PACIFIC JOURNAL OF MATHEMATICS
Vol. 164, No. 1, 1994

LATTICES OF LIPSCHITZ FUNCTIONS

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Let M be a metric space. We observe that $\mathrm{Lip}(M)$ has a striking lattice structure: its closed unit ball is lattice-complete and completely distributive. This motivates further study into the lattice structure of $\mathrm{Lip}(M)$ and its relation to M. We find that there is a nice duality between M and $\mathrm{Lip}(M)$ (as a lattice). We also give an abstract classification of all normed vector lattices which are isomorphic to $\mathrm{Lip}(M)$ for some M.

The set $\mathrm{Lip}(M)$ of bounded real-valued Lipschitz functions on a metric space M has been studied extensively (see [2] for some references) as either a Banach space or a Banach algebra. However, its natural lattice structure has been almost completely ignored, probably because it is not a Banach lattice: the "Riesz norm" law

$$|x| \le |y| \Rightarrow \|x\| \le \|y\|,$$

which connects lattice structure with norm, is not satisfied by either of the two norms customarily given to $\mathrm{Lip}(M)$. (Here $|x| = x \vee (-x)$.)

Nonetheless, the lattice structure of $\mathrm{Lip}(M)$ is intimately related to its most natural norm. Indeed, for any norm-bounded set of elements $\{x_\alpha\} \subset \mathrm{Lip}(M)$, the join $\bigvee x_\alpha$ exists and satisfies

$$(*) \qquad \left\|\bigvee x_\alpha\right\| \le \sup\{\|x_\alpha\|\}.$$

Since $-\bigvee x_\alpha = \bigwedge(-x_\alpha)$ (whenever either side exists), this implies a similar statement for meets and is equivalent to saying that the closed unit ball of $\mathrm{Lip}(M)$ is lattice-complete. What's more, the unit ball is completely distributive, which makes it very special from the lattice-theoretic point of view. We therefore feel that a study of $\mathrm{Lip}(M)$ which emphasizes its lattice structure is well warranted.

This paper begins such a study. Having identified the special lattice properties of $\mathrm{Lip}(M)$, we also find it interesting to examine the class of all normed vector lattices which share these properties. We will call these objects *Lip-spaces*. (Note: by "normed vector lattice" we simply mean a vector lattice which is equipped with a vector space norm. This is at variance with a usage of this term which requires that the norm be a Riesz norm.)

The outline of the paper is as follows. In the first two sections we make several definitions and give some simple results (with proofs mainly left to the reader) on Lip(M) and Lip-spaces. In the third section we state our main theorem on the relation between metric spaces and Lip-spaces, and prove the easy parts of it. The fourth section contains two theorems on the normed vector lattice structure of Lip(M), and in the fifth section we prove that every Lip-space is isomorphic to Lip(M) for some M. The hard parts of our main theorem are proven in §§IV and V.

We assume some familiarity with vector lattices; standard facts will be referred to [3]. Also see [1] for basic material on complete distributivity. This material is based upon work supported under a National Science Foundation graduate fellowship.

I. Throughout the paper, $\mathscr{B}_a(X)$ will denote the closed ball of radius a about the origin in a normed vector space X.

Let M and N be metric spaces. A *Lipschitz function* $f: M \to N$ is a map for which there exists a constant $k \in \mathbf{R}^+$ such that $\rho(f(p), f(q)) \leq k \cdot \rho(p, q)$ for all p, $q \in M$. We define $L(f)$ to equal the least such k.

Lip(M) is defined as the set of all bounded real-valued Lipschitz functions $x: M \to \mathbf{R}$ and is given the norm

$$\|x\| = \max(\|x\|_\infty, L(x)).$$

Lip(M) is a Banach space under $\|\cdot\|$, and it has a vector lattice structure with meet and join defined pointwise. Under this structure any norm-bounded set has a join, still taken pointwise, which satisfies (*) of the introduction. This implies that the closed unit ball $\mathscr{B}_1(\text{Lip}(M))$ is lattice-complete, in fact a complete sublattice (i.e. a subset closed under $\bigvee$ and $\bigwedge$) of the set of all functions $M \to [-1, 1] \subset \mathbf{R}$. As $[-1, 1]$ is completely distributive and this property is preserved by products and complete sublattices, it follows that $\mathscr{B}_1(\text{Lip}(M))$ is completely distributive.

If M is any metric space, let M' be the space M remetrized by

$$\rho^{M'}(p, q) = \min(2, \rho^M(p, q)).$$

Then Lip(M) and Lip(M') are identical (same elements, same operations, same norm). Also, if $\overline{M}$ is the completion of M then Lip(M) and Lip($\overline{M}$) are isometrically isomorphic, including lattice-isomorphic. Therefore we restrict attention to the class $\mathscr{M}^{(2)}$ of complete metric spaces with diameter ≤ 2.

For M, $N \in \mathscr{M}^{(2)}$, we say that a map $f: M \to N$ is *bi-Lipschitz* if there are positive constants k_1, k_2 such that

$$k_1 \cdot \rho(p, q) \le \rho(f(p), f(q)) \le k_2 \cdot \rho(p, q)$$

for all p, $q \in M$; thus "bi-Lipschitz and onto" is equivalent to "Lipschitz with Lipschitz inverse."

II. Let a *Lip-space* be a vector lattice X equipped with a vector space norm such that the closed unit ball $\mathscr{B}_1(X)$ is lattice-complete and completely distributive. This generalizes the properties of $\mathrm{Lip}(M)$; it should become clear in §III that this level of abstraction is appropriate.

