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EXTREMAL FUNCTIONS AND THE CHANG-MARSHALL INEQUALITY

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Answering a question of J. Moser, S.-Y. A. Chang and D. E. Marshall proved the existence of a constant C such that $\frac{1}{2\pi}\int_0^{2\pi} e^{|f(e^{i\theta})|^2}\,d\theta \le C$ for all functions f analytic in the unit disk with $f(0) = 0$ and Dirichlet integral not exceeding one. We show that there are extremal functions for the functionals $\Lambda_\alpha(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\,d\theta$ when $0 \le \alpha < 1$. We establish a variational condition satisfied by extremal functions. We show that the identity function $f(z) = z$ is a local maximum in a certain sense for the functionals Λ_α and conjecture that it is a global maximum.

1. Introduction. The Dirichlet space $\mathfrak{D}$ consists of those functions f analytic on the unit disk Δ which have finite Dirichlet integral

$$\|f\|_{\mathfrak{D}}^2 = \frac{1}{\pi}\iint_\Delta |f'(z)|^2\,dx\,dy.$$

We will always assume that $f(0) = 0$. It is well-known and easy to establish that $\mathfrak{D}$ is a Hilbert space under the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\iint_\Delta f'(z)\overline{g'(z)}\,dx\,dy,$$

and that, if $f(z) = \sum_{n=1}^\infty a_n z^n$ and $g(z) = \sum_{n=1}^\infty b_n z^n$, then

$$\langle f, g\rangle = \sum_{n=1}^\infty n a_n \overline{b_n}.$$

In particular,

$$\|f\|_{\mathfrak{D}}^2 = \sum_{n=1}^\infty n|a_n|^2.$$

For $\alpha \ge 0$ and $f \in \mathfrak{D}$, we define

$$\Lambda_\alpha(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\,d\theta.$$

This is known to be finite for all $\alpha \ge 0$ and all $f \in \mathfrak{D}$, and it can be shown that the quantity

$$I_\alpha = \sup\{\,\Lambda_\alpha(f) \mid \|f\|_{\mathfrak{D}} \le 1\,\}$$

is finite for $0 \leq \alpha \leq 1$ and infinite for $\alpha > 1$. A proof of this for $0 \leq \alpha < 1$ will be indicated below. For $\alpha > 1$, this follows from an estimate of A. Beurling **[1]**. Answering a question of J. Moser **[9]**, the finiteness of this quantity for the critical value $\alpha = 1$ was first established by S.-Y. A. Chang and D. E. Marshall **[5]**. A different proof of this fact was subsequently given by Marshall **[8]**, and this has been generalized by M. Essén **[7]**.

The proof of Chang and Marshall involves a delicate argument based on an estimate of Beurling **[1]**. Since we need Beurling's estimate in section 2, we describe it here. Let $f \in \mathfrak{D}$ satisfy $\|f\|_{\mathfrak{D}} \leq 1$, and for $t \geq 0$, let

$$E_t = \{\,\theta \mid |f(e^{i\theta})| > t\,\}.$$

If $|E_t|$ denotes the normalized Lebesgue measure of E_t, then Beurling's estimate is

$$|E_t| \leq e^{-t^2+1}.$$

It follows immediately that

$$\begin{aligned}\Lambda_\alpha(f) &= 1 + 2\alpha \int_0^\infty t e^{\alpha t^2} |E_t|\, dt \\ &\leq 1 + 2\alpha e \int_0^\infty t e^{-(1-\alpha)t^2}\, dt = 1 + e\frac{\alpha}{1-\alpha},\end{aligned}$$

if $0 \leq \alpha < 1$. We note that this proves that I_α is finite if $0 \leq \alpha < 1$, but, in light of the result of Chang and Marshall, the bound is far from precise.

