr{F}^*/\mathscr{G}) \to \mathbb{R}/\mathbb{Z}$$

where $\mathscr{G} = \mathrm{Map}(M, G)$ is the gauge group of the G-bundle $E \to M$.

It is well known that $\mathscr{F}/\mathscr{G} \approx \mathrm{Hom}(\pi_1(M), G)/G$ and the space $\mathrm{Hom}(\pi_1(M), G)/G$ carries a symplectic structure. Under this identification $\mathscr{F}^*/\mathscr{G}$ gets identified with the space $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(M), G)/G$ of conjugacy classes of irreducible representations of $\pi_1(M)$. Moreover when genus of $M \geq 3$, $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(M), G)/G$ is simply connected. Let ω be the symplectic form on $\mathscr{F}/\mathscr{G} = \mathrm{Hom}(\pi_1(M), G)/G$. For $\sigma \in \Omega(\mathscr{F}^*/\mathscr{G})$ choose a surface S in $\mathscr{F}^*/\mathscr{G}$ such that $\partial S = \sigma$. Since $\mathscr{F}^*/\mathscr{G}$ is simply connected when genus of $M \geq 3$ and ω has integral periods, $\overline{\int}_S \omega \in \mathbb{R}/\mathbb{Z}$ is independent of S. The main result of this paper (after suitable normalisation) is

THEOREM. $\overline{\chi}(\sigma) = \overline{\int}_S \omega$.

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1. Construction of the basic map. In this paper we suppose M is a compact Riemann surface of genus g, $G = \mathrm{SU}(2)$ with Lie algebra $\mathfrak{G} = \mathfrak{su}(2)$ and $E \to M$ is the trivial G-bundle on M. $\mathscr{C}$ is the space of all connections and $\mathscr{F}$ the subspace of flat connections on $E \to M$. We sometimes replace $\mathscr{C}$ (resp. $\mathscr{F}$) by $\mathscr{C}^*$ (resp. $\mathscr{F}^*$), the space of all (resp. flat) irreducible connections on $E \to M$. The space $\mathrm{Map}(M, G)$ of all maps from M to G is the gauge group and will be denoted by $\mathscr{G}$. D^2 is the closed unit disc in $\mathbb{R}^2$ and $\partial D^2 = S^1$ is the unit circle. $\Omega(\mathscr{F}) = \mathrm{Map}(S^1, \mathscr{F})$ is the loop space of $\mathscr{F}$.

Given a loop $\sigma: S^1 \to \mathscr{F}$ we extend σ to $\tilde{\sigma}: D^2 \to \mathscr{C}$ ($\mathscr{C}$ is contractible). On the trivial G-bundle $E \times D^2 \to M \times D^2$ define the connection form ϑ_σ as

$$\vartheta_\sigma|_{(e,t)} = \tilde{\sigma}(t)|_{(e)} \quad \forall\, (e, t) \in E \times D^2;$$

i.e., restriction of ϑ_σ on the subbundle $E \times \{t\} \to M \times \{t\}$ is the connection form $\tilde{\sigma}(t)$ $\forall\, t \in D^2$. Let $K(\vartheta_\sigma)$ be the curvature 2-form of ϑ_σ and C_2 be the second-Chern polynomial on $\mathfrak{G} = \mathfrak{su}(2)$. The specific formula for C_2 shows that

$$C_2(A) = \frac{1}{8\pi^2} \operatorname{trace}(A^2) \quad \text{for } A \in \mathfrak{su}(2).$$

Evaluation of C_2 on $K(\vartheta_\sigma)$ gives the closed 4-form $\overline{C_2(K(\vartheta_\sigma))}$ on $E \times D^2$ which projects to the closed 4-form $C_2(K(\vartheta_\sigma))$ on $M \times D^2$.