In a Lip-space every norm-bounded set $\{x_\alpha\}$ has a join and a meet, both of norm $\le \sup\{\|x_\alpha\|\}$.

PROPOSITION 1. *Every Lip-space is norm-complete.*

Proof. Let X be a Lip-space. First let $(x_i) \subset X$ be a sequence such that $\sum \|x_i\| < \infty$ and each $x_i \ge 0$; we will show that $\sum x_i$ exists. Let $s_n = x_1 + \cdots + x_n$; since $\|s_n\| \le \sum \|x_i\|$ for each n, $x = \bigvee_1^\infty s_n$ exists. Because the sequence (s_n) is increasing, $x = \bigvee_k^\infty s_n$ for each $k \in \mathbf{N}$. Also note that the law $(\bigvee x_\alpha) - y = \bigvee (x_\alpha - y)$ is generally valid in any vector lattice whenever one side is defined ([3], p. 56), so

$$\|x - s_k\| = \left\| \bigvee_{n=k}^{\infty} (s_n - s_k) \right\| \le \sup_{n \ge k}\{\|s_n - s_k\|\} \le \sum_{i=k+1}^{\infty} \|x_i\|,$$

and as the last sum goes to zero as $k \to \infty$, we have $\sum x_i = x$.

To show that the norm is complete, we must show that $\sum x_i$ exists for an arbitrary sequence $(x_i) \subset X$ such that $\sum \|x_i\| \le \infty$. In this case let $x_i^+ = x_i \vee 0$ and $x_i^- = (-x_i) \vee 0$ for all i; then $x_i = x_i^+ - x_i^-$ ([3, p. 57]). Now $x_i^\pm \ge 0$ and $\|x_i^\pm\| \le \|x_i\|$, so by the above both $y = \sum x_i^+$ and $z = \sum x_i^-$ exist. Setting $x = y - z$, we then have

$$\begin{aligned} \left\| x - \sum_{i=1}^{k} x_i \right\| &= \left\| \left(y - \sum_{i=1}^{k} x_i^+ \right) - \left(z - \sum_{i=1}^{k} x_i^- \right) \right\| \\ &\le \left\| \sum_{i=k+1}^{\infty} x_i^+ \right\| + \left\| \sum_{i=k+1}^{\infty} x_i^- \right\|, \end{aligned}$$

where both of the last two terms go to zero as $k \to \infty$. Thus $\sum x_i$ exists and equals x. □

Every Lip-space X is Archimedean since if $x \geq 0$ then

$$\left\| \bigwedge_{n=1}^{\infty} x/n \right\| = \left\| \bigwedge_{n=k}^{\infty} x/n \right\| \leq \|x\|/k$$

for any $k \in \mathbf{N}$, hence $\bigwedge_1^\infty x/n = 0$.

For each $a \in \mathbf{R}^+$ there is a *constant*

$$c_a \bigvee \{x : \|x\| \leq a\}$$

in X. We also define $c_0 = 0$ and $c_{-a} = -c_a$. Then for all $a \in \mathbf{R}$, $c_a = a \cdot c_1$; thus,

$$c_a = c_a + \bigwedge_{n=1}^{\infty} c_{1/n} = \bigwedge_{n=1}^{\infty} (c_a + c_{1/n}) = \bigwedge_{n=1}^{\infty} c_{a+1/n} = \bigwedge_{b>a} c_b .$$

We also have $\|c_a\| \leq |a|$, and if X has more than one element then $c_a > 0$ for any $a > 0$, and so $(1+\varepsilon)c_a \not\leq c_a$ for any $\varepsilon > 0$, hence $\|(1+\varepsilon)c_a\| > a$, hence $\|c_a\| \geq a$. Thus $\|c_a\| = |a|$ for any a unless the Lip-space is trivial.

Note that the closed unit ball of X is contained in but generally not equal to the interval $[c_{-1}, c_1]$. (E.g. consider the case $X = \mathrm{Lip}([0,1])$, when c_1 is the function which is constantly 1.) This is related to the fact that the norm is not a Riesz norm. However, any Lip-space can be made into a Riesz-normed vector lattice by introducing the new norm

$$\begin{aligned} \|x\|_\infty &= \inf\{a \in \mathbf{R}^+ : c_{-a} \leq x \leq c_a\} \\ &= \inf\{a \in \mathbf{R}^+ : |x| \leq c_a\}. \end{aligned}$$

We have $\|x\|_\infty \leq \|x\|$ and $\|c_a\|_\infty = |a|$ if X is nontrivial. In general the norm $\|\cdot\|_\infty$ is not complete and its unit ball, which equals $[c_{-1}, c_1]$, is not lattice-complete. (Note: the term "constant" and the symbol $\|\cdot\|_\infty$ are indeed consistent with the case $X = \mathrm{Lip}(M)$.)

The usual vector lattice notions of *normal homomorphism* and *band* have to be modified here because of the existence of a norm and the special role of norm-bounded joins and meets. Thus, we let a *seminormal homomorphism* of Lip-spaces be a bounded linear map $\phi: X \to Y$ which preserves constants and norm-bounded joins. An *isomorphism* of Lip-spaces is a seminormal homomorphism which is a (not necessarily isometric or onto) Banach space isomorphism. By the open mapping theorem, it is enough for ϕ to be a 1-1 seminormal homomorphism with norm-closed range. A *semiband* of a Lip-space X is a linear subspace Y which is closed under norm-bounded joins and such that

$$(y \in Y \text{ and } |x| \leq |y|) \text{ imply } x \in Y.$$

By essentially a standard argument ([3, pp. 101–103]), every semiband is the kernel of a seminormal homomorphism, and conversely. (The only nonstandard aspect of this argument appears in the forward direction when we must endow X/Y with a norm. Here it is enough to verify that Y is norm-closed; but Y is naturally a Lip-space, so this result follows from Proposition 1.)