We now show that $\Lambda_\alpha(f)$ is finite for any $f \in \mathfrak{D}$ and any $\alpha > 0$. By scaling, it is clear that $\Lambda_\alpha(f) < \infty$ if $\alpha\|f\|_{\mathfrak{D}}^2 < 1$. If $f \in \mathfrak{D}$ and $\epsilon > 0$, there is a polynomial p such that $\|f-p\|_{\mathfrak{D}}^2 < \epsilon$. But $|f|^2 \leq 2|f-p|^2 + 2|p|^2 \leq 2|f-p|^2 + 2\|p\|_\infty^2$, where $\|p\|_\infty = \sup\{\,|p(z)| \mid z \in \Delta\,\}$. Then

$$\Lambda_\alpha(f) \leq e^{2\alpha\|p\|_\infty} \frac{1}{2\pi} \int_0^{2\pi} e^{2\alpha|f-p|^2}\, d\theta,$$

and, by taking ϵ small enough, it follows that $\Lambda_\alpha(f)$ is finite. Chang and Marshall attribute this observation to J. Garnett. We also note that, since $x^n e^{x^2} \leq C_n e^{2x^2}$ for any positive integer and some constant C_n, the quantity

$$\frac{1}{2\pi} \int_0^{2\pi} p(|f(e^{i\theta})|) e^{\alpha|f(e^{i\theta})|^2}\, d\theta$$

is finite for any $\alpha > 0$, any $f \in \mathfrak{D}$, and any polynomial p.

It is natural to ask if there exist extremal functions for the functionals Λ_α in case $0 < \alpha \leq 1$. In the subcritical case $0 < \alpha < 1$, this will be established by an easy continuity argument in §2. The critical case $\alpha = 1$ appears to be much more difficult, and the remainder of this paper is devoted to studying this problem. In §3 we establish a variational criterion which must be satisfied by extremal functions which are sufficiently smooth. It will be seen that the identity function $f(z) = z$ satisfies this criterion. In §4 we show that in a certain sense the identity function is a local maximum for the functionals Λ_α. In §5 we discuss various other aspects of this problem.

Finally, we make two observations concerning the nature of extremal functions. The first of these is attributed by Chang and Marshall to J. Clunie.

Let $f(z) = \sum_{n=1}^\infty a_n z^n$ have finite Dirichlet integral, and let $f^*(z) = \sum_{n=1}^\infty |a_n| z^n$. Then clearly $\|f\|_{\mathfrak{D}} = \|f^*\|_{\mathfrak{D}}$, while

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^{2n}\,d\theta &= \sum_{k=1}^\infty \left| \sum_{j_1+\cdots+j_n=k} a_{j_1}\ldots a_{j_n}\right|^2 \\ &\leq \sum_{k=1}^\infty \left(\sum_{j_1+\cdots+j_n=k} |a_{j_1}|\ldots|a_{j_n}|\right)^2 \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f^*(e^{i\theta})|^{2n}\,d\theta\end{aligned}$$

for $n = 1, 2, \ldots$. It follows from the power series representation $\Lambda_\alpha(f) = \sum_{n=0}^\infty \alpha^n \frac{\|f\|_{2n}^{2n}}{n!}$ that $\Lambda_\alpha(f) \leq \Lambda_\alpha(f^*)$. Because of this, it is enough to consider functions with nonnegative coefficients.

The second observation uses a theorem of L. Carleson [**2**] to show that extremal functions must be of a certain form. Although we do not use this observation in the sequel, we think it is of sufficient interest to include here. It is clear that if f is extremal for the functional Λ_α, then $\|f\|_{\mathfrak{D}} = 1$. Since f belongs to the Hardy space H^2, f admits a Riesz factorization

$$f(z) = z^m B(z) S(z) F(z),$$

where m is a positive integer, $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $F(z)$ is an outer function [**6**, Thm. 2.8]. Let $g(z) = zF(z)$. Then Carleson's theorem guarantees that $\|g\|_{\mathfrak{D}} \leq \|f\|_{\mathfrak{D}}$, with equality if and only if $f = g$. On the other hand, $|f(e^{i\theta})| = |g(e^{i\theta})|$ almost everywhere, so $\Lambda_\alpha(f) = \Lambda_\alpha(g)$. Thus an extremal function must have the form $zF(z)$, where $F(z)$ is an outer function.

