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ON AMBIENTAL BORDISM

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Let M^m be a closed and oriented submanifold of a closed or oriented manifold N^n, such that $[M, i] = 0 \in \Omega_m(N)$, where $i: M \to N$ is the inclusion and $\Omega_m(N)$ is the mth oriented bordism group of N. If $n = m + 2$ or $m \leq 3$ or $m \leq 4$ and $n \neq 7$ then M bounds in N.

Introduction. Let us consider M^m a closed submanifold of N^n. In this paper, we study the possibility that there exists submanifold $W^{m+1} \subset N^n$ such that $\partial W = M$. If $M = S^m$ and $N = S^{m+2}$, such that a submanifold W is called a Seifert surface knot S^m. In **[5]**, Sato showed that every connected closed and oriented submanifold M^m of S^{m+2} is a boundary of an oriented surface of S^{m+2}.

In **[4]**, Hirsch studies the following problem: If a compact connected and oriented manifold M^m bounds, does there exist embedding from M^m into $\mathbb{R}^n$ which is a boundary in $\mathbb{R}^n$?

The answer is yes, if $n \geq 2m$.

The difference between the two problems is that, in our case, the embedding from M into N is fixed.

There is an obvious necessary condition for the existence of W, when M and N are oriented manifolds.

Let $\Omega_m(N)$ be the mth oriented bordism group of N **[2]**. If $i: M \to N$ is the inclusion map, we can define an element $[M, i]$ in $\Omega_m(N)$ and see that $[M, i] = 0$ if M bounds in N.

Generally, the converse in not true, but sometimes the vanishing of $[M, i]$ guarantees the existence of W, for example if the codimension $n - m$ is large.

We prove the following theorem.

THEOREM 5.2. *Let us suppose that $M^m \subset N^n$, $n > m + 1$, is such that $[M, i] = 0$ in $\Omega_m(N)$. Then M bounds in N if one of the following conditions occurs:*

(a) $n = m + 2$,
(b) $m \leq 3$,
(c) $m \leq 4$ *and* $n \neq 7$.

In his Doctoral thesis [1] the author proved that, when $n = 2m + 1$, and M and N are closed and oriented, a submanifold $M \subset N$ bounds in N if, and only if, $[M, i] = 0 \in \Omega_m(N)$.

1. A more general problem of ambiental bordism. Let

$$G \subset O(n-m-1), \qquad n > m+1,$$

be a closed transformation group and let $\gamma_G \to BG$ be the classifying fiber bundle of $(n-m-1)$-vector bundles which have a G-structure.

Let us consider MG the Thom space of γ_G. We have:

$$\pi_i(MG) = \begin{cases} 0, & i < n-m-1, \\ \mathbb{Z}, & i = n-m-1 \text{ and } G \subset \mathrm{SO}(n-m-1), \\ \mathbb{Z}_2, & i = n-m-1 \text{ and } G \not\subset \mathrm{SO}(n-m-1). \end{cases}$$

Let us consider now N^n to be a closed connected manifold which we assume to be oriented if $G \subset \mathrm{SO}(n-m-1)$. (If $G \not\subset \mathrm{SO}(n-m-1)$ we drop the orientability hypothesis.)

Let $M^m \subset N^n$ be a closed submanifold and let us suppose that the normal fiber bundle ν_M of M in N has a cross section s, nowhere zero, such that $\nu_M = \{s\} \oplus \xi$, where $\{s\}$ is a subbundle generated by s and ξ is a $(n-m-1)$-vector bundle endowed with a G-structure.

We shall say that a submanifold $W \subset N$ satisfies condition $(*)$ if it has the properties:

(i) $\partial W = M$ and s is the inward-pointing vector field on ∂W.

(ii) the normal fiber bundle ν_W has a G-structure which agrees with the given G-structure of ξ over M. (Observe that $\xi = \nu_W | M$.)

2. Primary obstruction to the existence of W. Let V be a closed tabular neighborhood of M in N, $A = \partial W$ and $X = N - \mathring{V}$. We can think s a function $s: M \to A$. Then $s(M)$ is a submanifold of A, whose normal fiber bundle is isomorphic to ξ. By the Thom construction there exists a function $f: A \to MG$ such that, if ∞ is the point at infinity to MG, then f is differentiable on $A - f^{-1}(\infty)$, f is transversal to BG and $f^{-1}(BG) = (M)$ [6].

We shall take $\pi_{m-n-1}(MG)$ as the cohomology coefficient group. Let $e \in H^{n-m-1}(MG)$ be the fundamental class of the space MG. We know that $f^*(e) = \alpha$, where α is the dual class of $s_*(\mu_M)$ and μ_M is the fundamental class of M.