Integrating $C_2(K(\theta_\sigma))$ along D^2 yields a closed 2-form on M ($\dim M = 2$) and thus defines a cohomology class in $H^2(M, \mathbb{R})$, i.e.

$$\left\{ \int_{D^2} C_2(K(\theta_\sigma)) \right\} \in H^2(M, \mathbb{R}) \approx \mathbb{R}.$$

LEMMA 1.1. *$\{\int_{D^2} C_2(K(\vartheta_\sigma))\}$ is independent of the extension of $\sigma: S^1 \to \mathscr{F}$ to $\tilde{\sigma}: D^2 \to \mathscr{C}$.*

Proof. Let $\tilde{\sigma}$, $\tilde{\sigma}'$ be two extensions of σ with corresponding connection forms ϑ_σ, ϑ'_σ and curvature forms $K(\vartheta_\sigma)$, $K(\vartheta'_\sigma)$ on the bundle $E \times D^2 \to M \times D^2$.

We claim $\int_{D^2} \overline{C_2(K(\vartheta_\sigma))} - \int_{D^2} \overline{C_2(K(\vartheta'_\sigma))}$ is an exact form on M. On $E \times D^2$ we have

$$dTC_2(\vartheta_\sigma) = \overline{C_2(K(\vartheta_\sigma))}, \qquad dTC_2(\vartheta'_\sigma) = \overline{C_2(K(\vartheta'_\sigma))}$$

where $TC_2(\vartheta_\sigma)$, $TC_2(\vartheta'_\sigma)$ are the Chern-Simons secondary forms with respect to ϑ_σ, ϑ'_σ respectively (cf. **[CS**, §3**]**).

Therefore

$$\int_{D^2} \overline{C_2(K(\vartheta_\sigma))} - \overline{C_2(K(\vartheta'_\sigma))} = \int_{D^2} d(TC_2(\vartheta_\sigma) - TC_2(\vartheta'_\sigma)).$$

By the Stokes theorem for integration along fibers (cf. **[GS**, Lemma 2.3**]**) we have (d denotes ext. differentiation in $E \times D^2$ and d_E in E)

$$\begin{aligned}\int_{D^2} & d(TC_2(\vartheta_\sigma) - TC_2(\vartheta'_\sigma)) \\ &= \int_{S^1} (TC_2(\vartheta_\sigma)|_{E\times S^1} - TC_2(\vartheta'_\sigma)|_{E\times S^1}) \\ &\quad + d_E \int_{D^2} (TC_2(\vartheta_\sigma) - TC_2(\vartheta'_\sigma)).\end{aligned}$$

But $\vartheta_\sigma = \vartheta'_\sigma$ on $E \times S^1$.

Therefore $TC_2(\vartheta_\sigma) = TC_2(\vartheta'_\sigma)$ on $E \times S^1$ and the first integral vanishes. Therefore

$$\int_{D^2} (\overline{C_2(K(\vartheta_\sigma))} - \overline{C_2(K(\vartheta'_\sigma))}) = d_E \int_{D^2} (TC_2(\vartheta_\sigma) - TC_2(\vartheta'_\sigma))$$

is exact as a form on E.

$$\Rightarrow \left\{ \int_{D^2} \overline{C_2(K(\vartheta_\sigma))} \right\} = \left\{ \int_{D^2} \overline{C_2(K(\vartheta'_\sigma))} \right\} \in H^2(E, \mathbb{R})$$

$$\Rightarrow \left\{ \int_{D_2} C_2(K(\vartheta_\sigma)) \right\} = \left\{ \int_{D_2} C_2(K(\vartheta'_\sigma)) \right\}$$

since $\pi^*: H^2(M, \mathbb{R}) \to H^2(E, \mathbb{R})$ is an isomorphism

and this proves the lemma.

We thus have a map

$$\Omega(\mathcal{F}) \xrightarrow{\chi} H^2(M, \mathbb{R}) \approx \mathbb{R}, \quad \sigma \mapsto \chi(\sigma) = \left\{ \int_{D^2} C_2(K(\vartheta_\sigma)) \right\} \ldots \tag{1.2}$$

where $\Omega(\mathcal{F})$ is the loop space of $\mathcal{F}$. It is easy to check that $\chi(\sigma \circ \sigma') = \chi(\sigma) + \chi(\sigma')$ where $\sigma \circ \sigma'$ is the composite of two loops in $\mathcal{F}$.

