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ORIENTED ORBIFOLD COBORDISM

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A complete set of invariants (generalized Pontrjagin numbers) for rational oriented orbifold cobordism is determined. Using these numbers we prove that for any odd dimensional oriented orbifold Q there is a nonzero multiple of Q which bounds another orbifold and that, unlike the manifold case, this need not be true for $4k+2$ dimensional orbifolds. In addition we construct generators for the rational orbifold cobordism ring and show that it is a free commutative ring on these.

Introduction. This paper establishes a foundation for oriented orbifold cobordism in a manner analogous to Thom's results for oriented manifolds. Thom's main theorem states that some multiple of an oriented compact manifold bounds another such manifold if and only if its Pontrjagin numbers are zero. Orbifolds are like manifolds, except that they locally look like $\mathbf{R}^n/G$ for G a finite group. For each H a finite subgroup of $\mathrm{SO}(n)$ we define H characteristic numbers of the compact oriented orbifold Q which account for its H singular set and are its Pontrjagin numbers when H is trivial. Our main theorem is that some multiple of Q bounds another compact oriented orbifold if and only if all its H characteristic numbers are zero for all finite subgroups H of $\mathrm{SO}(n)$.

Thom's corollary that every manifold with dimension not divisible by four rationally bounds does not translate exactly in the orbifold case. There is no reason to expect it to as the H singular set of even an odd dimensional orbifold may have nonzero Pontrjagin numbers. However, a careful consideration of orientations and the twisted cohomology of the classifying space of the centralizer of H, $C_{\mathrm{O}(d)}(H)$, shows that any odd dimensional orbifold rationally bounds another orbifold W with the same set of local groups. The result for $(4k+2)$-dimensional orbifolds is not as neat. We can isolate a certain class of finite subgroups of $\mathrm{SO}(4k+2)$, which includes all finite abelian subgroups, such that any $(4k+2)$-dimensional orbifold with all local groups in this class rationally bounds. This set does not include all finite subgroups as we construct a seventy-dimensional orbifold with nonzero characteristic numbers which therefore does not rationally bound.

The article begins with a preliminary section on orbifolds, orbibundles, orientations, and orbifold cobordism. The second section is devoted to defining the H characteristic numbers and proving that they rationally determine orbifold cobordism. The proof adopts the local techniques of Lee and Wasserman [8]. We follow this with an analysis of these numbers and the cohomology from which they arise, yielding the odd and $(4k+2)$-dimensional results. Using the work of the previous two sections, in the fourth section we construct orbifolds which form bases for the rational orbifold cobordism groups and show that $\Omega_{*,\mathrm{orb}} \otimes \mathbf{Q}$ is a free commutative algebra over $\Omega_* \otimes \mathbf{Q}$ with generators a certain subset of these orbifolds.

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1. Preliminaries.

DEFINITION 1.1. Let X be a topological space.

(i) An *orbifold chart on* X is a four-tuple $(U, G, \overline{U}, \pi)$ with U an open subset of X, $\overline{U}$ an open subset of $\mathbf{R}^n$, G a finite group of diffeomorphisms acting effectively on $\overline{U}$, and $\pi\colon \overline{U} \to U$ an open map onto U such that $\pi = h\pi_U$ where π_U is the natural orbit map and h is a homeomorphism from $\overline{U}/G$ to U.

(ii) Two charts $(U_i, G_i, \overline{U}_i, \pi_i)$, $i = 1, 2$, on X are *compatible* if for every $x \in U_1 \cap U_2$ there is an open set $U \subset \overline{U}_1$ with a lift $\overline{x}$ of x contained in U and a diffeomorphism $f\colon U \to \overline{U}_2$, called an *overlap map*, such that $\pi_2 f = \pi_1$.

(iii) An *orbifold atlas* $\mathscr{U}$ *on* X is a collection $\{(U_\alpha, G_\alpha, \overline{U}_\alpha, \pi_\alpha)\}_{\alpha\in\Lambda}$ of compatible charts on X such that $\bigcup_{\alpha\in\Lambda} U_\alpha = X$.

DEFINITION 1.2. A smooth *orbifold* Q is a Hausdorff space $|Q|$, called the *underlying space*, together with a maximal orbifold atlas $\mathscr{U}$ on $|Q|$.

DEFINITION 1.3. An *orbifold with boundary* W is defined by allowing $\overline{U}$ to be an open subset of $H^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n\colon x_n \geq 0\}$ in the above definition. The *orbifold boundary* of W, denoted $\partial_o W$, has underlying space $|\partial_o W| = \{w \in |W|\colon w \in \pi(\partial(\overline{U} \cap H^n))$ for some chart of $W\}$ and the obvious orbifold structure.

An *orbifold diffeomorphism* is defined by a homeomorphism $f\colon |Q| \to |Q'|$ such that $(U, G, \overline{U}, \pi)$ is in the maximal atlas for Q if and only if $(f(U), G, \overline{U}, f\pi)$ is a chart in the maximal atlas of Q'.

DEFINITION 1.4. Let $x \in |Q|$ and $(U, G, \overline{U}, \pi)$ be a chart for Q with $x \in U$ and $y \in \overline{U}$ such that $\pi(y) = x$. The *local group at* x is defined to be G_y, the isotropy group of y.

We have that G_y is isomorphic to the linear group formed by its differentials at y. If $(U_i, G_i, \overline{U}_i, \pi_i)$ are two charts with $x \in U_i$ and $\pi_i(y_i) = x$ and h is an overlap map sending y_1 to y_2 then conjugation by dh_{y_1} provides an isomorphism between dG_{y_1} and dG_{y_2}. It follows that the local group at x is independent of the choice of chart and is well defined up to conjugacy as a subgroup of $\mathrm{Gl}(n, \mathbf{R})$ where $n = \dim Q$.

DEFINITION 1.5. The set of conjugacy classes in $\mathrm{Gl}(n, \mathbf{R})$ of local groups of Q is denoted by $\mathscr{G}_Q$ and the class of a subgroup H in $\mathscr{G}_Q$ is denoted by (H). The set $\mathscr{G}_Q$ is partially ordered by $(H) \leq (H')$ if H is conjugate in $\mathrm{Gl}(n, \mathbf{R})$ to a subgroup of H'. Note that $\mathscr{G}_Q$ is finite if Q is compact.

Many of the concepts one would want in a theory of orbifolds are most precisely defined in the language of groupoids. This approach can be found in [7] and in more detail in [5].

DEFINITION 1.6. For $\mathscr{U} = \{(U_\alpha, G_\alpha, \overline{U}_\alpha, \pi_\alpha)\}_{\alpha \in \Lambda}$, an atlas for Q, there is the topological groupoid $\Gamma_{\mathscr{U}}$ whose set of objects is $\tilde{Q} \overset{\text{def}}{=} \{x : x \in \coprod_{\alpha \in \Lambda} \overline{U}_\alpha\}$ and whose morphisms are the germs of local diffeomorphisms which commute with the natural map π from $\tilde{Q}$ to $|Q|$. Thus the morphisms are generated by the germs of local groups and overlap maps. $\Gamma_{\mathscr{U}}$ has the "germ" topology ([5, 3.7]).

DEFINITION 1.7. Q is *orientable* if there is an orientation of $\tilde{Q}$ which is preserved by $\Gamma_{\mathscr{U}}$. A weaker notion is that of *local orientability* which means that all the local groups are subgroups of $\mathrm{Gl}(n, \mathbf{R})_+$.

Suppose F is a topological space and G is a topological group acting effectively on F. As a generalization of (F, G) bundles we have:

DEFINITION 1.8. An (F, G) *orbibundle* $E \xrightarrow{p} Q$ is defined as an (F, G) bundle $\tilde{E} \xrightarrow{\tilde{p}} \tilde{Q}$ with a $\Gamma_{\mathscr{U}}$ action (see [7, §2.2]) on $\tilde{E}$ through (F, G) bundle isomorphisms which extends the natural action on $\tilde{Q}$.

There is an induced map $\tilde{E}/\Gamma_{\mathscr{U}} \xrightarrow{p} |Q|$ of $\Gamma_{\mathscr{U}}$ quotients and $p^{-1}(U_\alpha) \cong \overline{U}_\alpha \times_{G_\alpha} F$ where G_α acts on F by elements of G. If F is an orbifold, $G < \mathrm{Diffeo}(F)$, $\tilde{E}$ is a smooth (F, G) bundle and

$\Gamma_{\mathscr{U}}$ acts smoothly on $\tilde{E}$, then $\tilde{E}/\Gamma_{\mathscr{U}}$ has a natural orbifold structure which we often denote by E.