In the above definitions, the clauses concerning norm-bounded joins imply similar conditions on norm-bounded meets by the law $-\bigvee x_\alpha = \bigwedge(-x_\alpha)$.

We also define a *sub-Lip-space* of a Lip-space X to be a linear subspace Y which contains the constants and is equipped with a norm such that $\mathscr{B}_1(Y)$ is a complete sublattice of $\mathscr{B}_1(X)$ which contains c_1 (the greatest element of $\mathscr{B}_1(X)$). It is clear that Y is itself a Lip-space whose constants are the same as those of X. However, the norm on Y need not be the restriction to Y of the norm on X.

III. We already have a way of going from a metric space M to a Lip-space $\mathrm{Lip}(M)$. Conversely, if X is a Lip-space let its *dual metric space* be the set $X^\sim$ of all seminormal homomorphisms $X \to \mathbf{R}$, with metric inherited from the Banach space dual X^*. Since every seminormal homomorphism $X \to \mathbf{R}$ sends $\mathscr{B}_1(X) \subset [c_{-1}, c_1]$ into $[-1, 1]$, they all belong to $\mathscr{B}_1(X^*)$ and hence the distance between any two is ≤ 2. Also, $X^\sim$ is complete: any Cauchy sequence (p_n) in $X^\sim$ has a limit p in X^* which evidently preserves constants; and if $\{x_\alpha\} \subset \mathscr{B}_a(X)$ and $\|p - p_n\| \leq \varepsilon$, then

$$\begin{aligned} \left|p\left(\bigvee x_\alpha\right) - \bigvee p(x_\alpha)\right| &\leq \left|p\left(\bigvee x_\alpha\right) - p_n\left(\bigvee x_\alpha\right)\right| \\ &\quad + \left|p_n\left(\bigvee x_\alpha\right) - \bigvee p_n(x_\alpha)\right| + \left|\bigvee p_n(x_\alpha) - \bigvee p(x_\alpha)\right| \\ &\leq a \cdot \varepsilon + 0 + a \cdot \varepsilon, \end{aligned}$$

hence p preserves norm-bounded joins, so $p \in X^\sim$. So we have $X^\sim \in \mathscr{M}^{(2)}$.

If M, $N \in \mathscr{M}^{(2)}$ and $f\colon M \to N$ is Lipschitz then we have a natural map $f_*\colon \mathrm{Lip}(N) \to \mathrm{Lip}(M)$ given by $f_*(x) = x \circ f$, and if X and Y are Lip-spaces and $\phi\colon X \to Y$ is a seminormal homomorphism, then we have a natural map $\phi_*\colon Y^\sim \to X^\sim$ given by $\phi_*(p) = p \circ \phi$. We can now state our main results.

MAIN THEOREM. *Let M, $N \in \mathscr{M}^{(2)}$ and let X and Y be Lip-spaces.*

(a) *M is naturally isometric to* $(\mathrm{Lip}(M))^{\sim}$.

(b) *X is naturally isomorphic to* $\mathrm{Lip}(X^{\sim})$ *and the natural map* $\tau: X \to \mathrm{Lip}(X^{\sim})$ *satisfies* $\max(\|x\|_{\infty}, \|x\|/3) \le \|\tau(x)\| \le \|x\|$ *for all* $x \in X$.

(c) *Let $f: M \to N$ be Lipschitz. Then*

- (i) *f_* is a seminormal homomorphism and* $\|f_*\| = \max(1, L(f))$
- (ii) *if $f(M)$ is dense in N then f_* is* 1-1, *and*
- (iii) *if f is bi-Lipschitz then f_* is onto.*

(d) *Let $\phi: X \to Y$ be a seminormal homomorphism. Then*

- (i) *ϕ_* is Lipschitz and* $\max(1, L(\phi_*)) \le \|\phi\|$,
- (ii) *if ϕ is onto then ϕ_* is bi-Lipschitz, and*
- (iii) *if ϕ is* 1-1 *then $\phi_*(Y^{\sim})$ is dense in $X^{\sim}$.*

In parts (c)(i) and (d)(i), the norm of a seminormal homomorphism is just taken to be its usual norm as a map between Banach spaces.

We make some observations on this theorem. First, it yields a complete classification of Lip-spaces up to isomorphism. By (b) every Lip-space is isomorphic to $\mathrm{Lip}(M)$ for some $M \in \mathscr{M}^{(2)}$, and by (c) and (d) $\mathrm{Lip}(M)$ is isomorphic to $\mathrm{Lip}(N)$ iff there is a bi-Lipschitz map from M onto N, for bi-Lipschitz plus dense range clearly implies onto. We also see that the Lip-spaces are precisely those normed vector lattices which are isomorphic to $\mathrm{Lip}(M)$ for some M.

In part (b) τ is generally not an isometry. Indeed, it is easy to construct a two-dimensional Lip-space X which provides a counterexample. The existence of such an X also implies that the inequality in part (d)(i) need not be an equality: for in this case $\phi = \tau^{-1}$ is not an isometry, hence $\|\tau^{-1}\| > 1$, while $(\tau^{-1})_* = (\tau_*)^{-1}$ is an isometry by part (a), hence $L((\tau^{-1})_*) = 1$.