2. The symplectic structure on $\mathcal{F}/\mathcal{G} \approx \mathrm{Hom}(\pi_1(M), G)/G$. The quotient $\mathcal{F}/\mathcal{G}$, i.e., the space of G-equivalence class of flat connections on $E \to M$ can be identified with $\mathrm{Hom}(\pi_1(M), G)/G$. We describe the symplectic structure on $\mathcal{F}/\mathcal{G}$ following the approach by Atiyah and Bott ([**AB**, [**W**]). $\mathcal{C}$ is an affine space with the space $\Lambda^1(M, \mathfrak{su}(2))$ of $\mathfrak{su}(2)$-valued 1-forms on M as its group of translations. In particular each tangent space $T_A(\mathcal{C})$ is identified with $\Lambda^1(M, \mathfrak{su}(2))$.

Let $B: \mathfrak{su}(2) \times \mathfrak{su}(2) \to \mathbb{R}$, $(X, Y) \mapsto \mathrm{trace}(XY)$ be the Killing form on $\mathfrak{su}(2)$. Then the pairing

$$(\eta, \mu) \mapsto \int_M B_*(\eta \wedge \mu) = \int_M \mathrm{trace}(\eta \wedge \mu)$$

($\eta, \mu \in \Lambda^1(M, \mathfrak{su}(2)) \approx T_A(\mathbb{C})$) defines an exterior 2-form ω on the infinite dimensional affine space $\mathcal{C}$. Since its definition does not involve A explicitly, it is invariant under the translations of $\mathcal{C}$ and is thus closed.

If d_A is the covariant differential corresponding to A then $A \in \mathcal{F}$ iff $d_A \circ d_A = 0$. Differentiating this equation with respect to a tangent vector $\eta \in \Lambda^1(M, \mathfrak{su}(2))$ one finds that the tangent vectors in $\mathcal{F}$ are precisely those $\eta \in \Lambda^1(M, \mathfrak{su}(2))$ with $d_A\eta = 0$, i.e. $T_A(\mathcal{F}) = Z^1(M, \mathfrak{su}(2))$.

The exterior 2-form ω on $\mathcal{C}$ restricts to a closed 2-form on $\mathcal{F}$. However on $\mathcal{F}$ this is degenerate. In fact the subspace of $T_A(\mathcal{F})$

which annihilates ω is precisely $B^1(M, \mathfrak{su}(2)) \subset Z^1(M, \mathfrak{su}(2))$. $B^1(M, \mathfrak{su}(2))$ is the image of $\Lambda^0(M, \mathfrak{su}(2)) = \mathrm{Map}(M, \mathfrak{su}(2))$ under $d_A(2)$. $\Lambda^0(M, \mathfrak{su}(2))$ is the Lie algebra of the gauge group $\mathcal{G} = \mathrm{Map}(M, \mathrm{SU}(2))$. ω restricts to a closed non-degenerate exterior 2-form on $\mathcal{F}/\mathcal{G}$ thus giving a symplectic structure on $\mathcal{F}/\mathcal{G}$, which is identified with

$$\mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2).$$

LEMMA 2.1. *When genus of $M \geq 3$, $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2)$ is simply connected.*

Proof. $\mathcal{F}^*/\mathcal{G} \approx \mathrm{Hom}^{\mathrm{irr}}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2)$ can be identified with the moduli space $\mathcal{M}_0^{\mathrm{st}}$ of stable vector bundles of rank 2 and trivial determinant on M by a theorem of Narasimhan and Seshadri [NS]. In fact by a theorem of Seshadri [S], $\mathcal{F}/\mathcal{G}$ is a complete complex algebraic variety—the moduli space $\mathcal{M}_0$ of (s-equivalence classes of) semistable vector bundles—in which $\mathcal{M}_0^{\mathrm{st}}$ sits as the smooth part. The singular part $\mathcal{M}_0 - \mathcal{M}_0^{\mathrm{st}} = K$ is a Kummer variety of complex dimension g (=genus of M).