Many of the usual notions for bundles carry through to orbibundles, including associated principal orbibundles and reduction of the structure group G, [**13**, §1]. Vector bundles are now generalized to $(\mathbf{R}^n, \mathrm{Gl}(n, \mathbf{R}))$-orbibundles, called orbivector bundles. Riemannian metrics on orbivector bundles are reductions of their structure group to $\mathrm{O}(n)$. An oriented orbibundle would be one where $\tilde{E}$ is an oriented (F, G) bundle and $\Gamma_{\mathscr{U}}$ preserves the orientation.

The most revelant examples for this paper include the tangent bundle to Q, TQ, (which is $T\tilde{Q}$ with $\Gamma_{\mathscr{U}}$ acting by differentials), its associated principal $\mathrm{Gl}(n, \mathbf{R})$ orbibundle, PQ, and, given a Riemannian metric g on Q, the associated principal $\mathrm{O}(n)$ orbibundle, P_gQ. We note that PQ, respectively P_gQ, are smooth manifolds and that there is a smooth right $\mathrm{Gl}(n, \mathbf{R})$, respectively $\mathrm{O}(n)$, action with finite isotropy groups whose orbit space is Q. Thus since any paracompact orbifold admits a Riemannian metric, we conclude that any compact orbifold is the quotient of a compact manifold (of higher dimension) by a smooth compact Lie group action with finite isotropy groups.

For the groupoid $\Gamma_{\mathscr{U}}$ there is a universal principal $\Gamma_{\mathscr{U}}$ bundle [**7**, §3.1] $E\Gamma_{\mathscr{U}} \xrightarrow{\tilde{\pi}} B\Gamma_{\mathscr{U}}$ and natural maps $\psi_{\mathscr{U}}: B\Gamma_{\mathscr{U}} \to |Q|$, $\hat{\psi}_{\mathscr{U}}: E\Gamma_{\mathscr{U}} \to \tilde{Q}$ making the appropriate diagram commutative. If $\mathscr{U}'$ is another atlas for Q, $B\Gamma_{\mathscr{U}'}$ is homotopy equivalent to $B\Gamma_{\mathscr{U}}$ by a homotopy h such that $h\psi_{\mathscr{U}} = \psi_{\mathscr{U}'}$ [**5**, Remark 8.1]. Thus we often denote $B\Gamma_{\mathscr{U}}$ by BQ and $\psi_{\mathscr{U}}$ by ψ. One model for BQ is $P_gQ \times_{\mathrm{O}(n)} E\mathrm{O}(n)$ ([**5**, Remark 8.2]). Also a map of orbifolds $\lambda: Q' \to Q$ is defined by a generalized homomorphism [**5**, §4.8; **7**, §2.3] $\hat{\lambda}$ between $\Gamma_{\mathscr{U}'}$ and $\Gamma_{\mathscr{U}}$ and so defines a homotopy class of maps $B\lambda: BQ' \to BQ$. Here $\mathscr{U}'$ and $\mathscr{U}$ are atlases for Q' and Q, respectively.

For V an (F, G) orbibundle over Q we denote by ψ^*V the bundle $\hat{\psi}^*_{\mathscr{U}}\tilde{E}/\Gamma_{\mathscr{U}}$. Then since $\Gamma_{\mathscr{U}}$ acts freely on $E\Gamma_{\mathscr{U}}$ we have the following.

LEMMA 1.9. *ψ^*V is an ordinary (F, G) bundle and so is classified by a map $f: BQ \to BG$.*

If the structure group of the orbibundle V reduces to $L < G$ then f lifts to a classifying map to BL. Also for the orbifold map $\lambda: Q' \to Q$, λ^*V is defined by the bundle $V \times_{\Gamma_{\mathscr{U}}} \hat{\lambda}$ over $\tilde{Q}'$ with $\Gamma_{\mathscr{U}'}$ acting on the right of $\hat{\lambda}$. Working through the definitions one checks that

$(B\lambda)^*\psi^*V = \psi'^*\lambda^*V$. Here $\psi'\colon BQ' \to |Q'|$ is the canonical map.

From [5, 6.4] and [7, 4.2.4] we have

LEMMA 1.10. *If Q is a locally orientable, n-dimensional orbifold then $|Q|$ is a rational homology manifold.*

Thus as in [16, §2] $|Q|$ has a twisted fundamental class $[Q] \in H_n(|Q|; \widehat{\mathbf{Q}})$, where $\widehat{\mathbf{Q}}$, or or_Q, denotes the orientation system of $|Q|$. We also have that if $E \xrightarrow{p} Q$, denoted ϕ, is a locally orientable $(\mathbf{R}^n/K, G)$-orbibundle with K finite and orientation preserving and $G < N_{\mathrm{Gl}(n,\mathbf{R})}(K)/K$, where $N_{\mathrm{Gl}(n,\mathbf{R})}(K)$ is the normalizer of K in $\mathrm{Gl}(n, \mathbf{R})$, then $H_n(|E_x|, |E_x - 0|; \mathbf{Q})$, $x \in |Q|$, defines an orientation system for ϕ, denoted or_ϕ or $\widehat{\mathbf{Q}}$. In general, in $H^n(|E|, |E_0|; p^*\,\mathrm{or}_\phi)$ there is a unique class, μ_ϕ, called the Thom class of ϕ such that $\bigcup \mu_\phi\colon H^k(|E|; \Gamma) \xrightarrow{\cong} H^{n+k}(|E|, |E_0|; \Gamma \otimes p^*\,\mathrm{or}_\phi)$ for any rational local system Γ. If $e \in H^n(|Q|; \mathrm{or}_\phi)$ is the class which gets mapped onto μ_ϕ by p^* then e is called the *twisted Euler class of* ϕ. Also if Q' is the orientation double cover of Q associated with ϕ then $H^*(|Q|; \mathrm{or}_\phi) \cong H^*(|Q'|; \mathbf{Q})^-$, the -1 eigenvectors of α^* where α is the involution of the cover.

LEMMA 1.11. *Suppose Q is a locally orientable orbifold. Then ψ^* is a rational cohomology isomorphism, both for trivial coefficients and between $H^*(|Q|; \mathrm{or}_Q)$ and $H^*(BQ; \psi^*\,\mathrm{or}_Q)$.*

Proof. The trivial coefficient case is proved in [5] and [7, 4.2.3]. One has that $H^*(BQ; \psi^*\,\mathrm{or}_Q)$ is isomorphic to the -1 eigenvectors of $B\alpha^*$ acting on $H^*(BQ'; \mathbf{Q})$ and these coincide with $H^*(|Q'|; \mathbf{Q})^-$ under the isomorphism ψ'^* where $\psi'\colon BQ' \to |Q'|$. □

Assume that ζ is an orbivector bundle over a locally orientable orbifold Q and that V is a $(\mathbf{R}^n/K, G)$ orbibundle over Q as above with $TQ \oplus V$ oriented.

DEFINITION 1.12. (i) The ith Pontrjagin class of ζ, $p_i(\zeta)$, is defined to be $(\psi^*)^{-1}(p_i(\psi^*(\zeta)))$.

(ii) Let $x \in H^*(BG; \widehat{\mathbf{Q}})$ where the coefficients are those of the canonical $(\mathbf{R}^n/K, G)$ bundle over BG; then $x(V) \in H^*(Q; \mathrm{or}_Q)$ is the class $(\psi^*)^{-1}(f^*x)$.

From the discussion following Lemma 1.9 we conclude

LEMMA 1.13. *Pontrjagin classes of orbivector bundles have all the usual properties, as in* [**12**]; *in particular, they respect direct sums, are zero for trivial bundles and natural with respect to orbifold maps. The other characteristic classes of orbibundles in the above definition have the latter two properties.*

DEFINITION 1.14. Suppose $|X| \subset |Q|$. We say that X is a k-dimensional *suborbifold* of Q if there is an atlas

$$\mathscr{U} = \{(U_\alpha, G_\alpha, \overline{U}_\alpha, \pi_\alpha)\}_{\alpha\in\Lambda}$$

for Q such that for every $\alpha \in \Lambda$ with $U_\alpha \cap |X| \neq \varnothing$, $\overline{U}_\alpha \cap \mathbf{R}^k$ is invariant under G_α, and $\pi_\alpha(\overline{U}_\alpha \cap \mathbf{R}^k) = U_\alpha \cap |X|$.

If $\Lambda' = \{\alpha \in \Lambda\colon U_\alpha \cap |X| \neq \varnothing\}$ then X has the orbifold structure given by the atlas $\mathscr{U}' = \{(U_\alpha \cap |X|, G_\alpha/K_\alpha, \overline{U}_\alpha \cap \mathbf{R}^k, \pi_{\alpha|\overline{U}_\alpha\cap R^k})\}_{\alpha\in\Lambda'}$, where K_α is the (normal) subgroup of G_α which acts as the identity on $\overline{U}_\alpha \cap \mathbf{R}^k$.

