PACIFIC JOURNAL OF MATHEMATICS
Vol. 165, No. 1, 1994

ON CLOSED HYPERSURFACES OF CONSTANT SCALAR CURVATURES AND MEAN CURVATURES IN S^{n+1}

SHAOPING CHANG

We consider in this note the following question: given a closed Riemann n-manifold of constant scalar curvature, how can it be minimally immersed in the round $(n+1)$-sphere? Our main result states that the immersion has to be isoparametric if the number of its distinct principal curvatures is three identically. This provides another piece of supporting evidence to a conjecture of Chern.

0. Introduction. Consider $\mathscr{M}^n_{\text{closed}}$ the set of all the closed minimal hypersurfaces of constant scalar curvatures R in the unit round $(n+1)$-sphere S^{n+1}. Let $\mathscr{R}_n \subset \mathbf{R}$ be the collection of all the possible values of such R's. Chern **[12]** posed the following:

Chern Conjecture. For any $n \geq 3$, $\mathscr{R}_n$ is a discrete subset of the real numbers.

This is a very interesting conjecture in the theory of minimal submanifolds in spheres. To attack this problem, it will be most helpful if one has a good guess on what $\mathscr{M}^n_{\text{closed}}$ is for each n. When $n = 3$, from his work on the exterior differential systems R. Bryant **[1]** proposed the following:

Bryant Conjecture. A piece of minimal hypersurface of constant scalar curvature in S^4 is isoparametric of type $g \leq 3$.

Here a hypersurface (not necessarily compact) M^n in S^{n+1} is said to be *isoparametric of type* g if it has constant principal curvatures $\lambda_1 < \cdots < \lambda_g$ with respective constant multiplicities $m_1, \ldots, m_g$. Such hypersurfaces with $g \leq 3$ are classified due to Cartan's work **[2]** in 1939.

Note that the Bryant conjecture is very strong because M^3 is not assumed to be closed. Nevertheless, there is good evidence that it may be true. In **[3]**, together with the works of Simons **[11]** and Peng-Terng **[10]**, the author was able to establish the Chern Conjecture when $n = 3$ by showing that each $M^3 \in \mathscr{M}^3_{\text{closed}}$ is an isoparametric hypersurface. Hence, $\mathscr{R}_3 = \{0, 3, 6\}$. Also, the Bryant Conjecture was verified when M^3 has multiple principal curvatures somewhere.

Therefore, we would like to pursue such a point of view for the study of $\mathscr{M}^n_{\text{closed}}$ in higher dimensions. Suppose that M^n also satisfies the following:

Condition (g): The number g of distinct principal curvatures is constant.

Recall that there is one minimal hypersurface among each family of isoparametric hypersurfaces (cf. [**9**]). All the closed minimal isoparametric hypersurfaces by definition are members of $\mathscr{M}^n_{\text{closed}}$ and satisfy Condition (g). Conversely, it is straightforward to check that any $M^n \in \mathscr{M}^n_{\text{closed}}$ satisfying Condition (g) with $g \leq 2$ has constant principal curvatures and thus is isoparametric. When $g = 3$, as a consequence of the main result of the present paper, one has the following:

THEOREM. *If* $M^n \in \mathscr{M}^n_{\text{closed}}$ *satisfies Condition* (g) *with* $g = 3$, *then* M^n *is either an equator* S^n, *a product of spheres* $S^p \times S^q$ *or a Cartan minimal hypersurface.*

REMARK. The Bryant conjecture will be established if one can exhibit such a theorem without assuming M^n to be compact.

We now state the following:

MAIN THEOREM. *A closed hypersurface* M^n *of constant scalar curvature* R *and constant mean curvature* H *in* S^{n+1} *is isoparametric provided it has* 3 *distinct principal curvatures everywhere.*

REMARK. When the principal curvatures are all non-simple, R. Miyaoka [7] exhibited that M^n is isoparametric even without assuming the scalar curvature is constant.

Acknowledgment. We wish to express our thanks to Dr. Y. Xu for his interest in the work and to Professor S. Y. Cheng for his valuable comments and continuous support and encouragement.