If $f: A \to MG$ extends to a map $\overline{f}: X \to MG$, then we can suppose, up to homotopy, that $\overline{f}$ is differentiable in $X - \overline{f}^{-1}(\infty)$ and that $\overline{f}$ is transversal to BG. Taking $W_1 = \overline{f}^{-1}(BG)$ we obtain a submanifold of X whose boundary is $s(M)$.

Let us observe that this submanifold can be extended to a submanifold W which satisfies condition $(*)$.

We conclude then that there exists W, satisfying $(*)$, if and only if f extends to X.

The class $\delta f^*(e)$ is the obstruction to the extension of f to the $(n-m)$-skeleton of X, where $\delta\colon H^{n-m-1}(A) \to H^{n-m}(X, A)$ is the coboundary operator.

Consider the commutative diagram:

$$\begin{array}{ccc} H^{n-m-1}(A) & \xrightarrow{\delta} & H^{n-m}(X, A) \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} \\ H_m(A) & \xrightarrow{s_*} & H_m(X) \cong H_m(N-M). \end{array}$$

We conclude that the primary obstruction to the extension of f, up to duality, is the element $s_*(\mu_M) \in H_m(N-M)$ (regarding s as function from M into $N-M$).

Hence, we have:

Proposition 2.1. *f extended to the $(n-m)$-skeleton of X if, and only if, $s_*(\mu_M) = 0$ in $H_m(N-M)$.*

Assuming that $s_*(\mu_M) = 0$, let us consider two cases:

1. $G = O(n-m-1)$.

Here, f extends up to the $(n-m+1)$-skeleton of X, because $\pi_{n-m}(MG) = 0$ and, if $n-m=2$, then f extends to all of X since $MO(1)$ is a $K(\mathbb{Z}_2, 1)$ space.

2. $G = \mathrm{SO}(n-m-1)$.

Since $\pi_{n-m+i}(MG) = 0$, $i = 0, 1, 2$, f extends up to the $(n-m+3)$-skeleton of X. Hence, if $\dim M \leq 3$, f extends.

On the other hand, if $n-m = 2$ or 3 then MG is a $K(\mathbb{Z}, 1)$ or $K(\mathbb{Z}, 2)$, respectively. In any case, f extends globally.

3. Oriented ambiental bordism. From now on, all manifolds and submanifolds will be considered to be oriented.

Theorem 3.1. *Let us suppose that:*

(a) $H_j(X) = 0$, $0 < j < m-3$.

(b) *The canonical homomorphism* $\pi_{n-1}(\mathrm{MSO}(n-m-1)) \xrightarrow{\varphi} \Omega_m$ *is injective.*

There exists W satisfying $(*)$ if, and only if, $s_*(\mu_M) = 0 \in H_m(X)$ and M is a boundary.

Proof. Let us use the notation $\pi_i = \pi_i(\mathrm{MSO}(n-m-1))$. If $s_*(\mu_M) = 0$, then f extends to the $(n-m)$-skeleton of X.

From hypothesis (a) and Lefschetz duality, we conclude that

$$H^j(X, A, \pi_{j-1}) = 0, \qquad n-m < j < n.$$

Let D be an open disk of $X - A$. Since X is orientable, $H^j(X - D, A, \pi_{j-1}) \cong H^j(X, A, \pi_{j-1}) = 0$, $n - m < j < n$. Hence, there exists an extension $\overline{f}\colon X - D \to Y$ of $f\colon A \to Y$, where $Y = \mathrm{MSO}(n-m-1)$.

Let us consider $S = \partial D$ and $h = \overline{f}|\partial D\colon S \to Y$. We may suppose that h is transversal to $\mathrm{BSO}(n-m-1)$ and let

$$M^m = h^{-1}(\mathrm{BSO}(n-m-1)).$$

Consider $\overline{W} = \overline{f}^{-1}(\mathrm{BSO}(n-m-1))$, a bordism between M_1 and $s(M)$. Since $s(M)$ is a boundary, M_1 also is.

We have also that $\psi([h]) = [M_1] = 0$ and since ψ is a monomorphism, h is homotopic to a constant map and so h extends over D.

The converse is straightforward. □

4. On the existence of normal vector fields homologous to zero in $N - M$. In the next section, we show that in certain situations it is possible to obtain a cross-section $s\colon M \to S(\nu_M)$ such that $s_*(\mu_M) = 0 \in H_m(N - M)$, where $S(\nu_M) \to M$ is the normal sphere bundle of M in N.