It is known [**AB**] that the moduli space $\mathcal{M}_1$ of stable vector bundles of rank 2 and degree 1 with fixed determinant is simply connected and has complex dimension $3g-3$. Let $\mathbf{P}$ be the projective Poincaré bundle over $\mathcal{M}_1 \times \{x\}$ for a fixed point x in $\mathcal{M}_1$. Since $\mathbf{P} \to \mathcal{M}_1 \times \{x\}$ is a nice fibration [**NRa**] with standard fibre as the projective space $\mathbf{P}^1$, it follows by looking at the homotopy exact sequence that $\mathbf{P}$ is simply connected and has complex dimension $3g-2$. There is also a global map $f\colon \mathbf{P} \to \mathcal{M}_0 \times \{x_0\}$ ($x_0 \in \mathcal{M}_0$) which is not a nice fibration. However, the restriction $f\colon \mathbf{P} - f^{-1}(K) \to \mathcal{M}_0^{\mathrm{st}} \times \{x_0\}$ is a nice fibration. We claim $\mathbf{P} - f^{-1}(K)$ is simply connected when $g \geq 3$. Assuming the claim, it follows again by looking at the homotopy exact sequence that $\mathcal{M}_0^{\mathrm{st}} \approx \mathcal{F}^*/\mathcal{G} \cong \mathrm{Hom}^{\mathrm{irr}}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2)$ is simply connected.

K is the Kummer variety of complex dimension g. If x is a smooth point of K, $f^{-1}(x)$ looks like two copies of the projective space $\mathbf{P}^{g-1}$ intersecting at a point. If x is a singular point of K then $f^{-1}(x)$ looks like a nonreduced $\mathbf{P}^{g-1}$. Therefore complex dimension of $f^{-1}(K) = g + g - 1 = 2g - 1$. Since complex dimension of $\mathbf{P} = 3g - 2$, and $\mathbf{P}$ is smooth, complex codimension of $f^{-1}(K) = (3g-2)-(2g-1) = g-1$. Clearly real codimension of $f^{-1}(K) \geq 3$ if

$g \geq 3$ and therefore $\mathbf{P} - f^{-1}(K)$ is simply connected and the lemma follows. □

It is also known that ω has integral periods. Given a loop $\sigma: S^1 \to \mathscr{F}^*/\mathscr{G}$ we assign $\overline{\omega}(\sigma) \in S^1$ as follows. Since $\mathscr{F}^*/\mathscr{G}$ is simply connected we can choose a surface S in $\mathscr{F}^*/\mathscr{G}$ which bounds the loop σ. Integrating ω on S gives a real number. Choosing another surface $\widetilde{S}$ in $\mathscr{F}^*/\mathscr{G}$ which bounds the loop σ and integrating on $\widetilde{S}$ give a real number which differs from $\int_S \omega$ by an integer since ω has integral periods, i.e.

$$\int_S \omega = \left(\int_{\widetilde{S}} \omega\right) \mod \mathbb{Z}.$$

Thus

$$\text{(2.2)} \qquad \begin{aligned} \overline{\omega}&: \Omega(\mathscr{F}^*/\mathscr{G}) \to S^1 = \mathbb{R}/\mathbb{Z} \ldots \\ \sigma &\mapsto \overline{\omega}(\sigma) = \left(\frac{1}{4\pi^2}\int_S \omega\right) \mod \mathbb{Z} \end{aligned}$$

is well defined.

3. The Coulomb connection on $\mathscr{C}^* \to \mathscr{C}^*/\mathscr{G}$. $\mathscr{C}^*$ is the space of irreducible connections on the trivial SU(2)-bundle $E \to M$. It is well known that

$$\mathscr{C}^* = \{A \in \mathscr{C} \mid d_A: \Lambda^0(M, \mathfrak{su}(2)) \to \Lambda^1(M, \mathfrak{su}(2)) \text{ is injective}\}.$$

The Poincaré metric on M and the metric given by the Killing form on $\mathfrak{su}(2)$ induces inner products on $\Lambda^0(M, \mathfrak{su}(2))$ and $\Lambda^1(M, \mathfrak{su}(2))$.

Let $d_A^*: \Lambda^1(M, \mathfrak{su}(2)) \to \Lambda^0(M, \mathfrak{su}(2))$ be the adjoint of d_A.

We now define a connection on $\mathscr{C}^*$: We take the horizontal space at $A \in \mathscr{C}^*$ to be the space

$$H_A = \operatorname{Ker} d_A^* = \{B \in \mathscr{C}, d_A^* B = 0\}.$$

Clearly $\operatorname{Ker} d_A^* \approx \Lambda^1(M, \mathfrak{su}(2))/(d_A(\Lambda^0(M, \mathfrak{su}(2)))) = T_{[A]}(\mathscr{C}^*/\mathscr{G})$ where $[A] \in \mathscr{C}^*/\mathscr{G}$ is the equivalence class of A under gauge group action.