1. Notations and the reduction of the proof. Throughout the paper, we use $A, B, C, \ldots$, for indices ranging from 1 to n and denote by δ_{AB} the Kronecker symbols.

For each point $x \in M^n$, let $\lambda(x)$, $\mu(x)$ and $\sigma(x)$ be the three distinct principal curvatures of multiplicities $p(x)$, $q(x)$ and $r(x)$, respectively, at x.

In order to establish the Main Theorem, we need to show that all the three continuous functions λ, μ and σ on M^n are indeed constant functions.

We first observe that all the three integer-valued functions p, q and r are constant integers.

Indeed, consider the following system of linear equations with p, q and r as unknowns:

$$(*) \qquad \begin{aligned} p + q + r &= n, \\ p\lambda + q\mu + r\sigma &= H, \\ p\lambda^2 + q\mu^2 + r\sigma^2 &= S, \end{aligned}$$

where S is the square length of the second fundamental form.

Since λ, μ and σ are distinct everywhere, we can solve for p, q and r in terms of λ, μ, σ and S, which are all continuous on M^n.

This shows that p, q and r are constant as desired since they need to be integers.

REMARK. By the same argument, one can see that for $\forall g$, Condition (g) always yields the constancy of the multiplicities.

Therefore, we can choose a local frame $\{e_i, e_\alpha, e_a\}$ where the indices i, α and a range from 1 to p, $p+1$ to $p+q$ and $p+q+1$ to $p+q+r$ $(= n)$, respectively, such that the second fundamental form $h = \sum_{A,B} h_{AB}\omega_A\omega_B$ is given by

$$(h) \qquad (h_{AB}) = \begin{pmatrix} \lambda I_p & & \\ & \mu I_q & \\ & & \sigma I_r \end{pmatrix}$$

where for each integer s, we denote by I_s the identity matrix of rank s, and $\{\omega_i, \omega_\alpha, \omega_a\}$ is the dual co-frame of $\{e_i, e_\alpha, e_a\}$.

Recall that the structure equations of M^n are given by the following:

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B,$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} R_{ABCD}\omega_C \wedge \omega_D,$$

where ω_{AB}'s denote the connection forms of M^n and R_{ABCD} the curvature tensor.

Define $\nabla h = \sum_{A,B,C} h_{ABC}\omega_A\omega_B\omega_C$ the covariant derivative of h by

$$(\nabla h) \qquad \sum_C h_{ABC}\omega_C = dh_{AB} + \sum_C h_{CB}\omega_{CA} + \sum_C h_{AC}\omega_{CB}.$$

Then, by virtue of (h), (∇h) can be interpreted as

$$(1.1) \qquad \sum_C h_{ijC}\omega_C = \delta_{ij}d\lambda,$$

$$(1.2) \qquad \sum_C h_{\alpha\beta C}\omega_C = \delta_{\alpha\beta}d\mu,$$

$$(1.3) \qquad \sum_C h_{abC}\omega_C = \delta_{ab}d\sigma,$$

$$(2.1) \qquad \sum_C h_{i\alpha C}\omega_C = (\lambda - \mu)\omega_{i\alpha},$$

$$(2.2) \qquad \sum_C h_{inC}\omega_C = (\lambda - \sigma)\omega_{in},$$

$$(2.3) \qquad \sum_C h_{\alpha a C}\omega_C = (\mu - \sigma)\omega_{\alpha a}.$$

Recall that h_{ABC} is symmetric in all the indices since the ambient space S^{n+1} is of constant curvature and (cf. [**4**])

$$(S) \qquad \sum_{A,B,C} h_{ABC}^2 = S(S-n) + H^2 - Hf$$

where $f = \sum_{A,B,C} h_{AB}h_{BC}h_{CA}$.

Note that $S = n(n-1) + H^2 - R$ (cf. [**4**]) is constant and all the principal curvatures λ, μ and σ are smooth functions on M^n.