Let $\overline{U}'_\alpha$ denote $\overline{U}_\alpha \cap \mathbf{R}^k$. If Q is a Riemannian orbifold then $T\overline{U}_{\alpha|\overline{U}'_\alpha}$ decomposes as $T\overline{U}'_\alpha \oplus \nu(\overline{U}'_\alpha)$ and this is invariant under $\Gamma_{\mathscr{U}}$ restricted to $\coprod_{\alpha\in\Lambda'} \overline{U}'_\alpha$. Therefore $\coprod_{\alpha\in\Lambda'} \nu(\overline{U}'_\alpha)/K_\alpha$ defines a normal bundle $\nu(X)$ over X. $|X|$ can be written as $\coprod |X_\beta|$ where (K_β) is constant for $x \in |X_\beta|$. Then $\nu(X_\beta)$ is a $(\mathbf{R}^k/K_\beta, N_{O(k)}(K_\beta)/K_\beta)$-orbibundle.

From the equivariant tubular neighborhood theorem [**2**] applied to P_gQ ([**6**, Proposition 2.1.2]) we obtain

PROPOSITION 1.15 (*orbifold tubular neighborhood theorem*). *Let Q' be a suborbifold of the paracompact orbifold Q^n. Then there is a neighborhood U of Q' in Q which is orbifold diffeomorphic to the total space of the normal bundle $\nu(Q')$ of Q'.*

Similarly one has an Orbifold Collaring Theorem which says that if W is a paracompact orbifold then $\partial_o W \times [0, 1]$ is orbifold diffeomorphic to some neighborhood of $\partial_o W$ in W. By using the Collaring Theorem and standard arguments from the manifold case [**4**, §1.3] one obtains the geometric theorems necessary for the development of orbifold cobordism. Namely, one can glue two orbifolds together along orbifold diffeomorphic orbifold boundaries to form a new orbifold and also that one can "straighten the angle" as for manifolds.

DEFINITION 1.16. Suppose Q_i, $i = 1, 2$, are two closed, oriented, n-dimensional orbifolds. We say that Q_1 *is oriented orbifold cobor-*

dant to Q_2, written $Q_1 \sim_o Q_2$, if there is an oriented orbifold W such that $Q_1 \amalg -Q_2 = \partial_o W$ with the inherited orientation.

Analogous to manifold cobordism, oriented orbifold cobordism is an equivalence relation on the set of closed, oriented, n-dimensional orbifolds. We denote the class of Q by $[Q]_o$ and the set of oriented orbifold cobordism classes of n-dimensional orbifolds by $\Omega_{n,\mathrm{orb}}$. $\Omega_{*,\mathrm{orb}}$ is a graded ring via disjoint union and the cartesian product.

2. H **characteristic numbers.** We assume henceforth that Q is a smooth, oriented, Riemannian n-dimensional orbifold and that H is a finite subgroup of $\mathrm{SO}(n)$ of degree d (Definition 3.1).

We motivate our definition of H characteristic numbers by first considering the case where H is maximal in $\mathscr{G}_Q$. Then the H singular set, Q^H (see below), is a manifold. It follows from a twisted bordism theory that some multiple m of the normal bundle ν_H of Q^H bounds another bundle E if and only if $\langle p_I(TQ^H) \cup f^*x, [Q^H]\rangle = 0$ for all $x \in H^*(BN_{\mathrm{O}(d)}(H)/H; \hat{\mathbf{Q}})$ and p_I a product of Pontrjagin classes with $4|I| + \dim x = \dim Q^H$ (Lemma 2.8). Here f is the classifying map of ν_H. A standard geometric construction using E then gives a cobordism between mQ and an orbifold Q' with $\mathscr{G}_{Q'} \subset \mathscr{G}_Q - (H)$ (Lemma 2.2).

For H nonmaximal our H characteristic numbers are orbifold characteristic numbers of the same form as above. To define these we construct an orbifold Q_H and its normal bundle ν_H which are the best orbifold approximation of the H singular set and a neighborhood of it in Q. As these numbers are orbifold cobordism invariants (Lemma 2.14), if all of them are 0 for all H we can successively apply the above cobordism to make Q rationally cobordant to a manifold whose Pontrjagin numbers are all zero and which thus rationally bounds. It then follows that some multiple of Q bounds.

DEFINITION 2.1. The H *singular set of* Q is $Q^H \overset{\text{def}}{=} \{x \in Q | (H) < (G_x)\}$.

Now suppose that H is maximal in $\mathscr{G}_Q$. Then for a chart $(U, H, \overline{U}, \pi)$ for Q with $U \cap Q^H \neq \varnothing$ we have $U \cap Q^H = \pi(\overline{U}^H)$ so Q^H is a manifold and a suborbifold. Also for $\phi\colon U \subset \overline{U}_1 \to \overline{U}_2$ an overlap map, $d\phi|_{U^H}$ is in $N_{\mathrm{O}(n)}(H)$ and hence the normal bundle of Q^H, ν_H, is an $(\mathbf{R}^d/H, N_{\mathrm{O}(d)}(H)/H)$-bundle.

LEMMA 2.2. *Suppose that* Q *is a closed orbifold and that* ν_H *bounds. That is, there is a compact manifold* Z *and a smooth*

$(\mathbf{R}^d/H, N_{O(d)}(H)/H)$-bundle ξ over Z such that (i) $\partial Z = Q^H$, (ii) $\xi|_{Q^H} = \nu_H$, *and* (iii) *the orientation of the bundle $(TZ \oplus \xi)|_{Q^H}$ is the same as that of $\nu(Q^H, Z) \oplus TQ^H \oplus \nu_H$. (Here the normal bundle of Q^H in Z, $\nu(Q^H, Z)$ has outward direction positive.) Then Q is oriented orbifold cobordant to an orbifold Q' with $\mathcal{G}_{Q'} \subset \mathcal{G}_Q - (H)$ via an orbifold W with $\mathcal{G}_W = \mathcal{G}_Q$.*

Proof. Let $h: U \to \mathbf{D}\nu_H$ be an orientation preserving orbifold diffeomorphism from a closed tubular neighborhood U of Q^H onto the total space of $\mathbf{D}\nu_H$. (In general if η is a $(\mathbf{R}^d/H, N_{O(d)}(H)/H)$-orbibundle then $\mathbf{D}\eta$ and $\mathbf{S}\eta$ denote the associated orbibundles with fiber $\mathbf{D}^d/H$, respectively $\mathbf{S}^{d-1}/H$.) Let $W = (Q \times I \cup \mathbf{D}\xi)/\sim$ where $\sim$ is defined by identifying $(x, 1)$ in $U \times \{1\}$ with $h(x) \in \mathbf{D}\nu_H \subset \mathbf{D}\xi$. Once we straighten the angle at $\mathbf{S}\nu_H$, W becomes an oriented orbifold with boundary $Q \amalg Q'$, where $Q' = (Q - (U - \partial U)) \times \{1\} \cup \mathbf{S}\xi/\sim'$ with $\sim'$ the above identification restricted to $\partial U \times \{1\}$. Since ξ is an $\mathbf{R}^d/H$ bundle over a manifold, $\mathcal{G}_{\mathbf{D}\xi} = \mathcal{G}_{\mathbf{R}^d/H} \subset \mathcal{G}_Q$ and $\mathcal{G}_{\mathbf{S}\xi} \subset \mathcal{G}_{\mathbf{R}^d/H} - (H)$ and hence $\mathcal{G}_W = \mathcal{G}_Q$ and $\mathcal{G}_{Q'} \subset \mathcal{G}_Q - (H)$. □

This leads us to consider the following twisted bordism theory. Suppose Υ is a $(\mathbf{R}^d/H, N_{O(d)}(H)/H)$-bundle over a CW complex B with finite skeleton in each dimension.

DEFINITION 2.3. Consider pairs (M^l, f), where M^l is a compact manifold and $f: M \to B$ with $TM \oplus f^*\Upsilon$ oriented. We say that two such pairs (M_i^l, f_i), $i = 1, 2$, with M_i^l closed, are *twisted cobordant*, denoted $(M_1, f_1) \sim_t (M_2, f_2)$, if there is a pair (W^{l+1}, g) as above such that (i) $\partial W = M_1 \amalg M_2$, (ii) $g|_{M_i} = f_i$, and (iii) $(TW \oplus g^*\Upsilon)|_{M_1}$ has the same orientation as $\nu(M_1) \oplus TM_1 \oplus f_1^*\Upsilon$ and $(TW \oplus g^*\Upsilon)|_{M_2}$ opposite the orientation of $\nu(M_2) \oplus TM_2 \oplus f_2^*\Upsilon$, where $\nu(M_i)$, the normal bundle of M_i in W, has outward direction as positive.