PROPOSITION 4.1. *The Euler class of the normal bundle of M^m in N^n is zero if and only if $i_*(\mu_M) \subset \operatorname{im} j_*$, where μ_M is the fundamental class of M and $i\colon M \to N$, $j\colon N - M \to N$ are inclusion maps.*

Proof. Let us consider $e \in H^{n-m}(M, \mathbb{Z})$, the Euler class of the normal bundle ν_M, and let $D_A\colon H^{n-m}(M : \mathbb{Z}) \to H_m(N, N - M; \mathbb{Z})$ be the Alexander duality. We have that $D_A(e) = \alpha_*(\mu_M)$ where α_* is induced by the inclusion map $\alpha\colon (N, N - M)$.

Using the exact sequence of pair $(N, N-M)$ it follows that $\alpha_*(\mu_M) = 0$ if, and only if, $i_*(\mu_M) \subset \operatorname{im} j_*$. □

COROLLARY 4.2. *If $M^m \subset N^n$ is homologous to zero, $n - m = 2$ or $n \geq 2m$, then M has a normal vector field that is nowhere zero.*

Proof. By Proposition 4.1 the Euler class of ν_M is zero. Then there is a nowhere zero normal vector field on the $(n-m)$-skeleton of M, which can be extended to all M, because $n-m \geq m$ or $\pi_i(R^2-0)=0$, $i>1$ in the case $n-m=2$. □

Let $\pi\colon E \to M^m$ be a differentiable $\mathrm{SO}(n+1)$-bundle with fiber S^n and base M^m (and oriented manifold).

If $s\colon M \to E$ is a cross-section, let θ_s be the Poincaré dual to $\bar{s}_*(\mu_M)$, where $\bar{s}=-s$ is the opposite cross-section to s.

Having fixed a cross-section $s_0\colon M \to E$, the following diagrams are commutative:

$$\begin{tikzcd}
{[M, E]} \arrow[d, "\xi"] \arrow[dr, "\varphi"] & & \\
H^n(M) \arrow[r, "\pi^*"] \arrow[d, "D"] & H^n(E) \arrow[d, "D"] & \\
H_{m-n}(M) \arrow[r, "\Delta"] & H_m(E) \arrow[r, "\pi_*"] & H_m(M)
\end{tikzcd}$$

where $[M, E]$ is the set of homotopy classes of cross-sections, $\xi([s]) = \bar{s}^*(\theta_{\bar{s}_0})$; $\varphi([s]) = \theta_{\bar{s}_0} - \theta_{\bar{s}}$, is Poincaré duality and last line is a portion of the generalized Gysin sequence.

We define $\psi\colon [M, E] \to H_m(E)$ by $\psi([s]) = s_{s_*}(\mu_M) - s_*(\mu_M)$ and observe that $\psi = D \circ \psi$.

If $m \leq n+1$ or $n=1$, then the function ξ is onto and so the image of ψ is the kernel of π_*.

This fact will be applied in the proof of Proposition 4.3 below, where the fiber bundle to be considered is $S(\nu_M) \to M$.

Proposition 4.3. *Let $M^m \subset N^n$, $n=m+2$ or $n \geq 2m$, be an oriented submanifold homologous to zero in an oriented manifold N. Then there exists a cross-section $r\colon M \to S(\nu_M)$ such that its image is homologous to zero in $H_m(N-m)$.*

Proof. Let $s_0\colon M \to S(\nu_M)$ be a cross-section that is nowhere zero (Corollary 4.2) and let us consider the commutative diagrams:

$$\begin{tikzcd}
 & H_m(S(\nu_M)) \arrow[r, "\pi_*"] \arrow[d, "l_*"] & H_m(M) \arrow[d, "i_*"] \\
H_m(M) \arrow[ur, "s_{0_*}"] \arrow[r, "s_*"'] & H_m(N-M) \arrow[r, "j_*"] & H_m(N)
\end{tikzcd}$$

where $s_* = l_*(s_{0_*})$ and l_* is induced by the inclusion $S(\nu_M) \to (N-M)$.

We have $j_* s_*(\mu_M) = i_* \pi_* s_{0_*}(\mu_M) = 0$ implying that $s_*(\mu_M)$ belongs to the kernel of j_* which is the image of $\partial\colon H_{m+1}(N, N-M) \to H_m(N-M)$.

Let us consider the following commutative diagram:

$$\begin{array}{ccc} H_{m+1}(D(\nu_M), S(\nu_M)) & \xrightarrow{\partial} & H_m(S(\nu_M)) \\ \Big\downarrow{\scriptstyle exc} & & \Big\downarrow{\scriptstyle j_*} \\ H_{m=1}(N, N-M) & \xrightarrow{\partial} & H_m(N-M). \end{array}$$

It follows that there exists an element $\mu \in H_m(S(\nu_M))$ such that $\mu \in \operatorname{Ker} \pi_*$ and $j_* = s_*(\mu_M)$.

Since $\operatorname{im} \psi = \ker \pi_*$, there exists a cross-section $r\colon M \to S(\nu_M)$ such that $\psi([r]) = \mu$.