Let $\Delta_A = d_A^* \circ d_A: \Lambda^0(M, \mathfrak{su}(2)) \to \Lambda^0(M, \mathfrak{su}(2))$ be the covariant Laplacian.

It is easily seen that the connection form of this connection at $A \in \mathscr{C}^*$ is given by $\Delta_A^{-1} \circ d_A^*$. (For more details refer to **[NR]**.) We call this connection form as the Coulomb connection. Clearly $\mathscr{F}^*/\mathscr{G}$ is contained in $\mathscr{C}^*/\mathscr{G}$. Pulling back the Coulomb connection to $\mathscr{F}^*/\mathscr{G}$

gives a connection on $\mathcal{F}^* \to \mathcal{F}^*/\mathcal{G}$. This restricted connection is also called the Coulomb connection.

4. Construction of the map $\overline{\chi}\colon \Omega(\mathcal{F}^*/\mathcal{G}) \to \mathbb{R}/\mathbb{Z}$. In §1, we can replace $\mathcal{F}$ by $\mathcal{F}^*$, the space of all irreducible flat connections and construct the map $\chi\colon \Omega(\mathcal{F}^*) \to \mathbb{R}$.

Given a loop $\sigma\colon [0, 1] \to \mathcal{F}^*/\mathcal{G}$ with $\sigma(0) = \sigma(1)$ we can lift it horizontally to a path $\tilde{\sigma}\colon [0, 1] \to \mathcal{F}^*$ using the Coulomb connection on $\mathcal{F}^* \to \mathcal{F}^*/\mathcal{G}$. Clearly $\tilde{\sigma}(0)$ and $\tilde{\sigma}(1)$ are gauge-equivalent connections, i.e, they lie in the same fibre over $\sigma(0)$. Since $\mathcal{G} = \mathrm{Map}(M, \mathrm{SU}(2))$ is connected, $\tilde{\sigma}(1)$ can be joined to $\tilde{\sigma}(0)$ by a path φ. The path $\tilde{\sigma}$ from $\tilde{\sigma}(0)$ to $\tilde{\sigma}(1)$ followed by the path φ from $\tilde{\sigma}(1)$ to $\tilde{\sigma}(0)$ defines a loop $\tilde{\sigma}_\varphi$ based at $\tilde{\sigma}(0)$ in $\mathcal{F}^*$ and $\chi(\tilde{\sigma}_\varphi) \in \mathbb{R}$. If φ' is another path joining $\tilde{\sigma}(1)$ and $\tilde{\sigma}(0)$ then $\chi(\tilde{\sigma}_{\varphi'})$ need not be equal to $\chi(\tilde{\sigma}_\varphi)$. However we claim $\chi(\tilde{\sigma}_\varphi) = \chi(\tilde{\sigma}_{\varphi'}) \bmod \mathbb{Z}$. We then set $\overline{\chi}(\sigma) = \overline{\chi(\tilde{\sigma}_\varphi)}$, where $\overline{\chi(\tilde{\sigma}_\varphi)}$ is the image of $\chi(\tilde{\sigma}_\varphi)$ in $\mathbb{R}/\mathbb{Z}$. To prove the claim we need the following lemma.

LEMMA 4.1. *Let $\eta \in \mathcal{F}$ be a fixed flat connection and $\psi\colon S^1 \to \mathcal{G} = \mathrm{Map}(M, \mathrm{SU}(2))$ (also thought of as a map $\psi\colon S^1 \times M \to \mathrm{SU}(2)$) be a loop in the gauge group. The action of $\mathcal{G}$ on $\mathcal{F}$ defines a loop ψ_η based at η in $\mathcal{F}$. Then $\chi(\psi_\eta) =$ degree of ψ.*

REMARK 4.2. Thus two homotopically equivalent loops in the same fibre (gauge orbit) of $\mathcal{F} \to \mathcal{F}/\mathcal{G}$ map under χ to the same integer.