2. Weak continuity of functionals. Let B denote the closed unit ball of $\mathfrak{D}$. Since B is weakly compact, in order to prove the existence of an extremal function for Λ_α, it is sufficient to show that Λ_α is weakly continuous on B. Since

$$\Lambda_\alpha(f) = \sum_{n=0}^{\infty} \alpha^n \frac{\|f\|_{2n}^{2n}}{n!}$$

for each $f \in B$. We can do this for $0 < \alpha < 1$ by showing that the L^p norms $f \to \|f\|_p$ are weakly continuous on B, and by establishing an estimate $\|f\|_{2n}^{2n} = O(n!)$ in order to get uniform convergence of the series above. This is done with the following three lemmas. In what follows, we use standard notation and ideas from the theory of Hardy spaces as found, for example, in [6]. In particular, we note that the norm of a function $f(z)$ in the Hardy space H^p coincides with the norm of its boundary function $f(e^{i\theta})$ in L^p.

LEMMA 1. *If $f \in \mathfrak{D}$, then $f \in H^p$, and*

$$\|f\|_p^p \le e\frac{p}{2}\Gamma\left(\frac{p}{2}\right)\|f\|_{\mathfrak{D}}.$$

Proof. Let $f \in \mathfrak{D}$ satisfy $\|f\|_{\mathfrak{D}} = 1$. Let $E_t = \{\theta \mid |f(e^{i\theta})| > t\}$. Then

$$\begin{aligned}\|f\|_p^p &= p\int_0^\infty t^{p-1}|E_t|\,dt \le ep\int_0^\infty t^{p-1}e^{-t^2}\,dt \\ &= e\frac{p}{2}\int_0^\infty u^{\frac{p}{2}-1}e^{-u}\,du = e\frac{p}{2}\Gamma\left(\frac{p}{2}\right),\end{aligned}$$

where we have used Beurling's estimate $|E_t| \le e^{-t^2+1}$. The lemma follows.

LEMMA 2. *If $(f_n)_{n=1}^\infty$ is a weakly null sequence in $\mathfrak{D}$, then* $\|f_n\|_2 \to 0$.

Proof. Since weakly null sequences are bounded, we may assume that $f_n \in B$ for each n. Writing $f_n(z) = \sum_{k=1}^\infty a_{n,k}z^k$, we have

$$\|f\|_{\mathfrak{D}}^2 = \sum_{k=1}^\infty k|a_{n,k}|^2,$$

and

$$\|f\|_2^2 = \sum_{k=1}^{\infty} |a_{n,k}|^2.$$

For any positive integer K, the Cauchy-Schwarz inequality yields

$$\sum_{k=K}^{\infty} |a_{n,k}|^2 \le \left(\sum_{k=K}^{\infty} \frac{1}{k}|a_{n,k}|^2\right)^{1/2} \left(\sum_{k=K}^{\infty} k|a_{n,k}|^2\right)^{1/2}$$

$$\le \frac{1}{\sqrt{K}} \left(\sum_{k=K}^{\infty} |a_{n,k}|^2\right)^{1/2},$$

so that

$$\sum_{k=K}^{\infty} |a_{n,k}|^2 \le \frac{1}{K}$$

for $n = 1, 2, \ldots$. On the other hand, because $(f_n)_{n=1}^{\infty}$ is weakly null, for each fixed k, $a_{n,k} \to 0$ as $n \to \infty$. Hence, given $\epsilon > 0$, we can choose K so that $\frac{1}{K} < \frac{\epsilon}{2}$, and then choose N so that if $n \ge N$ and $k < K$, then $|a_{n,k}| < \frac{\epsilon}{2K}$. Hence, if $n \ge N$, $\|f\|_2^2 < \epsilon$ and the lemma is proved.

LEMMA 3. *For each* p, $0 < p < \infty$, *the function* $f \to \|f\|_p$ *is weakly continuous on* B.

Proof. It is enough to show that if $f_n \to f$ weakly in B, then $\|f_n - f\|_p \to 0$. Let $g_n = \frac{1}{2}(f_n - f)$. Then $g_n \in B$, and $(g_n)_{n=1}^{\infty}$ is weakly null, so $\|g_n\|_2 \to 0$ by Lemma 2. If $0 < p < 2$, then $\|g_n\|_p \le \|g_n\|_2$ by Hölder's inequality, and the lemma is immediate. For $p > 2$, we use Hölder's inequality in the form

$$\|f\|_p \le \|f\|_2^{\frac{1}{p}} \|f\|_{2p-2}^{1-\frac{1}{p}}$$

together with Lemma 1 to complete the proof.