But $\psi([r]) = s_{0_*}(\mu_M) \to r_*(\mu_M)$ so $j_* r_*(\mu_M) = 0$ in $H_m(N-M)$. Hence, the image of $r\colon M \to S(\nu_M)$ is homologous to zero in $N-M$.

5. A theorem on ambiental bordism. Let us consider $\Omega_j(N)$ to be the jth bordism group of N.

If $H_j(N) = 0$, $0 < j < m-3$, it is possible using the bordism spectual sequence [2] to show that the function $\Omega_m(N) \to H_m(N) \oplus \Omega_m$, which associates to each pair $[M, f]$ the element $\mu([M, f]) + [M]$, is an isomorphism, where μ is the canonical homomorphism.

In the proof of Theorem 5.2, we are going to use the following lemma, which has been proved in [1] (the proof, if $q > m$, is due to Thom [6]).

LEMMA 5.1. *The homomorphism* $\varphi\colon \pi_{q+m}(\mathrm{MSO}(q)) \to \Omega_m$, $q \geq m$, *is an isomorphism.*

THEOREM 5.2. *Let us suppose* $M^m \subset N^n$, $n > m+1$, *is such that* $[M, i] = 0$ *in* $\Omega_m(N)$. *Then* M *bounds in* N *if one of the following conditions occurs:*

(a) $n = m+2$,
(b) $m \leq 3$,
(c) $m \leq 4$ *and* $n \neq 7$.

Proof. Any one of the conditions (a), (b) and (c), based on previous results, imply that normal bundle ν_M has a cross-section nowhere zero such that, considering s as a function from M into $N-M$, $s_*(\mu_M) = 0 \in H_m(N-M)$.

If (a) or (b) occurs, the theorem follows from case 2, already discussed in §2

If $n = 4$ and $n \geq 8$, we apply Theorem 3.1.

REMARK 1. If $n = m + 2$ or $m \leq 3$, then $[M, i] = 0 \in \Omega_m(N)$ if, and only if, M is homologous to zero in N.

REMARK 2. When $m = 4$ and $n \neq 7$, although we shall prove that $[M, i] = 0$ implies the existence of a normal section nowhere zero (Th. 5.3) we are not able to prove that there exists a normal vector field homologous to zero in $N - M$, which in this case would be sufficient to prove the conclusion of Theorem 5.2.

THEOREM 5.3. *Let us suppose* $M^4 \subset N^7$. *If* $[M, i] = 0$ *in* $\Omega_4(N)$ *then* ν_M *has a cross-section which is nowhere zero.*

Proof. There exists $W \subset N \times I$ such that $\partial W = M \times 0 \subset N \times I$ **[1]**.

Let ν_W and ν_M be the normal fiber bundles of W in $N \times I$ and of M in N, respectively. We can also suppose that $\nu_W | M \times 0 = \nu_M$.

Let us consider $\overline{W} \subset N \times \mathbb{R}$ to be the double of W and let $i\colon \overline{W} \to N \times \mathbb{R}$ and $j\colon N \times \mathbb{R} \to \overline{W} \to N \times \mathbb{R}$ be inclusion maps.

Since $i_*(\mu_{\overline{W}}) \subset \operatorname{im} j_*$, then $\overline{W}$ has a normal vector field which is nowhere zero in $N \times \mathbb{R}$ up to the 3-skeleton of $\overline{W}$.

Hence, there exists a 2-dimensional oriented vector bundle ξ over M such that $\nu_M | M^{(3)} = \xi \otimes \mathscr{E}^1$.

Let us consider e to be the Euler class of ξ in $H^2(M^{(3)})$ and let $\overline{e} \in H^2(M)$ be such that $io^*(\overline{e}) = e$, where $i\colon M^{(3)} \to M$ is the inclusion map.

Let $\overline{\xi}$ be a 2-dimensional vector bundle over M such that its Euler class is $\overline{e}$. Let us observe that $\overline{\xi} | M^{(3)} = \xi$.

Let f, $g\colon M \to \mathrm{BSO}(3)$ be classifying maps $\overline{\xi} \oplus \mathscr{E}^1$ and ν_M, respectively.

Since the Euler classes of $\overline{\xi} \oplus \mathscr{E}^1$ and of ν_M are equal, then their second Stiefel-Whitney classes are equal.

Let $\tilde{p}_1$ be the Pontryagin class of the classifying fiber bundle $\tilde{\gamma} \to \mathrm{BSO}(3)$ and let $\tilde{e}$ be the Euler class of $\tilde{\gamma}$. Since $f^*(\tilde{p}_1) = g^*(\tilde{p}_1)$. Hence, the vector bundles $\xi \oplus \mathscr{E}^1$ and ν_M are equivalent **[3]**. $\square$

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