Assuming the lemma we prove the claim

$$\chi(\tilde{\sigma}_\varphi) = \chi(\tilde{\sigma}_{\varphi'}) \mod \mathbb{Z}.$$

$\varphi^{-1}\varphi'$ defines a loop $\psi_{\tilde{\sigma}(0)}$ based at $\tilde{\sigma}(0)$ for appropriate $\psi\colon S' \to \mathcal{G}$. From the definition of χ, it follows that

$$\chi(\tilde{\sigma}_{\varphi'}) = \chi(\tilde{\sigma}_\varphi \circ \psi_{\tilde{\sigma}(0)}).$$

Therefore

$$\chi(\tilde{\sigma}_{\varphi'}) = \chi(\tilde{\sigma}_\varphi) + \chi(\psi_{\tilde{\sigma}(0)}) = \chi(\tilde{\sigma}_\varphi) + \text{degree } \psi$$
$$\Rightarrow \chi(\tilde{\sigma}_{\varphi'}) = \chi(\tilde{\sigma}_\varphi) \mod \mathbb{Z}.$$

Proof of Lemma 4.1. Let

$$\mu = \begin{pmatrix} i\mu_1 & \mu_2 + i\mu_3 \\ -\mu_2 + i\mu_3 & -i\mu_1 \end{pmatrix}$$

be the Maurer-Cartan form on $\mathrm{SU}(2)$.

$$d\mu = -\mu \wedge \mu \Rightarrow \begin{cases} d\mu_1 = -2\mu_2 \wedge \mu_3, \\ d\mu_2 = -2\mu_3 \wedge \mu_1, \\ d\mu_2 = -2\mu_1 \wedge \mu_2. \end{cases}$$

One knows that

$$\frac{1}{4\pi^2}\mu_1 \wedge \mu_2 \wedge \mu_3 \text{ is the volume form on } \mathrm{SU}(2).$$

Hence

$$(4.3) \qquad \frac{1}{4\pi^2}\int_{S^1 \times M} \psi^*\mu_1 \wedge \psi^*\mu_2 \wedge \psi^*\mu_3 = \text{degree of } \psi \ldots.$$

We first explicitly compute $\chi(\sigma)$ for any loop $\sigma\colon S^1 \to \mathscr{F}$.

For $t \in S^1$, let

$$\sigma(t) = \begin{pmatrix} i\alpha(t) & \beta(t) + i\gamma(t) \\ -\beta(t) + i\gamma(t) & -i\alpha(t) \end{pmatrix}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are real valued 1-forms on M for each $t \in S^1$.

$$\sigma(t) \in \mathscr{F} \Rightarrow d\sigma(t) = \frac{1}{2}[\sigma(t), \sigma(t)] = -\sigma(t) \wedge \sigma(t)$$

$$\Rightarrow \begin{cases} d\alpha(t) = -2\beta(t) \wedge \gamma(t), \\ d\beta(t) = -2\gamma(t) \wedge \alpha(t), \\ dr(t) = -2\alpha(t) \wedge \beta(t). \end{cases}$$

We extend σ to $\tilde{\sigma}\colon D^2 \to \mathscr{C}$ in the obvious way.

Let (s, t) be the polar coordinates on $D^2 = \{(s, t), 0 \le s \le 1, 0 \le t \le 2\pi\}$,

$$\tilde{\sigma}(s, t) = s\sigma(t) = \begin{pmatrix} is\alpha(t) & s\beta(t) + is\gamma(t) \\ -s\beta(t) + is\gamma(t) & -is\alpha(t) \end{pmatrix}.$$

The curvature $K(\vartheta^\sigma)$ of the connection form ϑ^σ on the bundle $E \times D^2 \to M \times D^2$ is given by

$$\begin{aligned} K(\vartheta^\sigma) &= d\vartheta^\sigma + \tfrac{1}{2}[\vartheta^\sigma, \vartheta^\sigma] \\ &= d\vartheta^\sigma + \vartheta^\sigma \wedge \vartheta^\sigma \\ &= d_E\vartheta^\sigma + d_{D^2}\vartheta^\sigma + \vartheta^\sigma \wedge \vartheta^\sigma \\ &= d_{D^2}\vartheta^\sigma + K(\tilde{\sigma}(s, t)) \end{aligned}$$

where $K(\tilde{\sigma}(s, t))$ is the curvature of $\tilde{\sigma}(s, t)$.