Let $[M, f]_t$ be the class of (M, f) and $\Omega_{l,t}(\Upsilon)$ the group of these cobordism classes. $\Omega_{*,t}(\Upsilon)$ is a left Ω_* module via $[N][M, f]_t = [N \times M, f\pi_2]_t$ where π_2 is projection onto the second factor and Ω_* denotes the oriented manifold cobordism ring.

LEMMA 2.4. $\Omega_{l,t}(\Upsilon) \otimes \mathbf{Q} \cong \bigoplus_{j=0}^{l} \Omega_j \otimes H_{l-j}(B; \hat{\mathbf{Q}})$.

Proof. When Υ is oriented $f^*\Upsilon$ has an induced orientation and hence M is oriented. Thus $\Omega_{l,t}(\Upsilon) = \Omega_l(B)$ and the lemma is a result of [4, Theorem 14.2]. Otherwise one traces arguments checking

orientations ([6, Lemma 4.2.3]) to obtain

$$\Omega_{l,t}(\Upsilon) \cong \pi_{l+k}(\mathcal{M}(\tilde{\gamma}^k \times_{\mathbf{Z}_2} \tilde{B}), \infty) \quad \text{for } k \geq l+2,$$

where (a) $\tilde{B}$ is the orientation double cover for Υ, and (b) $\mathbf{Z}_2$ acts by the diagonal using (a) and the orientation reversing action on $\tilde{\gamma}^k$. Here $\tilde{\gamma}^k$ denotes the canonical k-dimensional vector bundle over $\mathrm{BSO}(k)$ and $\mathcal{M}(\tilde{\gamma}^k \times_{\mathbf{Z}_2} \tilde{B})$ is the Thom space of the unoriented vector bundle $\tilde{\gamma}^k \times_{\mathbf{Z}_2} \tilde{B}$.

Standard arguments [12, 18.3], now involving the Thom isomorphism for local orientation systems, tell us that for $j < 2k-1$ there is an isomorphism between $\pi_j(\mathcal{M}(\tilde{\gamma}^k \times_{\mathbf{Z}_2} \tilde{B}), \infty) \otimes \mathbf{Q}$ and $H_{j-k}(\mathrm{BSO}(k) \times_{\mathbf{Z}_2} \tilde{B}; \hat{\mathbf{Q}})$. Calculations with -1 eigenvectors show that the latter is isomorphic to

$$\left(\bigoplus_{m=0}^{j-k} H_m(\mathrm{BO}(k); \mathbf{Q}) \otimes H_{j-k-m}(B; \hat{\mathbf{Q}})\right) \oplus \left(\bigoplus_{m=0}^{j-k} H_m(\mathrm{BO}(k); \hat{\mathbf{Q}}) \otimes H_{j-k-m}(B; \mathbf{Q})\right).$$

Since $j-k < k-1$ and $H_*(\mathrm{BO}(k); \hat{\mathbf{Q}})$ consists of sums of elements of the form $p_I e$ with the Euler class e in dimension k (Lemma 3.7), $H_m(\mathrm{BO}(k); \hat{\mathbf{Q}}) = 0$ when $0 \leq m \leq j-k$. Also $H_m(\mathrm{BO}(k); \mathbf{Q})$ is isomorphic to Ω_m in that range [12, Chapter 18] and the result follows. □

LEMMA 2.5. *$H_r(B; \hat{\mathbf{Q}})$ has a basis $\{x_{r,1}, \ldots, x_{r,\lambda_r}\}$ such that for each $1 \leq i \leq \lambda_r$ there is a singular manifold (M_i^r, f_i) with $f_{i*}[M_i^r] = x_{r,i}$.*

Proof. If Υ is oriented this is Theorem 15.3 [4]. Otherwise by that theorem $H_r(B', \mathbf{Q})$ has a basis $\{y_1, \ldots, y_{\mu_r}\}$ for which there are singular manifolds (P_i^r, g_i) with $g_{i*}[P_i^r] = y_i$. Here $\beta: B' \to B$ is the orientation double cover for the bundle Υ. One can then construct a basis $\{\tilde{x}_{r,1}, \ldots, \tilde{x}_{r,\lambda_r}\}$ for $H_r(B', \mathbf{Q})^-$ and singular manifolds $(M_i^r, \tilde{f}_i)$ with $\tilde{f}_{i*}[M_i^r] = \tilde{x}_{r,i}$, and let $f_i = \beta \tilde{f}_i$, $x_{r,i} = \beta_* \tilde{x}_{r,i}$. □

We proceed in a manner analogous to [4, Chapter II] to define invariants for this bordism theory. Let $x^{r,i} \in H^r(B; \hat{\mathbf{Q}})$ so that

$\langle x^{r,i}, x_{r,j}\rangle = \delta_{ij}$. Suppose $[M, f]_t \in \Omega_{l,t}(\Upsilon)$. For r an integer with $l \equiv r \bmod 4$ and I a partition of $(l-r)/4$ let $p_{I,r,j}[M, f]$ denote the number $\langle p_I TM \cup f^* x^{r,j}, [M]\rangle$. Standard arguments like those in [**4**, Chapter II, §17] yield that these numbers are zero if $[M, f]_t = 0$ and then define elements of $\mathrm{Hom}(\Omega_{l,t}(\Upsilon), \mathbf{Q})$.

Let $0 \le r_1 < r_2 < \cdots < r_\omega \le l$ be those integers with $l \equiv r_s \bmod 4$ and $H_{r_s}(B; \hat{\mathbf{Q}}) \neq 0$. Consider triples (s, k, K) where $1 \le s \le \omega$, $1 \le k \le \lambda_{r_s}$ and K is a partition of $(l-r_s)/4$. Let R be the square matrix $(p_{I,r_b,i}[V_J \times M_j^{r_m}, f_j\pi_2])$ where the rows are indexed lexicographically by triples (b, i, I) as above and the columns by (m, j, J). Here if $J = \{a_1, \ldots, a_k\}$ is a partition of v then V_J denotes the generator $\mathbf{C}P^{2a_1} \times \cdots \times \mathbf{C}P^{2a_k}$ of Ω_{4v}.

LEMMA 2.6. *The matrix R is nonsingular.*

Proof. For $1 \le b, m \le \omega$ let R_m^b be the submatrix of R with entries $p_{*,r_b,*}[V_* \times M_*^{r_m}, f_*\pi_2]$, and then, for $1 \le i \le \lambda_{r_b}$ and $1 \le j \le \lambda_{r_m}$ let $R_{m,j}^{b,i}$ be the submatrix of R_m^b with entries $p_{*,r_b,i}[V_* \times M_j^{r_m}, f_j\pi_2]$.

We first note that the matrix R_m^b is zero if $b > m$, since then $r_b > r_m$ and $f_j^* x^{r_b,i} \in H^{r_b}(M_j^{r_m}: \mathbf{Q})$ can only be zero for dimension reasons. If we show that R_b^b is nonsingular for all b then we will have that R is nonsingular. To do this we prove that $R_{b,j}^{b,i}$ is zero if $i \neq j$ and nonsingular if $i = j$. It can be shown that

$$p_{I,r_b,i}[V_J \times M_j^{r_m}, f_j\pi_2] = \sum_{K+L=I} \langle p_K(TM_j^{r_m}) \cup f_j^* x^{r_b,i}, [M_j^{r_m}]\rangle p_L[V_J].$$

Also $p_K(TM_j^{r_b}) \cup f_j^* x^{r_b,i} = 0$, unless $K = 0$, in which case

$$\langle p_0(TM_j^{r_b}) \cup f_j^* x^{r_b,i}, [M_j^{r_b}]\rangle = \langle x^{r_b,i}, f_{j*}[M_j^{r_b}]\rangle = \delta_{ij}.$$

Hence $R_{b,j}^{b,i}$ is the zero matrix if $i \neq j$, and when $i = j$ the matrix with entries $p_I[V_J]$, which by Theorem 16.8 [**10**] is nonsingular. □

LEMMA 2.7. *If all $p_{I,r,j}[M, f]$ are zero then some multiple of $[M, f]_t$ is zero.*

Proof. Since R is nonsingular the $p_{I,r_b,i}$ are linearly independent and thus by dimensional considerations from Lemma 2.4 form a basis for $\mathrm{Hom}(\Omega_{l,t}(\Upsilon), \mathbf{Q})$. From this it follows that if all the

$p_{I,r_b,i}[M, f]$ are zero then $[M, f]_t$ is zero in $\Omega_{l,t}(\Upsilon) \otimes \mathbf{Q}$. In addition we get that the $[V_J \times M_j^{r_m}, f_j\pi_2]_t$ are linearly independent and so form a basis of $\Omega_{l,t}(\Upsilon) \otimes \mathbf{Q}$. We call such a basis formed in this way a *preferred basis.* □

For H maximal in $\mathscr{G}_Q$, ν_H has a classifying map $f\colon Q^H \to BN_{O(d)}(H)/H$ and so our H characteristic numbers are defined by $\langle p_I(TQ^H) \cup f^*x, [Q^H]\rangle$ for $x \in H^{n-d-4s}(B(N_{O(d)}(H)/H); \hat{\mathbf{Q}})$ and I a partition $(i_1, \ldots, i_m)$ of s. These numbers are linear combinations of the $p_{I,r,j}[Q^H, f]$'s.