We have now established the following theorem

THEOREM 1. *For each* α, $0 < \alpha < 1$, *the nonlinear functional*

$$\Lambda_\alpha(f) = \frac{1}{2\pi} \int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\, d\theta$$

is weakly continuous on B. *Consequently* Λ_α *attains its maximum on* B.

3. A variational condition. If the function $f \in \mathfrak{D}$, $\|f\|_{\mathfrak{D}} = 1$, is extremal for some Λ_α, $0 < \alpha \le 1$, and if f has a certain degree of smoothness up to the boundary of Δ, then it is possible to establish a condition satisfied by f by means of a variational argument. In order for the integrals below to be defined we will assume that f' belong to the Hardy space H^1. In this case f will be continuous on the closed disk, and it will be sufficient to assume that the test functions ϕ also have derivatives in H^1.

PROPOSITION 1. *Suppose that $f \in \mathfrak{D}$, $\|f\|_{\mathfrak{D}} = 1$, $f' \in H^1$, and f is extremal for the functional Λ_α, for some α, $0 < \alpha \le 1$. Let ϕ be a function with $\phi' \in H^1$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 e^{\alpha|f(e^{i\theta})|^2} 2\Re\phi(e^{i\theta})\, d\theta$$
$$= \frac{S_\alpha(f)}{2\pi}\int_0^{2\pi} \left\{2\Re\phi(e^{i\theta})f'(e^{i\theta})\overline{f(e^{i\theta})} + \phi'(e^{i\theta})|f(e^{i\theta})|^2\right\} e^{i\theta}\, d\theta,$$

where

$$S_\alpha(f) = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 e^{\alpha|f(e^{i\theta})|^2}\, d\theta.$$

Proof. Write $\phi = u + iv$, and let $f_t = fe^{t\phi}$ for real t. Let

$$h(t) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\alpha\frac{|f_t(e^{i\theta})|^2}{\|f_t\|_{\mathfrak{D}}^2}\right\} d\theta.$$

Then, differentiating under the integral sign, we obtain

$$h'(t) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\alpha\frac{|f_t(e^{i\theta})|^2}{\|f_t\|_{\mathfrak{D}}^2}\right\} \frac{\partial}{\partial t}\frac{\alpha|f_t(e^{i\theta})|^2}{\|f_t\|_{\mathfrak{D}}^2}\, d\theta.$$

Because f is extremal, $h'(0) = 0$, and it will be necessary to evaluate the integrand when $t = 0$. Since $f_0 = f$, and $\|f\|_{\mathfrak{D}} = 1$, the first factor becomes $\exp\{\alpha|f(e^{i\theta})|^2\}$ when $t = 0$. Since $|f_t|^2 = |f|^2e^{2tu}$, we have

$$\frac{\partial}{\partial t}\frac{|f_t|^2}{\|f_t\|_{\mathfrak{D}}^2} = \frac{2u|f|^2e^{2ut}}{\|f_t\|_{\mathfrak{D}}^2} - \frac{|f_t|^2}{\|f_t\|_{\mathfrak{D}}^4}\frac{\partial}{\partial t}\|f_t\|_{\mathfrak{D}}^2,$$

and the first term becomes $2u|f|^2$ when $t = 0$. By Green's Theorem,

$$\|f_t\|_{\mathfrak{D}}^2 = \frac{1}{2\pi}\int_0^{2\pi} f_t'(e^{i\theta})\overline{f_t(e^{i\theta})}e^{i\theta}\, d\theta,$$

and so

$$\frac{\partial}{\partial t}\|f_t\|_{\mathfrak{D}}^2 = \frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{\partial f_t'}{\partial t}\overline{f_t} + f_t'\frac{\partial\overline{f_t}}{\partial t}\right\} e^{i\theta}\, d\theta.$$

But

$$f_t' = f'e^{t\phi} + t\phi' f e^{t\phi},$$

so

$$\frac{\partial f_t'}{\partial t} = \phi f' e^{t\phi} + \phi' f e^{t\phi} + t\phi\phi' e^{t\phi}.$$