It can be checked that $C_2(K(\vartheta^\sigma))$ is cohomologous to the form

$$(4.4) \qquad \tilde{\chi}(\sigma) = \frac{1}{4\pi^2}\int_{S^1} (\dot{\alpha}(t) \wedge \alpha(t) + \dot{\beta}(t) \wedge \beta(t) + \dot{\gamma}(t) \wedge \gamma(t))\, dt \ldots$$

where $\dot{\alpha}(t) = \frac{d}{dt}\alpha(t)$.

Thus

$$\chi(\sigma) = \left\{ \frac{1}{4\pi^2} \int_{S^1} (\dot\alpha(t) \wedge \alpha(t) + \dot\beta(t) \wedge \beta(t) + \dot\gamma(t) \wedge \gamma(t))\, dt \right\} \in H^2(M, \mathbb{R}) \approx \mathbb{R}.$$

Let

$$\eta = \begin{pmatrix} i\eta_1 & \eta_2 + i\eta_3 \\ -\eta_2 + i\eta_3 & -\eta_1 \end{pmatrix}$$

be an arbitrary but fixed flat connection.

Clearly $\psi_\eta(t) = \psi(t) \cdot \eta = \psi(t)^{-1}\eta \cdot \psi(t) + \psi(t)^*\mu \quad \forall\, t \in S^1$. $S^1 \xrightarrow{\psi_\eta} \mathscr{F}(t \mapsto \psi(t) \cdot \eta)$ defines a loop in $\mathscr{F}$.

After writing down the formula (4.4) for $\tilde\chi(\psi_\eta)$ it can be checked that

$$\overline{\chi}(\psi_\eta) = \frac{1}{2\pi^2} \int_{S^2} \psi^*\mu_1 \wedge \psi^*\mu_2 \wedge \psi^*\mu_3 + \text{exact}$$

$\Rightarrow \chi(\psi_\eta) =$ degree of ψ. This proves Lemma 4.1.

Thus $\chi\colon \Omega(\mathscr{F}^*) \to \mathbb{R}$ induces

$$\overline{\chi}\colon \Omega(\mathscr{F}^*/\mathscr{G}) \to \mathbb{R}/\mathbb{Z} = S^1 \ldots \tag{4.5}$$

5. Relation between the map $\overline{\chi}\colon \mathscr{F}^*/\mathscr{G} \to \mathbb{R}/\mathbb{Z}$ and the symplectic structure on $\mathscr{F}/\mathscr{G}$.

THEOREM 5.1. *Let $E \to M$ be the trivial* SU(2) *bundle over a compact Riemann surface M of genus ≥ 3, $\mathscr{F}$ (resp. $\mathscr{F}^*$) be the space of all (irreducible) flat connections and $\mathscr{G}$ be the gauge group. Let $\overline{\chi}\colon \Omega(\mathscr{F}^*/\mathscr{G}) \to S^1$ and $\overline{\omega}\colon \Omega(\mathscr{F}^*/\mathscr{G}) \to S^1$ be as defined in* (4.5) *and* (2.2) *respectively. Then*

$$\overline{\chi}(\sigma) = \overline{\omega}(\sigma) \quad \forall\, \sigma \in \mathscr{F}^*/\mathscr{G}.$$

Proof. Lift σ to a loop $\tilde\sigma$ in $\mathscr{F}^*$ as in §4; i.e. first lift σ to a path in $\mathscr{F}^*$ and join the end-points using a path in $\mathscr{G}$. As in §2, let ω be the exterior 2-form on the infinite dimensional affine space $\mathscr{C}$. Since $\mathscr{C}$ is contractible and ω is closed we can write $\omega = d\nu$ for some 1-form on $\mathscr{C}$ and $\int_S \omega = \int_{\tilde\sigma} \nu$ for any surface S which bounds $\tilde\sigma$ in $\mathscr{C}$.

Define ν as follows:

For $\eta \in \mathscr{C}$, $\nu_\eta\colon \Lambda^1(M, \mathfrak{su}(2)) \to \mathbb{R}$ is given by

$$\nu_\eta(\mu) = -\int_M \operatorname{tr}(\eta\Lambda\mu) \quad \text{for } \mu \in \Lambda^1(M, \mathfrak{su}(2)).$$

We claim

$$\text{(5.2)} \qquad d\nu = \omega \ldots.$$

We check $d\nu = \omega$ at $\eta \in \mathscr{C}$.