LEMMA 2.8. *If all the H characteristic numbers of Q are zero then some nonzero multiple of ν_H bounds as in Lemma 2.2.*

Proof. This follows from the previous lemma where Υ is the canonical $(\mathbf{R}^d/H, N_{O(d)}(H)/H)$ bundle over $BN_{O(d)}(H)/H$. If (M, f) bounds (W, g) we may assume that g is smooth and hence that $g^*\Upsilon$ is smooth. □

When H is nonmaximal Q^H need not be an orbifold as there may be crossings at those points with local groups larger than (H), (e.g. $\langle e^{2\pi i/p}\rangle \times \langle 1\rangle < \langle e^{2\pi i/p}\rangle \times \langle e^{2\pi i/p}\rangle$) which we break apart as follows. For every $x \in Q^H$ choose precisely one linear chart $(U_x, G_x, \overline{U}_x, \pi_x)$ with $H < G_x$ and $\pi_x(0) = x$. Let $[H_1^x], \ldots, [H_{k_x}^x]$ be those distinct conjugacy classes of G_x with H_i^x conjugate to H in $O(n)$ and $H_1^x = H$.

DEFINITION 2.9. $|Q_H| = \{(x, [H_i^x])\colon x \in Q^H, 1 \le i \le k_x\}$.

This construction is a refinement of $\tilde{\Sigma}X$ in [**9**, §1]. One can give explicit charts for Q_H [**6**, §3.1; **9**, §1]. Locally Q_H looks like $\overline{U}^{H_i^x}/(N_{G_x}(H_i^x)/K_i^x)$ where $K_i^x = \{k \in N_{G_x}(H_i^x)\colon ky = y \text{ for all } y \in \overline{U}_x^{H_i^x}\}$. The overlap maps are essentially restrictions of those of Q.

REMARK 2.10. (i) We can write Q_H as $\coprod_{0\le j\le r} Q_H^j$ where Q_H^j is the set of elements $(x, [H_i^x])$ of Q_H such that (K_i^x) is constant. We choose a representative K_j of this class so that $K_j > H$. We also assume that $K_0 = H$. Thus $Q_H^0 = \varnothing$ if $H \notin \mathscr{G}_Q$. (ii) If (H) is maximal in $\mathscr{G}_Q$, then $Q^H = Q_H = Q_H^0$. (iii) Since $\partial(\overline{U}_x^{K_i^x}) = \partial(\overline{U}_x)^{K_i^x}$, $\partial_o(Q_H^i) = (\partial_o Q)_H^i$.

We have that $T\overline{U}_{x|\overline{U}^{H_i^x}} = T\overline{U}_x^{H_i^x} \oplus \nu(\overline{U}_x^{H_i^x})$ and the differentials of the overlap maps and of the elements of $N_{G_x}(H_i^x)$ factors accordingly. Define the normal orbibundle of Q_H to be

$$\nu_H = \Big(\coprod_{x \in Q^H,\, 1 \le i \le k_x} \nu(\overline{U}_x^{H_i^x})/N_{G_x}(H_i^x) \Big) \Big/ \sim$$

where $\overline{(y, v)} \sim \overline{(\phi(y), d\phi_2 v)}$ for all appropriate overlap maps ϕ and $d\phi_2$ denotes the differential of ϕ in the normal direction. $\nu_{H|_{Q_H^j}}$, abbreviated ν_H^j, is a $(\mathbf{R}^d/K_j, N_{O(d)}(K_j)/K_j)$-orbibundle.

$T\nu_H$ is oriented as inherited from Q; however, neither ν_H nor Q_H need be orientable or even locally orientable as the following two examples show.

EXAMPLE 2.11. Let the dihedral group D_{2p} act on $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ by $\alpha(x_1, x_2, x_3) = (x_1, -x_2, -x_3)$ and $\beta(x_1, x_2, x_3) = \beta(z, x_3) = (e^{2\pi i/p} z, x_3)$. Then $Q = \mathbf{R}^3/D_{2p}$ is orientable, but $Q_{\mathbf{Z}_p} = \mathbf{R}^1/(x \sim -x)$ is not locally orientable at $\{0\}$.

EXAMPLE 2.12. Let Q' be the quotient of $\mathbf{S}^2 \times \mathbf{C}$ by the D_{2p} action: $\alpha(x, v) = (-x, \overline{v})$, $\beta(x, v) = (x, e^{2\pi i/p} v)$. Q' is orientable. $Q'_{\mathbf{Z}_p} = \mathbf{S}^2/D_{2p} = RP^2$ is locally orientable, but not orientable.

Note that in both of these examples $N_{O(2)}(\mathbf{Z}_p) = O(2)$. However, if $N_{O(d)}(H)$ is a subgroup of $SO(d)$ then ν_H has a natural orientation and so Q_H is oriented.

There is one exception to this induced orientation. In declaring $T\nu_H$ oriented as inherited from Q one is assuming that the chart $(U_x, G_x, \overline{U}_x, \pi_x)$ with $H < G_x$ in the definition of Q_H came from the orientation atlas of Q, which is always possible unless degree H is n. In the case where we cannot have $H < G_x$ we give the point x an orientation of -1 in Q_H. This makes sense since if H has degree n and is conjugate to H' in $O(n)$, but not $SO(n)$, one can show that $\mathbf{R}^n/H$ is oriented orbifold diffeomorphic, and hence cobordant, to $-\mathbf{R}^n/H'$.

DEFINITION 2.13. Let $\beta\colon B(N_{SO(d)}(H)/H) \to B(N_{O(d)}(H)/H)$ be the canonical orbit map. For $x \in H^{n-d-4s}(B(N_{O(d)}(H)/H); \widehat{\mathbf{Q}})$ and I a partition $(i_1, \ldots, i_m)$ of s define $p_{I,x}[Q]$ to be the number

(i) $\langle p_I(T_H^0) \cup x(\nu_H^0), [Q_H^0] \rangle$ if Q_H^0 is locally oriented,

(ii) $\frac{1}{2}\langle p_I(T\widetilde{Q}_H^0) \cup \beta^* x(\tilde{\nu}_H^0), [\widetilde{Q_H^0}] \rangle$ otherwise.

Note that the two definitions agree when Q_H^0 is locally orientable. The set of all such numbers, for all x and I, are called the H characteristic numbers of Q. Also when H is maximal these are the same as the numbers defined for that case.

LEMMA 2.14. *If $Q = \partial_o W$ where W is an oriented orbifold then $p_{I,x}[Q] = 0$ for all H, and pairs (I, x).*

Proof. By a standard argument and Lemma 1.13 $p_I(TQ_H^0) = p_I(i^*TW_H^0) = i^*p_I(TW_H^0)$. Similarly $x(\nu_H^0(Q)) = i^*x(\nu_H^0(W))$. Therefore $p_{I,x}[Q] = \langle i^*(p_I(TW_H^0) \cup x(\nu_H^0(W))), [Q_H^0]\rangle$ which in turn is zero since $[Q_H^0] = \partial[W_H^0]$ and $i_*\partial = 0$. □

Using $Q \times I$ with the appropriate metrics at the ends it follows from the above proposition that these numbers are independent of the metric on Q. Also it is easy to see that the $p_{I,x}$'s define additive homomorphisms from $\Omega_{*,\mathrm{orb}}$ to $\mathbf{Q}$. If H, H' are finite conjugate subgroups of $O(n)$, one can show [**6**, after Lemma 3.2.7] that via the conjugation the set of H characteristic numbers of Q is equal to the set of H' characteristic numbers of Q.