By differentiating both $(*)$ and $f = p\lambda^3 + q\mu^3 + r\sigma^3$, we have

$$pd\lambda + qd\mu + rd\sigma = 0,$$
$$p\lambda d\lambda + q\mu d\mu + r\sigma d\sigma = 0,$$
$$p\lambda^2 d\lambda + q\mu^2 d\mu + r\sigma^2 d\sigma = \frac{1}{3}df.$$

It follows that

$$(\#) \qquad \frac{pd\lambda}{\sigma - \mu} = \frac{qd\mu}{\lambda - \sigma} = \frac{rd\sigma}{\mu - \lambda} = \frac{df}{3D},$$

where $D = (\sigma - \mu)(\sigma - \lambda)(\mu - \lambda)$.

In the case when all the principal curvatures are non-simple, from the Miyaoka theorem [7], we immediately assert that M^n is isoparametric.

And the case when $p = q = r = 1$ was already verified by the author in [4]. It therefore suffices to show that all the principal curvatures are simple if so is one of them, say, $r = 1$.

To this aim, we need the following:

KEY LEMMA. *With the same notations as above. If $r = 1$ and $pq \geq 2$, then $h_{i\alpha n} = 0$, $\forall i, \alpha$.*

The proof of this lemma itself will be given in §2. We will finish the current section by showing how to achieve our aim from the Key Lemma.

Consider a point $x_0 \in M^n$ where $df = 0$, from (#) we have

$$d\lambda = d\mu = d\sigma = 0,$$

$$\text{i.e.}\quad h_{ijA} = h_{\alpha\beta A} = h_{abA} = 0, \quad \forall i, j, \alpha, \beta, a, b, A.$$

Now suppose otherwise that $r = 1$ and $pq \geq 2$.

From the Key Lemma, the left-hand side of (S) would vanish at x_0 and then

$$S(S - n) + H^2 - Hf(x_0) = 0.$$

When $H \neq 0$, since $df = 0$ at both maximum and minimum points of f, it would follow that $f = \frac{1}{H}(S(S - n) + H^2)$ identically. From (#), this in turn would yield that λ, μ and σ were constant and then M^n be isoparametric, contradicting the classification by Cartan.

When $H = 0$, it would follow that $S(S - n) = 0$ and then M^n be either an equator or a product of spheres, due to Chern-do Carmo-Kobayashi and Lawson [5, 6], contradicting the assumption that $g = 3$.

2. Proof of the Key Lemma. At each point $x \in M^n$, denote by Y the $p \times q$ matrix $(h_{i\alpha n}) \in M_{p\times q}$. We are supposed to show that $Y = 0$ everywhere if $r = 1$ and $pq \geq 2$.

We will employ the following [8]:

THEOREM [*Otsuki*, 1970]. *Let M^n be a hypersurface immersed in an $(n + 1)$-dimensional Riemannian manifold of constant curvature such that the multiplicities of principal curvatures are all constant.*

Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. Moreover, if the *multiplicity of a principal curvature is greater than* 1, *then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

Now, without loss of generality, assume that $q \geq 2$.

Applying the Otsuki theorem to μ and noting that $d\lambda = \frac{\mu-\sigma}{\sigma-\lambda}\, d\mu$ and $d\sigma = \frac{\mu-\lambda}{\lambda-\sigma}\, d\mu$ from (#), we have

$$\lambda_\alpha = \mu_\alpha = \sigma_\alpha = 0, \quad \forall \alpha.$$

Case 1. $p = 1$.

Rewrite (1.1) → (2.3) as

$$\text{(I.1)} \qquad h_{11C} = \lambda_C, \quad h_{\alpha\alpha C} = \mu_C, \quad h_{nnC} = \sigma_C, \quad \forall \alpha, C,$$

$$\text{(I.2)} \qquad h_{\alpha\beta C} = 0, \quad \forall \alpha \neq \beta,$$

$$\text{(II.1)} \qquad \omega_{1\alpha} = \frac{1}{\lambda - \mu}(\mu_1 \omega_\alpha + h_{1\alpha n}\omega_n),$$

$$\text{(II.2)} \qquad \omega_{1n} = \frac{1}{\lambda - \sigma}\left(\lambda_n \omega_1 + \sum_\beta h_{1\beta n}\omega_\beta + \sigma_1 \omega_n\right),$$

$$\text{(II.3)} \qquad \omega_{\alpha n} = \frac{1}{\mu - \sigma}(h_{1\alpha n}\omega_1 + \mu_n \omega_\alpha).$$

Recall that the curvature tensor of M^n is given by $R_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + h_{AC}h_{BD} - h_{AD}h_{BC}$.