Hence,

$$\left.\frac{\partial f_t'}{\partial t}\right|_{t=0} = \phi f' + \phi' f,$$

and so

$$\left.\frac{\partial f_t'}{\partial t}\overline{f_t}\right|_{t=0} = \phi\overline{f}f' + \phi' f\overline{f}.$$

Also,

$$\frac{\partial f_t}{\partial t} = \phi f e^{t\phi},$$

so that

$$\left.f_t'\frac{\partial \overline{f_t}}{\partial t}\right|_{t=0} = f'\overline{\phi f}.$$

Consequently, with $z = e^{i\theta}$,

$$\begin{aligned}\left.\frac{\partial}{\partial t}\|f_t\|_{\mathfrak{D}}^2\right|_{t=0} &= \frac{1}{2\pi}\int_0^{2\pi}\left(\phi\overline{f}f' + \phi' f\overline{f} + \overline{\phi f}f'\right) z\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(2u\overline{f}f' + \phi' f\overline{f}\right) z\, d\theta.\end{aligned}$$

Hence,

$$\left.\frac{\partial}{\partial t}\frac{|f_t|^2}{\|f_t\|_{\mathfrak{D}}^2}\right|_{t=0} = 2u|f|^2 - \frac{|f|^2}{2\pi}\int_0^{2\pi}\left(2u\overline{f}f' + \phi' f\overline{f}\right) z\, d\theta,$$

and the condition $h'(0) = 0$ becomes, on applying Fubini's Theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} 2u|f|^2 e^{\alpha|f|^2}\, d\theta = \frac{S_\alpha(f)}{2\pi}\int_0^{2\pi}\left(2u\overline{f}f' + \phi' f\overline{f}\right) z\, d\theta.$$

That completes the proof.

Corollary 1. *Under the same assumptions on f and ϕ, we have*

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}|f|^2 e^{\alpha|f|^2}(\phi + \overline{\phi})\, d\theta \\ &\quad = \frac{S_\alpha(f)}{2\pi}\int_0^{2\pi}\left\{(\phi+\overline{\phi})f'\overline{f}z + \phi(\overline{zf'}f - zf'\overline{f})\right\} d\theta.\end{aligned}$$

Proof. This follows on integrating

$$\int_0^{2\pi} \phi'(e^{i\theta}) f(e^{i\theta}) \overline{f(e^{i\theta})} e^{i\theta}\, d\theta$$

by parts, noting that $\frac{\partial}{\partial\theta} f(e^{i\theta}) = ie^{i\theta} f'(e^{i\theta})$, and that $\frac{\partial}{\partial\theta}\overline{f(e^{i\theta})} = -ie^{-i\theta}\overline{f'(e^{i\theta})}$.

COROLLARY 2. *Under the same assumptions on* f *and* ϕ, *we have*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 e^{\alpha|f(e^{i\theta})|^2} \phi(e^{i\theta})\, d\theta = \frac{S_\alpha(f)}{2\pi}\int_0^{2\pi} e^{-i\theta}\overline{f'(e^{i\theta})} f(e^{i\theta})\phi(e^{i\theta})\, d\theta.$$

Proof. From Corollary 1, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f|^2 e^{\alpha|f|^2}(\phi+\overline{\phi})\, d\theta = \frac{S_\alpha(f)}{2\pi}\int_0^{2\pi}\left\{(\phi+\overline{\phi})f'\overline{f}z + \phi(\overline{zf'}f - zf'\overline{f})\right\} d\theta.$$

Replacing ϕ by $i\phi$ and then dividing by i gives

$$\frac{1}{2\pi}\int_0^{2\pi} |f|^2 e^{\alpha|f|^2}(\phi-\overline{\phi})\, d\theta = \frac{S_\alpha(f)}{2\pi}\int_0^{2\pi}\left\{(\phi-\overline{\phi})f'\overline{f}z + \phi(\overline{zf'}f - zf'\overline{f})\right\} d\theta.$$

Adding these two equations completes the proof.