THEOREM 2.15. *If all the H characteristic numbers of Q are zero for all finite subgroups H of $\mathrm{SO}(n)$ then some multiple of Q orientably bounds another orbifold W with $\mathscr{G}_Q = \mathscr{G}_W$.*

Proof. Let H be maximal in $\mathscr{G}_Q$. Since the H characteristic numbers are all zero, by Lemmas 2.2 and 2.8 some multiple r_1 of Q is cobordant to an orbifold Q_1 with $\mathscr{G}_{Q_1} \subset \mathscr{G}_Q - (H)$ via an orbifold W_1 with $\mathscr{G}_{W_1} = \mathscr{G}_Q$. Since $Q_1 \sim_o r_1 Q$, by the discussion following Lemma 2.14 all the H' characteristic numbers of Q_1 must also be zero for all $H' \in \mathscr{G}_{Q_1}$.

Let $s = \mathrm{card}(\mathscr{G}_Q)$. Proceeding as above we get an oriented Q_s with $Q_s \sim_o r_{s-1}Q_{s-1} \sim_o \cdots \sim_o r_{s-1}\cdots r_1 Q$ and $\mathscr{G}_{Q_s} = \{e\}$. Thus Q_s is a manifold whose Pontrjagin numbers are all zero and so $2Q_s$ bounds a manifold W_s. Therefore $2r_{s-1}r_{s-2}\cdots r_1 Q$ orientably bounds the orbifold

$$W = 2r_{s-1}r_{s-2}\cdots r_2 W_1 \cup \cdots \cup_{2r_{s-1}Q_{s-1}} 2W_{s-1} \cup_{2Q_s} W_s$$

and $\mathscr{G}_W = \mathscr{G}_Q$. □

3. Analysis of the numbers. In this section we first collect a few definitions concerning real linear groups and representations. Lemmas

3.3 and 3.4 show how the centralizer $C_{O(d)}(H)$ decomposes as a product of orthogonal, unitary and symplectic groups and thus by Lemma 3.7 how $H^*(BC_{O(d)}(H); \hat{\mathbf{Q}})$ is generated by Pontrjagin, Chern, and twisted Euler classes. This enables us to prove that $H^s(BC_{O(d)}(H); \hat{\mathbf{Q}}) = 0$ if $s+d$ is odd (Lemma 3.8) and then, since $H^*(BN_{O(d)}(H)/H; \hat{\mathbf{Q}}) < H^*(BC_{O(d)}(H); \hat{\mathbf{Q}})$ (Lemma 3.9), that every odd-dimensional orbifold rationally bounds. The $4j+2$ case often requires further information about $N_{O(d)}(H)/H$ as we indicate by considering special cases in Theorem 3.11. Finally we use a group considered in this theorem (a certain representation of S_8) to construct a $(4j+2)$-dimensional oriented orbifold which does not rationally bound.

DEFINITION 3.1. A subgroup K of $\mathrm{Gl}(k, \mathbf{R})$ defines a real faithful representation of the abstract group K. K is *irreducible* if this representation is irreducible. Suppose K_i, $i = 1, 2$, are two linear groups and $\phi_i: G \to \mathrm{Gl}(k_i, \mathbf{R})$ are two representations of a group G with $K_i = \varphi_i(G)$. Let $\beta = (\varphi_1, \varphi_2)$. By $K_1 \oplus_\beta K_2$ we mean the linear group $(\varphi_1 \oplus \varphi_2)(K)$. We will omit the subscript β when the resulting ambiguity has no effect. K^m means $\phi(K)^{\oplus m}$ for some faithful representation ϕ of K. If K is a subgroup of $\mathrm{Gl}(k, \mathbf{R})$ and has no trivial summands then the *degree* of K is k.

DEFINITION 3.2. A real irreducible linear group is of *symplectic* (*or* $\mathbf{H}$) *type* if it has degree $4k$ and is conjugate in $\mathrm{Gl}(4k, \mathbf{R})$ to a subgroup of $\mathrm{Gl}(k, \mathbf{H})$. It is of *complex* (*or* $\mathbf{C}$) *type* if it has degree $2k$ and is conjugate in $\mathrm{Gl}(2k, \mathbf{R})$ to a subgroup of $\mathrm{Gl}(k, \mathbf{C})$ but is not of symplectic type. Otherwise it is of *real* (*or* $\mathbf{R}$) *type*. A real linear group is of real, complex, or symplectic type if all its irreducible components are of that type.

For $L = \mathbf{R}, \mathbf{C}, \mathbf{H}$ and i any positive integer let $L(i) = \mathrm{O}(i), \mathrm{U}(i), \mathrm{Sp}(i)$, and $d(L) = 1, 2, 4$, respectively. Let H be a finite subgroup of $\mathrm{O}(d)$ of degree d. Schur's Lemma , Theorems 1.10 and 6.7 of Chapter II, [3], imply the following two lemmas.

LEMMA 3.3. *If H is irreducible of type L then $C_{O(md)}(H^m) \cong L(m)$. Here we are identifying $L(1) < L(d/d(L))$, with the group of scalar matrices in $L(d/d(L))$, and thus with a subgroup of $\mathrm{O}(d)$. Note that under this identification $\mathrm{O}(1) < \mathrm{SO}(d)$ iff d is even.*

LEMMA 3.4. *In general we can write H as $H = \bigoplus_{i=1}^r K_i^{m_i}$ where K_i are nonequivalent irreducible representations of H in $\mathrm{O}(d_i)$. Then $C_{O(d)}(H) \cong \times_{i=1}^r C_{O(m_i d_i)}(K_i^{m_i})$.*

REMARK 3.5. From the above lemmas we conclude that $C_{O(d)}(H) < SO(d)$ if and only if H has no irreducible direct summand of odd degree.

DEFINITION 3.6. If all the irreducible components of H are of even, respectively odd degree then H is of *even*, respectively, *odd type*. Note that an odd type group must be of real type.

LEMMA 3.7. (i) $H^*(BL(m); \mathbf{Q})$ *is a polynomial algebra with generators*

(a) *the symplectic Pontrjagin classes* x_i *of* $\dim 4i$, $i = 1, \ldots, m$, *when* $L = \mathbf{H}$,

(b) *the Chern classes* c_i *of* $\dim 2i$, $i = 1, \ldots, m$, *when* $L = \mathbf{C}$,

(c) *the Pontrjagin classes* p_i *of* $\dim 4i$, $i = 1, \ldots, [m/2]$, *when* $L = \mathbf{R}$.

(ii) $H^*(BO(2m+1); \hat{\mathbf{Q}}) = 0$ *and* $H^*(BO(2m); \hat{\mathbf{Q}})$ *is the graded vector space with basis* $p_{i_1} \cdots p_{i_j} e$ *where* $1 \leq i_k \leq m$ *for* $1 \leq k \leq j$ *and* e *is the twisted Euler class of dimension* $2m$.

Proof. (i) is a standard result found in [**1**, **8**]. The classes in (ii) are the -1 eigenvectors of α^* where α is the orientation reversing involution on $BSO(j)$. □

LEMMA 3.8. *If* $H^s(BC_{O(d)}(H); \hat{\mathbf{Q}})$ *is nonzero then* $d+s$ *is even.*

Proof. Write H as in Lemma 3.4. Then by that lemma

$$H^*(BC_{O(d)}(H); \hat{\mathbf{Q}}) \cong \bigotimes_{1 \leq i \leq r} H^*(BC_{O(m_i d_i)}(K_i^{m_i}); \hat{\mathbf{Q}}).$$

If H is of even type then by Remark 3.5 the orientation systems of the canonical bundles over the $BC_{O(m_i d_i)}(K_i^{m_i})$'s are all trivial. Then by Lemmas 3.3, 3.4 and 3.7(i) each nonzero $H^s(BC_{O(m_i d_i)}(K_i^{m_i}); \hat{\mathbf{Q}})$ is even dimensional and so $s+d$ is even.

If H is of odd type $H^*(BC_{O(m_i d_i)}(K_i^{m_i}); \hat{\mathbf{Q}}) \cong H^*(BO(m_i); \hat{\mathbf{Q}})$ and we conclude [**6**, §5.1] that $H^*(BC_{O(d)}(H); \hat{\mathbf{Q}})$ is 0 if some m_i is odd and otherwise $\mathbf{Q}\langle p_I e_1 \cdots e_r \rangle$, the graded rational vector space with basis $p_I e_1 \cdots e_r$, the e_i's denoting the Euler classes (of $\dim m_i$) and p_I any appropriate product of Pontrjagin classes. Thus if $H^s(BC_{O(d)}(H); \hat{\mathbf{Q}}) \neq 0$, $d+s = \sum_{i=1}^r d_i m_i + 4|I| + \sum_{i=1}^r m_i$, which is divisible by four.