However, if $X = \mathrm{Lip}(M)$ for some $M \in \mathscr{M}^{(2)}$, then X is indeed isometrically isomorphic to $\mathrm{Lip}(X^{\sim})$ since $X^{\sim}$ is isometric to M by part (a). It follows that, in part (d)(i), if also $Y = \mathrm{Lip}(N)$ for some $N \in \mathscr{M}^{(2)}$, then $\max(1, L(\phi_*)) = \|\phi_{**}\| = \|\phi\|$ by part (c)(i) and the fact that X is isometric to $\mathrm{Lip}(X^{\sim})$ and Y is isometric to $\mathrm{Lip}(Y^{\sim})$.

Several parts of this theorem can be proven immediately:

(c)(i). The only nontrivial part is determining $\|f_*\|$; of course, this calculation implies that f_* is bounded. Assume f is nonconstant; the constant case is trivial.

For any $x \in \mathrm{Lip}(N)$, $\|f_*(x)\|_\infty = \|x \circ f\|_\infty \le \|x\|_\infty$ and

$$\begin{aligned} L(f_*(x)) &= \sup_{\substack{p,q\in M\\ p\neq q}} \frac{|x(f(p)) - x(f(q))|}{\rho(p,q)} \\ &= \sup_{\substack{p,q\in M\\ f(p)\neq f(q)}} \frac{|x(f(p)) - x(f(q))|}{\rho(f(p),f(q))} \cdot \frac{\rho(f(p),f(q))}{\rho(p,q)} \\ &\le L(x)\cdot L(f). \end{aligned}$$

Hence

$$\|f_*(x)\| \le \max(\|x\|_\infty, L(x)\cdot L(f)) \le \|x\|\cdot \max(1, L(f)),$$

so $\|f_*\| \le \max(1, L(f))$.

Conversely, for any $\varepsilon > 0$ there exist $p_0, q_0 \in M$, $p_0 \neq q_0$, such that $\rho(f(p_0), f(q_0)) \ge (L(f)-\varepsilon)\rho(p_0, q_0)$. Defining $x \in \mathrm{Lip}(N)$ by $x(p) = 1 - \rho(p, f(q_0))$, we have $\|x\| = 1$ and

$$\begin{aligned} \|f_*(x)\| \ge L(f_*(x)) &\ge \frac{|f_*(x)(p_0) - f_*(x)(q_0)|}{\rho(p_0,q_0)} \\ &= \frac{\rho(f(p_0), f(q_0))}{\rho(p_0,q_0)} \ge L(f) - \varepsilon. \end{aligned}$$

So $\|f_*\| \ge L(f)$; also $f_*(c_1) = c_1$ implies $\|f_*\| \ge 1$, and we are done.

(c)(ii). Suppose x, $y \in \mathrm{Lip}(N)$ and $x \neq y$; then x and y are continuous functions on N so $\{p \in N : x(p) \neq y(p)\}$ is a (nonempty) open subset of N. So if $f(M)$ is dense in N there exists $p_0 \in M$ such that $x(f(p_0)) \neq y(f(p_0))$, i.e. $f_*(x)(p_0) \neq f_*(y)(p_0)$, and thus $f_*(x) \neq f_*(y)$.

(c)(iii). To prove this part we need to invoke the well-known fact ([**5**, p. 244]) that if N is a metric space and $N' \subset N$, then every Lipschitz function $x: N' \to \mathbf{R}$ extends to a Lipschitz function $\tilde{x}: N \to \mathbf{R}$ with $\|\tilde{x}\| = \|x\|$.

Now if $f: M \to N$ is bi-Lipschitz then any $x \in \mathrm{Lip}(M)$ lifts to a Lipschitz function on $f(M)$, which can then be extended to a function $y \in \mathrm{Lip}(N)$ by the result just quoted. Then $f_*(y) = x$ and we conclude that f_* is onto.

(d)(i). $\|\phi\| \ge 1$ since $\phi(c_1) = c_1$. Now let p, $q \in Y^\sim$ and suppose $\rho(\phi_*(p), \phi_*(q)) > k$. Then there exists $x \in \mathscr{B}_1(X)$ such that $|\phi_*(p)(x) - \phi_*(q)(x)| > k$, hence there exists $y = \phi(x) \in \mathscr{B}_{\|\phi\|}(Y)$ such that $|p(y) - q(y)| > k$. It follows that $\|\phi\|\cdot\rho(p,q) > k$. We conclude that $\rho(\phi_*(p), \phi_*(q)) \le \|\phi\|\cdot\rho(p,q)$, so ϕ_* is Lipschitz and $L(\phi_*) \le \|\phi\|$.

(d)(ii). We just showed that ϕ_* is Lipschitz for any ϕ. Now suppose $\phi\colon X \to Y$ is onto. By the open mapping theorem there exists $a \in \mathbf{R}^+$ with $\mathscr{B}_a(Y) \subset \phi(\mathscr{B}_1(X))$. If p, $q \in Y^\sim$ and $\rho(p, q) > k$, then there must exist $y \in \mathscr{B}_1(Y)$ with $|p(y) - q(y)| > k$; then $\phi(x) = ay$ for some $x \in \mathscr{B}_1(X)$, so we have

$$|\phi_*(p)(x) - \phi_*(q)(x)| = |p(ay) - q(ay)| > a \cdot k,$$

and we conclude that $\rho(\phi_*(p), \phi_*(q)) \geq a \cdot \rho(p, q)$. So ϕ_* is bi-Lipschitz. □

It remains to prove (a), (b), and (d)(iii). Part (a) will be proven in the next section, and the other two parts will be the content of the last section.