In the next corollary $\mathscr{H}_0^p$ denotes the set of boundary functions $f(e^{i\theta})$ of functions $f \in H_0^p$, where H_0^p denotes the set of functions in H^p which vanish at the origin [cf. **6**, §3.2].

COROLLARY 3. *If* f *is as above, then*

$$|f|^2 e^{\alpha|f|^2} - S_\alpha(f) f\overline{zf'} \in \mathscr{H}_0^1.$$

It is easy to see that the identity function $f(z) = z$ satisfies the condition of Corollary 3. However, the argument above is not delicate enough to yield a uniqueness result. Indeed, if we consider the functionals

$$J_n(f) = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^{2n}\, d\theta,$$

for $n = 1, 2, \ldots$, then essentially the same argument shows that an extremal function f with $f' \in H^1$ must satisfy

$$|f|^{2n} - J_n(f) f \overline{z f'} \in \mathscr{H}_0^1 ,$$

and again this condition is fulfilled by the identity function. Let

$$\mu_n = \sup\{ J_n(f) \mid \|f\|_{\mathfrak{D}} \leq 1 \}.$$

Because the J_n are weakly continuous on the unit ball B of $\mathfrak{D}$, extremal functions exist for each n. For $n = 1$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$, we have

$$J_1(f) = \sum_{n=1}^{\infty} |a_n|^2 \leq \sum_{n=1}^{\infty} n|a_n|^2 = \|f\|_{\mathfrak{D}}^2 ,$$

with equality if and only if $f(z) = a_1 z$. Hence $\mu_1 = 1$ and the identity function is extremal. For $n = 2$ a more delicate argument again shows that $\mu_2 = 1$ and the identity function is extremal. However, since there are unbounded functions in $\mathfrak{D}$, $\mu_n \to \infty$ as $n \to \infty$. But $J_n(f) = 1$ for $f(z) = z$, so the identity function cannot be extremal for large n.

4. A local maximum. In this section we show that, in a certain sense, the identity function is a local maximum for each of the functionals Λ_α, $0 < \alpha \leq 1$. For simplicity of exposition we restrict ourselves to functions with nonnegative coefficients. The idea is to analyze Λ_α along curves

$$f_t(z) = z \cos t + h(z) \sin t , \qquad 0 \leq t \leq \frac{\pi}{2} ,$$

where $h \in \mathfrak{D}$, $\|h\|_{\mathfrak{D}} = 1$, and $h(z) = \sum_{n=2}^{\infty} b_n z^n$ has nonnegative coefficients. Since z is orthogonal to $h(z)$, $\|f_t\|_{\mathfrak{D}} = 1$, and it is clear that any function in $\mathfrak{D}$ with norm one and nonnegative coefficients has such a representation.

For reasons which will become clear shortly, it will be convenient to treat the function $h(z) = \frac{1}{\sqrt{2}} z^2$ as a special case. In this case we have

$$f_t(z) = z \cos t + \frac{1}{\sqrt{2}} z^2 \sin t , \qquad 0 \leq t \leq \frac{\pi}{2} .$$

We let $\phi_\alpha(t) = e^{-\alpha} \Lambda_\alpha(f_t)$.

Proposition 2. *The function* $\phi_\alpha(t)$ *is strictly decreasing for* $0 \le t \le \frac{\pi}{2}$, $0 < \alpha \le 1$.

Proof. Clearly,

$$|f_t(e^{i\theta})|^2 = 1 - \frac{\sin^2 t}{2} + \sqrt{2}\sin t\cos t\cos\theta.$$

Hence

$$\phi_\alpha(t) = e^{-\alpha\frac{\sin^2 t}{2}} \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{2}\alpha\sin t\cos t\cos\theta}\,d\theta.$$

But, for $n = 0, 1, 2, \dots$,

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n+1}\theta\,d\theta = 0,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \frac{(2n)!}{2^{2n}(n!)^2}.$$

Hence

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{2}\alpha\sin t\cos t\cos\theta}\,d\theta \\
&\quad = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(\sqrt{2}\sin t\cos t)^n \frac{1}{2\pi}\int_0^{2\pi}\cos^n\theta\,d\theta \\
&\quad = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} 2^n(\sin^2 t\cos^2 t)^n \frac{(2n)!}{2^{2n}(n!)^2} \\
&\quad = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(n!)^2}\left(\frac{\sin^2 t\cos^2 t}{2}\right)^n.
\end{aligned}$$