If H is of neither type we can write $H \cong H_1 \oplus H_2$ where H_1 is of odd type and H_2 is of even type and combine the above results for our lemma. □

LEMMA 3.9. *Let H be a finite subgroup of $\mathrm{SO}(d)$. Then*

(i) *The projection $\pi: N_{\mathrm{O}(d)}(H) \to N_{\mathrm{O}(d)}(H)/H$ induces an isomorphism $B\pi^*: H^*(B(N_{\mathrm{O}(d)}(H)/H); \widehat{\mathbf{Q}}) \to H^*(B(N_{\mathrm{O}(d)}(H)); \widehat{\mathbf{Q}})$*

(ii) *The inclusion $i: C_{\mathrm{O}(d)}(H) \to N_{\mathrm{O}(d)}(H)$ induces an isomorphism of $H^*(B(N_{\mathrm{O}(d)}(H)); \widehat{\mathbf{Q}})$ onto a subgroup of $H^*(B(C_{\mathrm{O}(d)}(H)); \widehat{\mathbf{Q}})$.*

Proof. (i) Consider the fibration

$$BH \to BN_{\mathrm{O}(d)}(H) \to B(N_{\mathrm{O}(d)}(H)/H)$$

of classifying spaces. Since H is finite $H^*(BH; \mathbf{Q}) = 0$. Also H is orientation preserving, and so the result follows by the Serre spectral sequence.

(ii) $N_{\mathrm{O}(d)}(H)/C_{\mathrm{O}(d)}(H)$ is finite as it embeds in $\mathrm{Aut}(H)$. Thus we get the covering space $N_{\mathrm{O}(d)}(H)/C_{\mathrm{O}(d)}(H) \to BC_{\mathrm{O}(d)}(H) \xrightarrow{Bi} BN_{\mathrm{O}(d)}(H)$ and (ii) from transfer theory. □

THEOREM 3.10. *If Q is an odd dimensional oriented orbifold then some multiple of Q bounds an oriented orbifold W with $\mathscr{G}_W = \mathscr{G}_Q$.*

Proof. The H characteristic number $p_{I,x}[Q]$ is defined when $4|I| + \dim x = \dim Q - \deg H$. Thus, if $\dim Q$ is odd, $\dim x + \deg H$ is odd, and by the previous two lemmas $x = 0$. Consequently all the orbifold characteristic numbers of Q are zero and our result follows from Theorem 2.15. □

We now turn to the $(4j+2)$-dimensional case. Let $\mathscr{H}_f$ be the set of all finite subgroups of $\mathrm{SO}(m)$ where m is allowed to vary over all positive integers. Let $\mathscr{K}$ denote the set of elements H of $\mathscr{H}_f$ such that $H^s(BN_{\mathrm{O}(d)}(H); \widehat{\mathbf{Q}})$ is nonzero implies $s+d$ is divisible by four. Here d is the degree of H. As in Theorem 3.10 if $\dim Q$ is not divisible by four and $\mathscr{G}_Q \subset \mathscr{K}$ then some multiple of Q bounds.

THEOREM 3.11. *Suppose $H \in \mathscr{H}_f$.*

(i) *If H is of odd type then $H \in \mathscr{K}$.*

(ii) *$H_1, H_2 \in \mathscr{K}$ implies $H_1 \times H_2 \in \mathscr{K}$.*

(iii) *If H is of complex type and conjugate to a group which is invariant under complex conjugation then $H \in \mathscr{K}$.*

(iv) *If H is a direct sum of symplectic type linear groups and groups of the form K^m with K irreducible, of real type and* $\deg K$ *odd or* $\deg K^m$ *divisible by four, then H is in $\mathscr{H}$.*

(v) *All abelian elements of $\mathscr{SO}_f$ belong to $\mathscr{H}$.*

(vi) *$\mathscr{H}$ is a proper subset of $\mathscr{SO}_f$.*

Proof. (i) This follows from the proof of Lemma 3.8.

(ii) Let $d_i = \deg H_i$. $N_{\mathrm{O}(d_1+d_2)}(H_1 \times H_2)$ has $C_{\mathrm{O}(d_1+d_2)}(H_1 \times H_2)$ $(\cong C_{\mathrm{O}(d_1)}(H_1) \times C_{\mathrm{O}(d_2)}(H_2))$ and therefore $N_{\mathrm{O}(d_1)}(H_1) \times N_{\mathrm{O}(d_2)}(H_2)$ as a finite index subgroup. By transfer $H^*(BN_{\mathrm{O}(d_1+d_2)}(H_1 \times H_2); \widehat{\mathbf{Q}})$ is contained in $H^*(BN_{\mathrm{O}(d_1)}(H_1); \widehat{\mathbf{Q}}) \otimes H^*(BN_{\mathrm{O}(d_2)}(H_2); \widehat{\mathbf{Q}})$.

(iii) Write H as in Lemma 3.4 where now each K_j is of complex type. We may also assume that this direct sum is closed under complex conjugation in $\mathrm{O}(d)$, which is conjugation by the diagonal matrix, α, with $(1, -1, \ldots, 1, -1)$ down the diagonal. Thus $N_{\mathrm{O}(d)}(H)$ contains the semidirect product $(\times_{j=1}^{s} U(m_j)) \times_\theta \langle\alpha\rangle$ and hence $H^*(BN_{\mathrm{O}(d)}(H); \widehat{\mathbf{Q}}) < H^*(B((\times_{j=1}^{s} U(m_j)) \times_\theta \langle\alpha\rangle); \widehat{\mathbf{Q}})$. Since $\alpha \in \mathrm{SO}(d)$ if and only if d is divisible by four, we can show that $H^*(B((\times_{j=1}^{s} U(m_j)) \times_\theta \langle\alpha\rangle); \widehat{\mathbf{Q}})$ is the graded vector space with basis consisting of those $c_{I_1} \cdots c_{I_s}$ with $\dim c_{I_1} \cdots c_{I_s} \equiv d \bmod 4$, where c_{I_j} is any product of Chern classes in $H^*(BU(m_j); \mathbf{Q})$.

(iv) Write H as $K_1 \oplus K_2 \oplus K_3$ where K_1 is of symplectic type, K_2 is of real and even type, and K_3 is of real and odd type. Let $d_i = \deg K_i$ and s_i be an integer such that $H^{s_i}(BC_{\mathrm{O}(d_i)}(K_i); \widehat{\mathbf{Q}}) \neq 0$. Both d_1 and d_2 are divisible by four, as are s_1 and s_2 from Lemma 3.7 (i) and Remark 3.5. By the proof of Lemma 3.8 $d_3 + s_3$ is also divisible by four.

(v) Every irreducible abelian group in $\mathscr{SO}_f$ is either of complex type and acts on $\mathbf{C}$, or $\mathbf{Z}_2$ and acts on $\mathbf{R}$. Hence we can write any abelian group H in $\mathscr{SO}_f$ as $H_1 \oplus H_2$ where H_1 is of complex type of degree d_1 and H_2 is a direct sum of $\phi_j((\mathbf{Z}_2)^m)$'s with ϕ_j irreducible. By taking a conjugate in $\mathrm{O}(d_1)$ we may assume that H_1 is closed under conjugation by the matrix α_{d_1}, as in (iii) and so the matrix $\alpha_{d_1} \oplus \mathrm{id}_{d_2}$ is an element of $N_{\mathrm{O}(d)}(H)$. Also ϕ_j has degree one, so we combine (iii) and (iv) to obtain our result.

(vi) For $H \in \mathscr{SO}_f$ irreducible of real, even type

$$H^*(BC_{\mathrm{O}(md)}(H^m); \widehat{\mathbf{Q}}) = \mathbf{Q}[p_I].$$

If md is not divisible by four there is not enough information to

draw the conclusion of (iv). For example, there may be no orientation reversing involution in $N_{O(md)}(H^m)$ to make our coefficient system nontrivial and hence kill off the Pontrjagin classes. If one had such an involution of an orientation preserving linear group H with $C_{O(d)}(H) = C_{SO(d)}(H)$ it would have to be an outer automorphism of H. This tells us that a possible counterexample might be S_n, $n \neq 2, 6$, as these groups are complete. In fact S_8 [**11**, character table] does have an irreducible, faithful character χ of degree seventy with $\chi(\sigma) = 10$ for σ a transposition. Furthermore all symmetric group representations have real type so this can be realized by a subgroup H of $O(70)$. If A is a transposition in H $\operatorname{tr}(A) = 10$ and $\deg H = 70$ implies that A is orientation preserving. As H is generated by transpositions we conclude that H is orientation preserving and hence an element of $\mathscr{HO}_f$. Since H is complete, $H^*(B(N_{O(70)}(H)/H); \hat{\mathbf{Q}}) = H^*(BC_{O(70)}(H); \mathbf{Q}) = \mathbf{Q} = \mathbf{Q}(p_0)$. □

COROLLARY 3.12. *There is a* $(4j+2)$*-dimensional oriented orbifold which does not rationally bound.*