IV. Theorem 1. *Let $M \in \mathscr{M}^{(2)}$ and let Y be a semiband in* Lip(M). *Then there is a closed subspace $M' \subset M$ such that Y consists of precisely those elements of* Lip(M) *which are zero on M'; that is, $Y = \ker \iota_*$, where $\iota\colon M' \to M$ is the inclusion map.*

Proof. Let x be the largest element of Y of norm ≤ 1; then $x \geq |x|$ so x is positive, i.e. $x(p) \geq 0$ for all $p \in M$. Define $M' = x^{-1}(0)$. If $y \in Y$ then $|y|/\| |y| \| \leq x$, hence $y(M') = 0$. To prove the converse, we will show that x is the largest element of Lip(M) of norm ≤ 1 which is zero on M'; then $y \in \text{Lip}(M)$ and $y(M') = 0$ will imply $|y|/\| |y| \| \leq x$, hence $y \in Y$. Thus, we must show that $x(p) = \rho(p, M') \wedge 1$ for all $p \in M$. (In case $M' = \varnothing$, we naturally set $\rho(p, M') = 2$ for all p.)

Certainly $x(p) \leq \rho(p, M') \wedge 1$ for all $p \in M$. Now pick $p_0 \in M$ and $\lambda < 1$ and suppose $x(p_0) < 1$. For countable ordinals α, we use transfinite induction to construct elements $p_\alpha \in M$ such that

$$(**) \qquad \beta < \alpha \Rightarrow x(p_\beta) - x(p_\alpha) \geq \lambda \cdot \rho(p_\beta, p_\alpha) > 0.$$

The construction goes as follows. Given p_α such that $x(p_\alpha) > 0$, to construct $p_{\alpha+1}$ first define the function $y(p) = (x(p_\alpha) - \lambda \cdot \rho(p, p_\alpha)) \vee 0$. If $y \leq x$ then $y \in Y$, hence $y/\|y\| \leq x$. However, since $\|y\| < 1$ and $y(p_\alpha) = x(p_\alpha)$ this is a contradiction.

Therefore $y \not\leq x$ and so there must exist a point $p_{\alpha+1} \in M$ such that $x(p_{\alpha+1}) < x(p_\alpha) - \lambda \cdot \rho(p_{\alpha+1}, p_\alpha)$. Assuming condition $(**)$ holds for all $\beta < \alpha$, we have

$$x(p_\beta) - x(p_\alpha) \geq \lambda \cdot \rho(p_\beta, p_\alpha),$$

and

$$x(p_\alpha) - x(p_{\alpha+1}) > \lambda \cdot \rho(p_\alpha, p_{\alpha+1}),$$

hence

$$x(p_\beta) - x(p_{\alpha+1}) > \lambda \cdot \rho(p_\beta, p_{\alpha+1}).$$

Also, for $\beta \leq \alpha$, $\rho(p_\beta, p_{\alpha+1}) > 0$ since $p_\beta = p_{\alpha+1}$ would imply $x(p_\beta) = x(p_{\alpha+1})$. So $(**)$ is still satisfied after $p_{\alpha+1}$ has been constructed.

For limit ordinals α, the net $(x(p_\beta))_{\beta<\alpha}$ is decreasing and bounded below, hence is Cauchy; therefore by $(**)$ the net $(p_\beta)_{\beta<\alpha}$ is also Cauchy and we may define p_α to be its limit. In this case, for any $\beta < \alpha$ the first part of $(**)$ is satisfied by continuity and the satisfaction of the second part is clear.

Since the numbers $x(p_\alpha)$ are strictly decreasing, the construction must stop eventually, but the only way this can happen is if $x(p_{\alpha_0}) = 0$ for some α_0. In summary, under the assumption that $x(p_0) < 1$ we have proven that

$$x(p_0) = x(p_0) - x(p_{\alpha_0}) \geq \lambda \cdot \rho(p_0, p_{\alpha_0}) \geq \lambda \cdot \rho(p_0, M').$$

Since this is so for any $\lambda < 1$ we conclude that $x(p_0) \geq \rho(p_0, M')$. Thus for any $p_0 \in M$, $x(p_0) \geq \rho(p_0, M') \wedge 1$, as desired. □

By Theorem 1, it is clear that the maximal semibands of $\text{Lip}(M)$ are precisely the kernels of seminormal homomorphisms $\text{Lip}(M) \to \mathbf{R}$. Thus the following corollary implies that every $M \in \mathscr{M}^{(2)}$ is in a natural bijection with the set of maximal semibands of $\text{Lip}(M)$.

COROLLARY. *For any* $M \in \mathscr{M}^{(2)}$, *the natural map* $\sigma: M \to (\text{Lip}(M))^\sim$ *is an isometry of* M *onto* $(\text{Lip}(M))^\sim$. (Main Theorem part (a)).

Proof. The map σ takes $p \in M$ to the function $x \mapsto x(p)$. To show it is an isometry, choose $p_0, q_0 \in M$. For any $x \in \mathscr{B}_1(\text{Lip}(M))$ we have

$$|\sigma(p_0)(x) - \sigma(q_0)(x)| = |x(p_0) - x(q_0)| \leq \rho(p_0, q_0),$$

hence $\rho(\sigma(p_0), \sigma(q_0)) \leq \rho(p_0, q_0)$. Conversely, consider the function $x \in \mathscr{B}_1(\text{Lip}(M))$ defined by $x(p) = 1 - \rho(p, q_0)$. We have

$$|\sigma(p_0)(x) - \sigma(q_0)(x)| = |x(p_0) - x(q_0)| = \rho(p_0, q_0),$$

and hence $\rho(\sigma(p_0), \sigma(q_0)) \geq \rho(p_0, q_0)$. So σ is an isometry.