Thus,

$$\phi_\alpha(t) = e^{-\alpha\frac{\sin^2 t}{2}} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(n!)^2}\left(\frac{\sin^2 t\cos^2 t}{2}\right)^n.$$

Differentiating $\phi_\alpha(t)$ with respect to t gives

$$\begin{aligned}
\phi'_\alpha(t) &= -\alpha \sin t \cos t e^{-\alpha \frac{\sin^2 t}{2}} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(n!)^2} \left(\frac{\sin^2 t \cos^2 t}{2} \right)^n \\
&\quad + e^{-\alpha \frac{\sin^2 t}{2}} \sum_{n=0}^{\infty} \frac{n\alpha^{2n}}{(n!)^2} \left(\frac{\sin^2 t \cos^2 t}{2} \right)^n \sin t \cos t (\cos^2 t - \sin^2 t) \\
&= \sin t \cos t e^{-\alpha \frac{\sin^2 t}{2}} \left[-\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(n!)^2} \left(\frac{\sin^2 t \cos^2 t}{2} \right)^n \right. \\
&\qquad \left. + (\cos^2 t - \sin^2 t) \sum_{n=1}^{\infty} \frac{n\alpha^{2n}}{(n!)^2} \left(\frac{\sin^2 t \cos^2 t}{2} \right)^{n-1} \right] \\
&= \sin t \cos t e^{-\alpha \frac{\sin^2 t}{2}} \left[-\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(n!)^2} \left(\frac{\sin^2 t \cos^2 t}{2} \right)^n \right. \\
&\qquad \left. + (\cos^2 t - \sin^2 t) \sum_{n=0}^{\infty} \frac{\alpha^{2n+2}}{(n!)^2 (n+1)} \left(\frac{\sin^2 t \cos^2 t}{2} \right)^n \right].
\end{aligned}$$

But $\cos^2 t - \sin^2 t = \cos 2t$, so

$$\phi'_\alpha(t) = -\sin t \cos t e^{-\alpha \frac{\sin^2 t}{2}} \sum_{n=0}^{\infty} \left(1 - \frac{\alpha \cos 2t}{n+1} \right) \frac{\alpha^{2n+1}}{(n!)^2} \left(\frac{\sin^2 t \cos^2 t}{2} \right)^n .$$

But $1 - \frac{\alpha \cos 2t}{n+1} > 0$ unless $n = 0$, $\alpha = 1$, and $t = 0$. That completes the proof.

Now if $h(z) = \sum_{n=2}^{\infty} b_n z^n$, we have

$$\|h\|_2^2 = \sum_{n=2}^{\infty} b_n^2 \leq \frac{1}{2} \sum_{n=2}^{\infty} n b_n^2 = \frac{1}{2},$$

with equality if and only if $h(z) = \frac{1}{\sqrt{2}} z^2$. Having disposed of this case in Proposition 2, we may assume that $\|h\|_2^2 < \frac{1}{2}$. Once again we let $\phi_\alpha(t) = e^{-\alpha} \Lambda_\alpha(f_t)$.

Proposition 3. *For each h there is a T such that $\phi_\alpha(t)$ is decreasing for $0 < t < T$.*

Proof. If we write $k(z) = zh(\overline{z}) + \overline{z}h(z)$, we obtain

$$|f_t(z)|^2 = 1 + \left(|h(z)|^2 - 1 \right) \sin^2 t + k(z) \frac{\sin 2t}{2}.$$

On the other hand, applying Taylor's Theorem to the function

$$q_\alpha(t) = e^{\alpha\left[(\lambda-1)\sin^2 t+\mu\frac{\sin 2t}{2}\right]}$$

gives

$$q_\alpha(t) = 1 + \alpha\mu t + \alpha\left(\frac{\alpha\mu^2}{2} + \lambda - 1\right)t^2 + g_\alpha(\lambda, \mu, \tau)q_\alpha(\tau)t^3,$$

where $g_\alpha(\lambda, \mu, t)$ is a polynomial in α, λ, and μ, with coefficients depending only on $\sin 2t$ and $\cos 2t$, and τ is a number between 0 and t. Applying this to $e^{\alpha|f_t|^2}$ gives

$$e^{\alpha|f_t(z)|^2} = e^\alpha\left(1 + \alpha k(z)t + \alpha\left(\frac{k^2(z)}{2} + |h(z)|^2 - 1\right)t^2 \right.$$
$$\left. + g_\alpha(|h(z)|^2, k(z), \tau)e^{\alpha|f_\tau(z)|^2}t^3\right).$$