Differentiating (II.3) and applying equations (II.1)–(II.3) and the structure equations of M^n to the resulting equation, we compute

$$\begin{aligned}
\text{LHS} = d\omega_{\alpha n} &= \omega_{\alpha 1} \wedge \omega_{1n} + \sum_\beta \omega_{\alpha\beta} \wedge \omega_{\beta n} - (1 + \mu\sigma)\omega_\alpha \wedge \omega_n \\
&= -\frac{1}{\lambda - \mu}(\mu_1 \omega_\alpha + h_{1\alpha n}\omega_n) \\
&\quad \wedge \frac{1}{\lambda - \sigma}\left(\lambda_n \omega_1 + \sum_\beta h_{1\beta n}\omega_\beta + \sigma_1 \omega_n\right) \\
&\quad + \sum_\beta \omega_{\alpha\beta} \wedge \frac{1}{\mu - \sigma}(h_{1\beta n}\omega_1 + \mu_n \omega_\beta) - (1 + \mu\sigma)\omega_\alpha \wedge \omega_n,
\end{aligned}$$

$$\begin{aligned}
\text{RHS} = &\left(d\frac{1}{\mu-\sigma}\right)\wedge(h_{1\alpha n}\omega_1+\mu_n\omega_\alpha)\\
&+\frac{1}{\mu-\sigma}(dh_{1\alpha n}\wedge\omega_1+h_{1\alpha n}d\omega_1+d\mu_n\wedge\omega_\alpha+\mu_n d\omega_\alpha)\\
= &\left(d\frac{1}{\mu-\sigma}\right)\wedge(h_{1\alpha n}\omega_1+\mu_n\omega_\alpha)\\
&+\frac{1}{\mu-\sigma}\left[dh_{1\alpha n}\wedge\omega_1+h_{1\alpha n}\left(\sum_\beta\omega_{1\beta}\wedge\omega_\beta+\omega_{1n}\wedge\omega_n\right)\right.\\
&\qquad+d\mu_n\wedge\omega_\alpha\\
&\qquad\left.+\mu_n\left(\omega_{\alpha 1}\wedge\omega_1+\sum_\beta\omega_{\alpha\beta}\wedge\omega_\beta+\omega_{\alpha n}\wedge\omega_n\right)\right].
\end{aligned}$$

Picking up only those terms of the type of $\omega_\beta\wedge\omega_n$, we get

$$\begin{aligned}
\text{LHS} = &-\frac{1}{(\lambda-\mu)(\lambda-\sigma)}\left(\mu_1\omega_\alpha\wedge\sigma_1\omega_n+h_{1\alpha n}\omega_n\wedge\sum_\beta h_{1\beta n}\omega_\beta\right)\\
&-(1+\mu\sigma)\omega_\alpha\wedge\omega_n,\\
\text{RHS} = &-\frac{\mu_n-\sigma_n}{(\mu-\sigma)^2}\omega_n\wedge\mu_n\omega_\alpha\\
&+\frac{1}{\mu-\sigma}\left[h_{1\alpha n}\left(\sum_\beta\frac{h_{1\beta n}}{\lambda-\mu}\omega_n\wedge\omega_\beta+\sum_\beta\frac{h_{1\beta n}}{\lambda-\sigma}\omega_\beta\wedge\omega_n\right)\right.\\
&\qquad\left.+\left(\mu_{nn}\omega_n\wedge\omega_\alpha+\frac{\mu_n}{\mu-\sigma}\omega_\alpha\wedge\omega_n\right)\right].
\end{aligned}$$