Now we show σ is onto. Let ϕ: $\mathrm{Lip}(M) \to \mathbf{R}$ be a seminormal homomorphism. By Theorem 1, there exists $M' \subset M$ such that $\ker\phi = \ker\imath_*$, where $\imath: M' \to M$ is the inclusion map. Since $\mathrm{Lip}(M') \cong \mathrm{Lip}(M)/\ker\phi \cong \mathbf{R}$, $\mathrm{Lip}(M')$ is one-dimensional, and M' must consist of a single point p_0: if M' had more than one point then for any $q \in M'$ the function $f(p) = \rho(p, q)$ in $\mathrm{Lip}(M')$ would be nonconstant so $\mathrm{Lip}(M')$ could not be one-dimensional. Identifying $\mathrm{Lip}(M')$ with $\mathbf{R}$, the map $\imath_*$ is then of the form $\imath_*(x) = x(p_0)$, i.e. $\imath_* = \sigma(p_0)$; and $\ker\imath_* = \ker\phi$ and $\imath_*(c_1) = \phi(c_1) = 1$ imply $\imath_* = \phi$. So $\phi = \sigma(p_0)$ and we are done. □

The following theorem, which we need for §V, is reminiscent of the Stone-Weierstrass theorem. Note, however, that if the hypothesis that Y contains the constants is removed, then Y may be nowhere near all of $\mathrm{Lip}(M)$. For example, $Y \subset \mathrm{Lip}([0, 1])$ could be the set of functions of the form $x \mapsto ax$ $(a \in \mathbf{R})$; this is a one-dimensional subspace which separates points in the manner required by the theorem.

THEOREM 2. *Let $M \in \mathscr{M}^{(2)}$ and let Y be a sub-Lip-space of* $\mathrm{Lip}(M)$ *which has the property that for any p, $q \in M$ and any $\varepsilon > 0$ there is an element $y \in \mathscr{B}_{1+\varepsilon}(Y)$ which satisfies $|y(p) - y(q)| = \rho(p, q)$. Then $Y = \mathrm{Lip}(M)$ as sets and $\mathscr{B}_1(\mathrm{Lip}(M)) \subset \mathscr{B}_3(Y)$.*

Proof. Let $x \in \mathscr{B}_1(\mathrm{Lip}(M))$ and $\varepsilon > 0$; we will show $x \in \mathscr{B}_{3+2\varepsilon}(Y)$. We begin by showing that for any p, $q \in M$ there exists $y \in \mathscr{B}_{3+2\varepsilon}(Y)$ such that $x(p) = y(p)$ and $x(q) = y(q)$. Since $|x(p) - x(q)| \leq \rho(p, q)$, by hypothesis there exists $y_0 \in \mathscr{B}_{1+\varepsilon}(Y)$ such that $|y_0(p) - y_0(q)| \geq |x(p) - x(q)|$; multiplying y_0 by a suitable scalar of norm ≤ 1, we get an element $y_1 \in \mathscr{B}_{1+\varepsilon}(Y)$ such that $|y_1(p) - y_1(q)| = |x(p) - x(q)|$, and the sign of the scalar may be chosen so that $y_1(p) - y_1(q) = x(p) - x(q)$. Now $|x(p) - y_1(p)| \leq \|x\|_\infty + \|y_1\|_\infty \leq 2 + \varepsilon$, so $y = y_1 + c_{x(p)-y_1(p)}$ is in $\mathscr{B}_{3+2\varepsilon}(Y)$; and it satisfies $x(p) = y(p)$, $x(q) = y(q)$.

Now for each $p \in M$ define

$$y_p = \bigvee\{y \in \mathscr{B}_{3+2\varepsilon}(Y) : y(p) = x(p)\};$$

we have $y_p \in \mathscr{B}_{3+2\varepsilon}(Y)$, $y_p(q) \geq x(q)$ for all $q \in M$ by the result of the last paragraph, and $y_p(p) = x(p)$. Thus $\bigwedge\{y_p : p \in M\}$ is in $\mathscr{B}_{3+2\varepsilon}(Y)$ and equals x. As this holds for any $\varepsilon > 0$, we conclude that $x \in \mathscr{B}_3(Y)$. So $\mathscr{B}_1(\mathrm{Lip}(M)) \subset \mathscr{B}_3(Y)$, and this implies that every element of $\mathrm{Lip}(M)$ is in Y. □

V. The final theorem is the most sophisticated in the paper. Its proof relies on the fact **[4]** that any completely distributive complete lattice L has the following property:

$$(\dagger) \qquad \text{for all } y \not\leq z \in L \text{ there exist } y' \not\leq y \text{ and } z' \not\geq z \text{ such that } L = [0_L, z'] \cup [y', 1_L].$$

Here 0_L and 1_L are the least and greatest elements of L, respectively.

LEMMA. *Let $u \leq v$ be elements of a vector lattice, with $u \not\leq 0$. Then*
(a) $u \wedge av \not\leq 0$ *for any* $0 < a < 1$ *and*
(b) $u + v \not\leq 0$.