Since $|k(z)| \le 2|h(z)|$, and $|f_\tau|^2 \le 1 + |h|^2$, the coefficient of t^3 is dominated by $p(|h|^2)e^{\alpha|h|^2}$, where p is some polynomial. As remarked in the introduction, there is a constant C_h such that

$$\left|\frac{1}{2\pi}\int_0^{2\pi} p(|h(e^{i\theta})|^2)e^{\alpha|h(e^{i\theta})|^2}\,d\theta\right| \le C_h.$$

Next, for $z = e^{i\theta}$, it is easy to see that

$$k(z) = 2\sum_{n=1}^{\infty} b_{n+1}\cos n\theta,$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi} k(e^{i\theta})\,d\theta = 0,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} k^2(e^{i\theta})\,d\theta = 2\sum_{n=2}^{\infty} b_n^2 = 2\|h\|_2^2.$$

Thus, integrating $e^{\alpha|f_t(e^{i\theta})|^2}$ gives

$$\phi_\alpha(t) = 1 + \alpha\left\{\|h\|_2^2(\alpha+1) - 1\right\}t^2 + c(t)t^3,$$

where $|c(t)| \leq C_h$. Since $\|h\|_2^2 < \frac{1}{2}$ by assumption, it follows that $\phi_\alpha(t)$ is decreasing for t close to 0.

5. Concluding remarks. Beurling [1] proved that his estimate is sharp in the sense that for the functions

$$B_a(z) = \left(\log \frac{1}{1-az}\right) \Big/ \sqrt{\log \frac{1}{1-a^2}}\,, \qquad 0 < a < 1,$$

there is a constant c independent of a such that

$$|E_t| \geq ce^{-t^2} \qquad \text{if } t = \sqrt{\log \frac{1}{1-a^2}}\,,$$

and Chang and Marshall [5] proceed by comparing $f \in \mathfrak{D}$, $\|f\|_{\mathfrak{D}} = 1$, with an appropriate B_a. We note that

$$B_a(z) = \sum_{n=1}^{\infty} \frac{a^n z^n}{n\sqrt{N_a}}\,,$$

where

$$N_a = \log \frac{1}{1-a^2}.$$

Since, as $a \to 0$, $\frac{1}{a}\sqrt{N_a} \to 1$, it follows that $B_a(z) \to z$ weakly as $a \to 0$. On the other hand, it can be shown that $B_a(z) \to 0$ weakly as $a \to 1$. It is of interest to investigate the behavior of $\Lambda_1(B_a)$. Numerical experiments suggest that $\Lambda_1(B_a)$ is decreasing and concave for $0 < a < 1$, and that $\lim_{a\to 0} \Lambda_1(B_a) = e$ and that $\lim_{a\to 1} \Lambda_1(B_a) = 0$. It is also possible to prove that $J_1(B_a)$ is decreasing and concave for $0 < a < 1$. However, numerical experiments suggest that this is not the case for $J_n(B_a)$ for larger n, and recently the second author has succeeded in proving that $\limsup_{a\to 1} \Lambda_1(B_a) > 1$.

Finally, let

$$S_{n,\alpha}(f) = \sum_{k=0}^{n} \frac{\alpha^k J_k(f)}{k!}$$

denote the partial sums in the series expansion of $\Lambda_\alpha(f)$. We remark that, for fixed α, if the identity function $f(z) = z$ is extremal for $S_{n,\alpha}$ for large n, then it would also be extremal for Λ_α. Similarly, if $f(z) = z$ is extremal for Λ_α for α close to 1, it would also be extremal for Λ_1. We conjecture that the identity function is extremal in all of these cases.

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