For $\mu_1, \mu_2, \in T_\eta(\mathscr{C}) = \Lambda^1(M, \mathrm{su}(2))$ (extend μ_1, μ_2 to vector fields in the obvious way).

$$d\nu(\mu_1, \mu_2) = \tfrac{1}{2}(\mu_1\nu(\mu_2) - \mu_2\nu(\mu_1) - \nu([\mu_1, \mu_2]));$$

since $\mathscr{C}$ is affine, we can assume $[\mu_1, \mu_2] = 0$ at η

$$\mu_1\nu(\mu_2) = d\nu(\mu_2)(\mu_1)$$

where $\nu(\mu_2)$ is treated as a function

$$\nu(\mu_2)\colon \mathscr{C} \to \mathbb{R},$$
$$\nu(\mu_2)(\varphi) = \int_M \mathrm{tr}(\mu_2 \wedge \varphi).$$

Since $\nu(\mu_2)$ is a linear function $d\nu(\mu_2) = \nu(\mu_1)$ so that $\mu_1\nu(\mu_2) = -\int_M \mathrm{tr}(\mu_2 \wedge \mu_1)$. Similarly $\mu_2\nu(\mu_1) = -\int_M \mathrm{tr}(\mu_1 \wedge \mu_2)$.

Therefore

$$\begin{aligned}\frac{1}{2}\{\mu_1\nu(\mu_2) - \mu_2\nu(\mu_1)\} &= -\frac{1}{2}\int_M \{\mathrm{tr}(\mu_2 \wedge \mu_1) - \mathrm{tr}(\mu_1 \wedge \mu_2)\} \\ &= -\int_M \mathrm{tr}(\mu_2 \wedge \mu_1) \quad \text{since } \mathrm{tr}(\mu_2 \wedge \mu_1) = -\mathrm{tr}(\mu_1 \wedge \mu_2) \\ &= +\int_M \mathrm{tr}(\mu_1 \wedge \mu_2).\end{aligned}$$

Therefore $d\nu(\mu_1, \mu_2) = \int_M \mathrm{tr}(\mu_1 \wedge \mu_2) = \omega(\mu_1, \mu_2)$ and this proves (5.2).

Clearly

$$\begin{aligned}\int_{\tilde\sigma} \nu = \int_{S^1} \nu_{\tilde\sigma(t)}(\dot{\tilde\sigma}(t))\,dt &= -\int_{S^1} \mathrm{tr}(\tilde\sigma(t) \wedge \dot{\tilde\sigma}(t))\,dt \\ &= \int_{S^1} \mathrm{tr}(\dot{\tilde\sigma}(t) \wedge \tilde\sigma(t))\,dt \\ &= \int_{S^1} (\dot\alpha(t) \wedge \alpha(t) + \dot\beta(t) \wedge \beta(t) + \dot\gamma(t) \wedge \gamma(t)\,dt)\end{aligned}$$

where

$$\tilde\sigma(t) = \begin{pmatrix} i\alpha(t) & \beta(t) + i\gamma(t) \\ -\beta(t) + i\gamma(t) & -i\alpha(t) \end{pmatrix}.$$

Hence $\int_{\tilde\sigma} \nu = 4\pi^2\chi(\tilde\sigma) \Rightarrow \chi(\tilde\sigma) = \frac{1}{4\pi^2}\int_{\tilde\sigma} \nu = \frac{1}{4\pi^2}\int_S \omega \Rightarrow \overline{\chi}(\sigma) = \overline{\omega}(\sigma)$ and this proves the theorem.

REMARK 5.3. In **[RSW]**, the authors prove the existence of a natural hermitian line bundle on $\mathscr{F}/\mathscr{G}$. Restricted to $\mathscr{F}^*/\mathscr{G}$, this line bundle carries a natural connection whose curvature is (up to a factor of i) the standard symplectic form. It is easy to check that $\overline{\omega}$: $\Omega(\mathscr{F}^*/\mathscr{G}) \to S^1$ is then (up to a constant) the holonomy of this connection.

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