Compare the coefficients of $\omega_\beta\wedge\omega_n$ and note that $-\frac{1}{(\mu-\sigma)(\lambda-\mu)}+\frac{1}{(\mu-\sigma)(\lambda-\sigma)}=-\frac{1}{(\lambda-\mu)(\lambda-\sigma)}$, we find $\forall\alpha,\ \beta$,

$$\begin{aligned}
&-\frac{\mu_1\sigma_1}{(\lambda-\mu)(\lambda-\sigma)}\delta_{\alpha\beta}+\frac{h_{1\alpha n}h_{1\beta n}}{(\lambda-\mu)(\lambda-\sigma)}-(1+\mu\sigma)\delta_{\alpha\beta}\\
&\quad=\frac{\mu_n(\mu_n-\sigma_n)}{(\mu-\sigma)^2}\delta_{\alpha\beta}-\frac{h_{1\alpha n}h_{1\beta n}}{(\lambda-\mu)(\lambda-\sigma)}-\frac{\mu_{nn}}{\mu-\sigma}\delta_{\alpha\beta}+\frac{\mu_n}{(\mu-\sigma)^2}\delta_{\alpha\beta}.
\end{aligned}$$

Hence

$$2h_{1\alpha n}h_{1\beta n}=z\delta_{\alpha\beta}$$

where z is a smooth function on M^n defined as

$$z = (1+\mu\sigma)(\lambda-\mu)(\lambda-\sigma) + \mu_1\sigma_1 + \frac{\mu_n(\mu_n-\sigma_n)}{(\mu-\sigma)^2}(\lambda-\mu)(\lambda-\sigma) - \frac{\mu_{nn}}{\mu-\sigma}(\lambda-\mu)(\lambda-\sigma) + \frac{\mu_n}{(\mu-\sigma)^2}(\lambda-\mu)(\lambda-\sigma).$$

Let Y^t denote the transpose of Y. In the form of matrix, the above equation reads as

$$Y^tY = \frac{1}{2}zI_q.$$

Since $Y \in M_{1\times q}$ with $q \geq 2$, it follows that $Y = 0$ everywhere as desired.

Case 2. $p \geq 2$.

Arguing as before, we further have

$$\lambda_i = \mu_i = \sigma_i = 0, \quad \forall i = 1, \dots, p.$$

And equations (2.1)–(2.3) now read as

$$\text{(II.1)}' \qquad \omega_{i\alpha} = \frac{1}{\lambda-\mu}h_{i\alpha n}\omega_n,$$

$$\text{(II.2)}' \qquad \omega_{in} = \frac{1}{\lambda-\sigma}\left(\lambda_n\omega_i + \sum_\beta h_{i\beta n}\omega_\beta\right),$$

$$\text{(II.3)}' \qquad \omega_{\alpha n} = \frac{1}{\mu-\sigma}\left(\sum_j h_{j\alpha n}\omega_j + \mu_n\omega_\alpha\right).$$

Similarly, by differentiating (II.1)$'$ we have

$$\begin{aligned}\text{LHS} &= \sum_j \omega_{ij}\wedge\omega_{j\alpha} + \sum_\beta \omega_{i\beta}\wedge\omega_{\beta\alpha} + \omega_{in}\wedge\omega_{n\alpha} - (1+\lambda\mu)\omega_i\wedge\omega_\alpha \\ &\sim \frac{1}{\lambda-\sigma}\left(\lambda_n\omega_i + \sum_\beta h_{i\beta n}\omega_\beta\right)\wedge\left(-\frac{1}{\mu-\sigma}\right)\left(\mu_n\omega_\alpha + \sum_j h_{j\alpha n}\omega_j\right) \\ &\quad - (1+\lambda\mu)\omega_i\wedge\omega_\alpha,\end{aligned}$$