Proof. (a) If $u \wedge av \leq 0$ for some $0 < a < 1$ then

$$au^+ \leq u^+ \wedge av^+ = (u \wedge av) \vee 0 = 0,$$

hence $u \leq 0$, contradiction. (Recall $u^+ = u \vee 0$.)
(b) $u + v \geq 2u$, but $2u \not\leq 0$. □

THEOREM 3. *If X is any Lip-space, then the natural map $\tau: X \to$ $\mathrm{Lip}(X^\sim)$ is an isomorphism of X onto* $\mathrm{Lip}(X^\sim)$, *and it satisfies* $\max(\|x\|_\infty, \|x\|/3) \leq \|\tau(x)\| \leq \|x\|$ *for all* $x \in X$. (Main Theorem part (b).)

Proof. The proof proceeds in four steps. In steps 1–3 we fix an element $w \not\leq 0$ in $\mathscr{B}_1(X)$.

Step 1. *There exist elements y_n, z_n $(n \in \mathbf{N})$ in $\mathscr{B}_1(X)$ such that $y_1 = w$ and $z_1 = c_1$ and with the properties*
(a) $\mathscr{B}_1(X) \subset [c_{-1}, z_n] \cup [y_n, c_1]$ *for all* n;
(b) *if $m < n$ then $y_m \not\leq z_n$ (and therefore $y_m \geq y_n$)*;
(c) *$y_n \not\leq 0$ (and therefore $z_n \geq 0$) for all n; and*
(d) *$z_n \not\geq c_{1/n}$ (and therefore $y_n \leq c_{1/n}$) for all $n > 1$.*

We construct y_n and z_n by induction. First let $y_1 = w$ and $z_1 = c_1$. Having constructed y_i and z_i for $1 \leq i \leq n-1$ satisfying (a)–(d), let $y = 0$ and $z = y_{n-1} \wedge c_{1/n}$. Then $y \not\geq z$ by part (a) of the lemma with $u = y_{n-1}$, $v - c_{1/(n-1)}$, and $a = 1 - 1/n$. So find y', $z' \in \mathscr{B}_1(X)$ satisfying $(\dagger)$ for $L = \mathscr{B}_1(X)$ and set $y_n = y'$, $z_n = z'$. It is clear that y_n and z_n satisfy (a), (c), and (d). We also have $z_n \not\geq y_{n-1}$, and then $z_n \not\geq y_m$ for any $m < n-1$ since $y_m \geq y_{n-1}$ by the inductive assumption. So (b) holds too.

Step 2. *There exists an element* $w' \in \mathscr{B}_1(X)$, $w' \geq 0$, *with the properties*

(a) $\mathscr{B}_1(X) \subset [-c_1, w'] \cup [-w', c_1]$,
(b) $c_a \not\leq w'$ *for any* $a > 0$, *and*
(c) $w \not\leq w'$.

Define $w' = \bigwedge_1^\infty z_n$. Clearly $w' \geq 0$.

First we prove (a). For any $x \in \mathscr{B}_1(X)$, if $x \leq z_n$ for all n then $x \leq w'$. Otherwise $x \not\leq z_n$ for some n, hence $x \geq y_n$; so we must show $y_n \geq -w'$. For any $m \in \mathbf{N}$, $y_m + y_n \not\leq 0$ by part (b) of the lemma, so

$$0 \not\geq y_m + y_n \Rightarrow -y_n \not\geq y_m \Rightarrow -y_n \leq z_m.$$

Hence $-y_n \leq w'$, hence $y_n \geq -w'$ as desired. This proves part (a).

For part (b), recall that $c_{1/n} \not\leq z_n$ hence $c_{1/n} \not\leq w'$ for any $n > 1$. For any $a > 0$ we can find $n > 1$ such that $1/n \leq a$, hence $c_{1/n} \leq c_a$, hence $c_a \not\leq w'$.

Part (c) is obvious: $w = y_1 \not\leq z_2$, so $w \not\leq w'$.

Step 3. *There is a seminormal homomorphism* $p: X \to \mathbf{R}$ *such that* $p(w) > 0$.

Define p by

$$\begin{aligned} p(x) &= \sup\{a \in \mathbf{R} : x \geq c_a + \lambda w' \text{ for some } \lambda \in \mathbf{R}\} \\ &= \inf\{a \in \mathbf{R} : x \leq c_a + \lambda w' \text{ for some } \lambda \in \mathbf{R}\}. \end{aligned}$$

We first show that this definition is consistent and that $p(x)$ is finite for all x. Fix $x \in X$. For any $a \in \mathbf{R}$ let $\lambda = |a| + \|x\|$; then $(x - c_a)/\lambda \in \mathscr{B}_1(X)$, hence

$$(x - c_a)/\lambda \geq -w' \quad \text{or} \quad (x - c_a)/\lambda \leq w',$$

i.e.

$$(\ddagger) \qquad x \geq c_a - (|a| + \|x\|)w' \quad \text{or} \quad x \leq c_a + (|a| + \|x\|)w'.$$

This holds for any a, so $\sup \geq \inf$. Conversely, if $b < a$ and $c_a + \lambda w' \leq x \leq c_b + \mu w'$ for some λ, μ then for $k = \max(|\lambda|, |\mu|)$ we have

$$c_a - kw' \le c_b + kw' \Rightarrow c_{a-b} \le 2kw' \Rightarrow c_{(a-b)/2k} \le w',$$

a contradiction. This implies that $\sup \le \inf$, and so the definition of $p(x)$ is consistent. Also, $c_{-\|x\|} \le x \le c_{\|x\|}$ implies $p(x)$ is finite.