Proof. Let $V = \mathbf{D}^{70}/H$, with H as in the proof of (vi) and the standard orientation on $\mathbf{D}^{70}$. V has orbifold boundary $S = \mathbf{S}^{69}/H$. Since $S = \partial_o V$, Lemma 2.14 and Theorem 2.15 imply that some multiple k of S bounds an oriented orbifold W with $\mathscr{G}_W = \mathscr{G}_S$. Also $\mathscr{G}_S \subset \mathscr{G}_V - (H)$. We form the oriented orbifold $Q^{70} = kV \cup_{kS} W$ and note that the only H characteristic numbers are rational multiples of $p_0[Q^H]$, the orientation weighted number of points labeled H. There are k such points—the origins of the k V's—and they all have the same orientation. Thus $p_0[Q^H]$ is nonzero and so by Lemma 2.14 no multiple of Q bounds. □

4. Generators of $\Omega_{*,\mathrm{orb}} \otimes \mathbf{Q}$. Following the proof of Corollary 3.12 we construct a basis P_n for $\Omega_{n,\mathrm{orb}} \otimes \mathbf{Q}$ and show that $\Omega_{*,\mathrm{orb}} \otimes \mathbf{Q}$ is a free commutative algebra over $\Omega_* \otimes \mathbf{Q}$ with generators a certain subset of the bases P_n, $n \geq 0$.

Let $\mathscr{K}_n$ be the set of conjugacy classes (H) in $O(n)$ of those finite subgroups H of $SO(n)$ with $H^{n-4s-d}(B(N_{O(d)}(H)/H); \hat{\mathbf{Q}}) \neq 0$ for some $s \geq 0$, where $d = \deg H$. Note that $\mathscr{K}_{2m+1}$ is empty by Lemma 3.8.

For each $(H) \in \mathscr{K}_n$ let Γ_H denote the canonical $(\mathbf{R}^d/H, N_{O(d)}(H)/H)$ bundle over $B(N_{O(d)}(H)/H)$ and $\mathscr{G}_H = \mathscr{G}_{\mathbf{R}^d/H}$. Consider the vector space $\Omega_{n-d,t}(\Gamma_H) \otimes \mathbf{Q}$ and its basis $[V_J \times M_j^{r_m}, f_j \pi_2]_t$

indexed by (m, j, J), as in §2 and the proof of Lemma 2.7. Let Υ denote $(f_j\pi_2)^*\Gamma_H$. Since $S\Upsilon = \partial_o \mathbf{D}\Upsilon$ Lemma 2.14 and Theorem 2.15 imply that a nonzero multiple k of $S\Upsilon$ bounds an oriented orbifold W with $\mathscr{G}_W = \mathscr{G}_H - (H)$. We let $Q(H, m, j, J) = k\mathbf{D}\Upsilon \cup_{kS\Upsilon} W$ and define a linear map

$$\Phi\colon S \overset{\text{"def"}}{=} \bigoplus_{\mathscr{K}_n} \Omega_{n-d,t}(\Gamma_H) \otimes \mathbf{Q} \to \Omega_{n,\text{orb}} \otimes \mathbf{Q}$$

by mapping $k[V_J \times M_j^{r_m}, f_j\pi_2]_t$ in $\Omega_{n-d,t}(\Gamma_H)\otimes\mathbf{Q}$ to $Q(H, m, j, J)$.

PROPOSITION 4.1. *Φ is a rational vector space isomorphism.*

Proof. If we view S as the vector space with basis indexed by H characteristic classes $p_{I,r_b,i}$ with H ranging over $\mathscr{K}_n$ we see by Lemma 2.14 and Theorem 2.15 that there is an injective linear map λ from $\Omega_{n,\text{orb}} \otimes \mathbf{Q}$ to S sending an orbifold to its H characteristic numbers. We note that $\lambda\Phi|_{\Omega_{n-d,t}(\Gamma_H)\otimes\mathbf{Q}}$ is injective. Indeed, for $x \in H^{n-d-4|I|}(B(N_{\mathrm{O}(d)}(H)/H); \widehat{\mathbf{Q}})$, and $u = \sum_i^k \alpha_i[V_{J_i} \times M_{j_i}^{r_{m_i}}, f_{j_i}\pi_2] \in \Omega_{n-d,t}(\Gamma_H) \otimes \mathbf{Q}$,

$$p_{I,x}(\Phi(u)) = \sum_{i=1}^{k} \alpha_i \langle p_I(TV_{J_i} \times M_{j_i}^{r_{m_i}}) \cup (f_{j_i}\pi_2)^* x, [V_{J_i} \times M_{j_i}^{r_{m_i}}]\rangle.$$

By Lemma 2.6 if the α_i's are not all zero there is an x for which $p_{I,x}(\Phi(u)) \neq 0$. For any $(K) \in \mathscr{K}_n$ $\lambda\Phi$ maps the finite dimensional vector space $S_K = \bigoplus_{(H)\in\mathscr{G}_K} \Omega_{n-d,t}(\Gamma_H) \otimes \mathbf{Q}$ to itself. Also by the above paragraph and the fact that for $(H) \in \mathscr{G}_K$ $Q(H, m, j, J)_K$ is the empty set when $(H) \neq (K)$ and $k(V_J \times M_j^{r_m})$ when $(H) = (K)$, we see that $\lambda\Phi$ so restricted is an isomorphism of S_K. Consequently $\lambda\Phi$ and hence Φ are isomorphisms. □

Let P_n be the set of all $Q(H, m, j, J)$ where $(H) \in \mathscr{K}_n$ and $[V_J \times M_j^{r_m}, f_j\pi_2]_t$ is a preferred basis for $\Omega_{n-d,t}(\Gamma_H) \otimes \mathbf{Q}$. By the above lemma P_n is a basis for $\Omega_{n,\text{orb}} \otimes \mathbf{Q}$.

$Q(H, m, j, J)$ is dependent on k and W but if we choose another pair k' and W' and let $Q'(H, m, j, J) = k'\mathbf{D}\Upsilon \cup_{k'S\Upsilon} W'$ we see that $kQ'(H, m, j, J) \sim_o k'Q(H, m, j, J) \amalg X$ where $\mathscr{G}_X \subset \mathscr{G}_H - (H)$.

Let $O_n = \{Q(H, m, j, 0) \in P_n\colon$ H is not a direct product (as a linear group)$\}$.

PROPOSITION 4.2. *$\Omega_{*,\mathrm{orb}} \otimes \mathbf{Q}$ is a free commutative algebra over $\Omega_* \otimes \mathbf{Q}$ with generating set $\{O_n, n \geq 0\}$.*

Proof. By the preceding discussion we may assume that

$$Q(H, m, j, J) = V_J \times Q(H, m, j, 0).$$

Also $\Omega_{2k+1,\mathrm{orb}} \otimes \mathbf{Q} = 0$, so $[Q]_o[Q']_o = [Q']_o[Q]_o$ in $\Omega_{*,\mathrm{orb}} \otimes \mathbf{Q}$. By the proof of Theorem 3.11 (ii) $H_*(B(N_{\mathrm{O}(d+d')}(H \times H')/(H \times H')); \hat{\mathbf{Q}})$ is a subgroup of $H_*(B(N_{\mathrm{O}(d)}(H)/H); \hat{\mathbf{Q}}) \otimes H_*(B(N_{\mathrm{O}(d')}(H')/H'); \hat{\mathbf{Q}})$. Then by Lemma 2.4 some multiple of $[M_j^{r_m}, f_j]_t \in \Omega_{r_m,t}(\Gamma_{H\times H'})$ lifts to $\Omega_{r_m,t}(\Gamma_H \times \Gamma_{H'})$ and so can be written as

$$\sum \alpha_i [N_{j_i}^{n_i} \times P_{j_i'}^{l_i}, \beta(g_{j_i} \times h_{j_i'})]_t.$$

Here $[N_{j_i}^{n_i}, g_{j_i}]_t$ is a preferred basis element for $\Omega_{n_i,t}(\Gamma_H)$, $[P_{j_i'}^{l_i}, h_{h_i'}]_t$ one for $\Omega_{l_i,t}(\Gamma_{H'})$ and

$$\begin{aligned}\beta\colon\ & B(N_{\mathrm{O}(d)}(H)/H) \times B(N_{\mathrm{O}(d')}(H')/H') \\ & \to B(N_{\mathrm{O}(d+d')}(H \times H')/(H \times H'))\end{aligned}$$

is the natural map. Hence as at the beginning of the proof we may assume that $[Q(H \times H', r, j, 0)]_o$ is equal to $\sum \alpha_i [Q(H, n_i, j_i, 0)]_o \cdot [Q(H', l_i, j_i', 0)]_o$. □

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