$$
\begin{aligned}
\mathrm{RHS} &= \left(d\frac{1}{\lambda-\mu}\right) \wedge h_{i\alpha n}\omega_n \\
&\quad + \frac{1}{\lambda-\mu}\left[dh_{i\alpha n} \wedge \omega_n + h_{i\alpha n}\left(\sum_j \omega_{nj} \wedge \omega_j + \sum_\beta \omega_{n\beta} \wedge \omega_\beta\right)\right] \\
&\sim \frac{h_{i\alpha n}}{\lambda-\mu}\left(-\frac{1}{\lambda-\sigma}\sum_{j,\beta} h_{j\beta n}\omega_\beta \wedge \omega_j - \frac{1}{\mu-\sigma}\sum_{j,\beta} h_{j\beta n}\omega_j \wedge \omega_\beta\right) \\
&= \sum_{j,\beta} \frac{h_{i\alpha n}h_{j\beta n}}{\lambda-\mu}\left(\frac{1}{\lambda-\sigma} - \frac{1}{\mu-\sigma}\right)\omega_j \wedge \omega_\beta \\
&= -\sum_{j,\beta} \frac{h_{i\alpha n}h_{j\beta n}}{(\lambda-\sigma)(\mu-\sigma)}\omega_j \wedge \omega_\beta ,
\end{aligned}
$$

where for any two given 2-forms ψ and ψ', by $\psi \sim \psi'$ we mean $\psi \equiv \psi' \pmod{\omega_n}$, i.e., $\psi - \psi' = \omega \wedge \omega_n$ for some 1-form ω.

Now, by picking up those terms of the type of $\omega_j \wedge \omega_\beta$ we have

$$
\begin{aligned}
&-\frac{1}{(\lambda-\sigma)(\mu-\sigma)}\left(\lambda_n\omega_i \wedge \mu_n\omega_\alpha + \sum_\beta h_{i\beta n} \wedge \sum_j h_{j\alpha n}\omega_j\right) \\
&- (1+\lambda\mu)\omega_i \wedge \omega_\alpha \\
&\qquad = -\sum_{j,\beta} \frac{h_{i\alpha n}h_{j\beta n}}{(\lambda-\sigma)(\mu-\sigma)}\omega_j \wedge \omega_\beta .
\end{aligned}
$$

Then,

$$
2h_{i\alpha n}h_{j\beta n} = \overline{z}\delta_{ij}\delta_{\alpha\beta} , \quad \forall i, j, \alpha, \beta ,
$$

where $\overline{z} = \lambda_n\mu_n + (1+\lambda\mu)(\lambda-\sigma)(\mu-\sigma)$.

In particular,

$$
h_{i\alpha n}h_{i\beta n} = \frac{1}{2}\overline{z}\delta_{\alpha\beta} , \quad \forall i, \alpha, \beta .
$$

Again, since $q \geq 2$ we have $h_{i\alpha n} = 0$, $\forall i$, α, i.e. $Y = 0$ everywhere.

This establishes the Key Lemma and thus completes the proof of the Main Theorem.

References

[1] R. Bryant, private conversation.

[2] E. Cartan, *Sur des familles remarquables d'hypersurfaces isoparametriques dans les espaces spheriques*, Math. Z., **45** (1939), 335–367.

[3] S. Chang, *On minimal hypersurfaces with constant scalar curvatures in* S^4, J. Differential Geom., **37** (1993), 523–534.

[4] ——, *A closed hypersurface of constant scalar curvature and constant mean curvature in* S^4 *is isoparametric*, Comm. in Analysis and Geometry, **1** (1993), 71–100.

[5] S. S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional analysis and related fields, Springer, Berlin, Heidelberg, New York, 1970, pp. 59–75.

[6] H. B. Lawson, *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math., **89** (1969), 187–191.

[7] R. Miyaoka, *Complete hypersurfaces in the space form with three principal curvatures*, Math. Z., **179** (1982), 345–354.

[8] T. Otsuki, *Minimal hypersurfaces in a Riemannian manifold of constant curvature*, Amer. J. Math., **92** (1970), 145–173.

[9] C. K. Peng and C. L. Terng, *Minimal hypersurface of spheres with constant scalar curvature*, Ann. of Math. Stud., No. 103 (1983), 177–198.

[10] ——, *The scalar curvature of minimal hypersurfaces in spheres*, Math. Ann., **266** (1983), 105–113.

[11] J. Simons, *Minimal varieties in a Riemannian manifold*, Ann. of Math., **88** (1968), 62–105.

[12] S. T. Yau, *Problem section*, Ann. of Math. Stud., No. 102 (1982), 693.

Received June 2, 1992.

University of Utah
Salt Lake City, UT 84112-1107