Now we show p is linear. For any $a > 0$,

$$x \ge c_b + \lambda w' \Leftrightarrow ax \ge c_{ab} + a\lambda w'$$

hence $p(ax) = ap(x)$; and

$$x \ge c_b + \lambda w' \Leftrightarrow -x \le c_{-b} - \lambda w'$$

hence $p(-x) = -p(x)$. Also, for any $x_1, x_2 \in X$

$$(x_1 \le c_a + \lambda w' \text{ and } x_2 \le c_b + \mu w') \Rightarrow x_1 + x_2 \le c_{a+b} + (\lambda + \mu)w',$$

hence $p(x_1 + x_2) \le p(x_1) + p(x_2)$, and $p(x_1 + x_2) \ge p(x_1) + p(x_2)$ similarly.

It is clear that p preserves constants. Since p clearly also preserves order, it follows that p takes $\mathscr{B}_1(X) \subset [c_{-1}, c_1]$ into $[-1, 1]$, hence p is bounded. Now we show p preserves norm-bounded joins. Let $\{x_\alpha\} \subset X$ be a norm-bounded set. Since p preserves order, $\bigvee p(x_\alpha) \le p(\bigvee x_\alpha)$. Conversely, using (‡) we have

$$\begin{aligned}\bigvee p(x_\alpha) < a \Rightarrow p(x_\alpha) < a \quad & \text{for all } \alpha \\ \Rightarrow x_\alpha \not\ge c_a - (|a| + \|x_\alpha\|)w' \quad & \text{for all } \alpha \\ \Rightarrow x_\alpha \le c_a + (|a| + \|x_\alpha\|)w' \quad & \text{for all } \alpha \\ \Rightarrow \bigvee x_\alpha \le c_a + (|a| + \sup\{\|x_\alpha\|\})w' & \\ \Rightarrow p\left(\bigvee x_\alpha\right) \le a. &\end{aligned}$$

Thus $\bigvee p(x_\alpha) = p(\bigvee x_\alpha)$ and p is a seminormal homomorphism.

Finally we show $p(w) > 0$. Suppose to the contrary that $p(w) < 1/n$ for all $n \in \mathbf{N}$. Then by (‡),

$$w \le c_{1/n} + (1/n + \|w\|)w' \quad \text{for all } n,$$

hence

$$w \le \left(\bigwedge_n (c_1 + w')/n\right) + \|w\|w' = \|w\|w' \le w',$$

a contradiction.

Step 4. *Proof of Theorem*. Let $x \in X$ and suppose $a > 0$ is such that $x \not\le c_a$. Let $w = (x - c_a)/\|x - c_a\|$; then we can find a seminormal homomorphism $p: X \to \mathbf{R}$ such that $p(w) > 0$ by steps 1–3. Then

$$\|\tau(x)\| \geq \|\tau(x)\|_\infty \geq |\tau(x)(p)| = |p(x)| = \|x - c_a\| p(w) + a \geq a.$$

Since any $a < \|x\|_\infty$ satisfies $x \not\leq c_a$ or $-x \not\leq c_a$, this shows $\|\tau(x)\| \geq \|x\|_\infty$.

Also,

$$\|\tau(x)\|_\infty = \sup_{p \in X^\sim} |\tau(x)(p)| = \sup_{p \in X^\sim} |p(x)| \leq \|x\|$$

and

$$L(\tau(x)) = \sup_{\substack{p, q \in X^\sim \\ p \neq q}} \frac{|\tau(x)(p) - \tau(x)(q)|}{\rho(p, q)} = \sup_{\substack{p, q \in X^\sim \\ p \neq q}} \frac{|p(x) - q(x)|}{\rho(p, q)} \leq \|x\|$$

so $\|\tau(x)\| \leq \|x\|$.

We have seen that τ is 1-1, so define a norm $|||\cdot|||$ on $\tau(X)$ by $|||\tau(x)||| = \|x\|$. With this norm $\tau(X)$ is a sub-Lip-space of $\mathrm{Lip}(X^\sim)$ which satisfies the condition of Theorem 2. We conclude that τ is onto and $\|x\| = |||\tau(x)||| \leq 3\|\tau(x)\|$ for any $x \in X$. As τ is clearly a seminormal homomorphism, the proof is complete. □

COROLLARY. *Let X and Y be Lip-spaces and let $\phi: X \to Y$ be a seminormal homomorphism. If ϕ is* 1-1 *then $\phi_*(Y^\sim)$ is dense in $X^\sim$.* (Main Theorem part (d)(iii).)

Proof. Suppose $\phi_*(Y^\sim)$ is not dense in $X^\sim$ and let $p_0 \in X^\sim$ be such that $r = \rho(p_0, \phi_*(Y^\sim)) > 0$. Define $y \in \mathrm{Lip}(X^\sim)$ by $y(q) = (r - \rho(p_0, q)) \vee 0$; so $y(\phi_*(Y^\sim)) = 0$. Then since $\tau_X: X \to \mathrm{Lip}(X^\sim)$ is onto, there exists $x \in X$ with $\tau_X(x) = y$. For any $p \in Y^\sim$ we then have

$$p(\phi(x)) = \phi_*(p)(x) = y(\phi_*(p)) = 0,$$

hence $\phi(x) = 0$ since $\tau_Y: Y \to \mathrm{Lip}(Y^\sim)$ is 1-1. But $x \neq 0$, so ϕ must not be 1-1. □

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Received March 20, 1992 and in revised form March 30, 1993.

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