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THE STRUCTURE OF sl(2, 1)-SUPERSYMMETRY: IRREDUCIBLE REPRESENTATIONS AND PRIMITIVE IDEALS

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We give a detailed study of the enveloping algebra of the Lie superalgebra sl(2, 1), including classification of irreducible Harish-Chandra modules, completeness of finite dimensional irreducible, explicit computation of center, and classification of primitive ideals.

Introduction and main results. Lie superalgebras are important both in physics and in mathematics [**5**]. In physics, they are used e.g. to unify fermions and bosons in a unique picture (one irreducible representation of the structure) via supersymmetry. In mathematics, their enveloping algebras provide a class of very interesting noetherian algebras. Much information is known about enveloping algebras of Lie algebras (e.g., [**4**]), but for superalgebras there is a lot to do (see e.g. [**2**] for a pioneering work, and [**13**] for a very nice survey of results obtained up to now). Let us restrict to the simple case; then a natural distinction does appear between simple superalgebras with an enveloping algebra which is a domain and others. The first case is exactly the series $\mathrm{osp}(1, 2n)$, which are also the only semi-simple simple superalgebras [**8**]. The simplest model of this case is $h = \mathrm{osp}(1, 2)$; $U(h)$ was completely studied in [**16**], including explicit computation of Prim $U(h)$. The simplest model of the second case is $g = \mathrm{sl}(2, 1)$, and the purpose of the present paper is a complete study of $U(g)$. We shall give a classification of irreducible Harish-Chandra modules, a detailed computation of the center $Z(g)$ of $U(g)$, and a classification of $\mathrm{Prim}\, U(g)$.

Let us recall known results: finite dimensional irreducible representations of $g = \mathrm{sl}(2, 1)$ are known [**18**], and also unitary irreducible are classified ([**6**], [**7**]). Moreover, finite dimensional representations provide a complete set of representations [**2**], but are generally not fully reducible.

A fundamental result of our paper is the fact that finite dimensional irreducible provide a complete set, because of information that can be deduced on $U(g)$. Actually, we deduce an explicit determination of the center $Z(g)$, which shows that $Z(g)$ is not a finitely generated algebra, a big difference with usual properties of simple Lie algebras, and even with $h = \mathrm{osp}(1, 2)$! Also we deduce some "structural" identities between central elements of $U(g)$ and $U(h)$, which are of interest since they "contain" the reduction of $U(g)$-modules into $U(h)$-modules. We then study irreducible g-modules, and establish a bijection with irreducible $g_{\bar{0}}$-modules, following an idea of **[9]**. Restriction to Harish-Chandra case is easily done, and classification of irreducible Harish-Chandra modules follows. We point out the natural introduction of two cases: the first one (regular case) has to be treated via induction from $g_{\bar{0}}$ (as suggested in **[9]**); the second one (degenerate case) is strange, since degenerate irreducible are still irreducible when restricted to the subalgebra h (some kind of special Gelfand-Zetlin trick!).

Finally, we give a classification of $\mathrm{Prim}\, U(g)$. As mentioned for representations, primitive ideals are either regular or degenerate, and these two classes appear to be quite different. Roughly speaking, degenerate primitive are "big ideals" (though generally minimal primitive!), and corresponding quotients are actually primitive quotients of $U(h)$. Once more, as in the case of h, the metaplectic case is singular, and leads to a very interesting primitive quotient of $U(g)$, obtained as an extension of the Weyl algebra by a parity.

Before giving a precise description of our results, let us mention some new results of several authors, which were announced after our paper was accepted, and which are sometimes parallel to ours:

First, a classification of $\mathrm{Prim}\, U(g)$, for $g = \mathrm{sl}(2, 1)$, was announced by I. Musson **[14]**, based on his results of **[12]**. This classification is obtained by techniques which are different from ours.

Second, a bijection between $\mathrm{Prim}\, U(g)$ and $\mathrm{Prim}\, U(g_{\bar{0}})$, for classical simple g of type I (including $g = \mathrm{sl}(2, 1)$), has been announced by E. Letzer **[10]**, also based on **[12]**. This bijection is not a lattice isomorphism, so it gives less information (in the case of $g = \mathrm{sl}(2, 1)$) than our results, or Musson's results **[14]**. In any case, an explicit description of primitive ideals of $g = \mathrm{sl}(2, 1)$ (e.g. in terms of generators) is still to be done, and we think that our techniques can be a milestone for such a description.

Third (and suggested by our Theorem 3) it has been announced independently by I. Musson **[15]** and E. Letzer **[11]** that if g is classical simple and $g \neq b(n)$, then finite dimensional irreducible provide a complete set; this result is of great interest, since information is known (in general) about irreducible, but very little is known (and probably very little is to be known!) about the general finite dimensional case.

Let us now give a precise description of our main results (unexplained notations are to be found in §§(0) and (I)).

Section (I) is a description of g, and of some subalgebras of g, which will be used in the paper. We also introduce corresponding Casimir elements.

Given an irreducible g-module V, let V_0 (resp. V_0') $= \{v \in V / \Delta_\pm V$ (resp. $E_\pm v) = 0\}$. In §(II), we first prove ((II.1.2) and (II.1.4.(3))):

THEOREM 1. (1) *V_0 (resp. V_0') is an irreducible $g_{\bar{0}}$-module.*
(2) *$V \to V_0$ (resp. V_0') is a one-to-one mapping from $\Pi(g)$ onto $\Pi(g_{\bar{0}})$.*

The proof uses induction techniques, which were developed in **[9]** for finite dimension, but which happen to work in general for g.

We then distinguish between degenerate irreducible g-modules ($\mathscr{C} = 0$), and regular ones ($\mathscr{C} \neq 0$), and obtain (II.1.5), (II.1.6) and (II.1.7)):

THEOREM 2. (1) *Let W be an irreducible $g_{\bar{0}}$-module, and $X = \mathrm{Ind}_{g_+ \uparrow g} W$. Then if $\mathscr{C} \neq 0$ in X, X is an irreducible g-module.*
(2) *Let V be an irreducible g-module, then:*

- *if V is degenerate, V is an irreducible h-module.*
- *if V is regular, $V = \mathrm{Ind}_{g_+ \uparrow g} V_0$.*

We also specify the $g_{\bar{0}}$-reduction of irreducible g-modules V, which is very dependent on the fact that V is regular or degenerate (II.1.5), (II.1.6). We then apply Theorem 1 to irreducible Harish-Chandra g-modules, and obtain a complete classification (II.2.2). We specify s-reduction of this type of g-modules in the degenerate case (II.2.3), and regular case (II.2.4). Note that some reductions do contain indecomposable non-irreducible s-modules (II.2.5).

Let $\Pi_r(g)$ (resp. $\Pi_r^f(g)$) be the set of irreducible regular (resp. finite dimensional irreducible regular) representations of g.

In §(III) we prove (III.1):

THEOREM 3. *$\Pi_r^f(g)$ is a complete set of representations of g.*

It was known that finite dimensional representations of g provide a complete set **[2]**. Nevertheless, the only well-known finite dimensional representations are irreducible ones, and finite dimensional representations are generally not fully reducible, so our result is a real improvement, and proves to be useful.

In §IV, we compute the center $Z(g)$ of $U(g)$, and obtain (IV.4.1):

THEOREM 4. *$Z(g)$ has basis $\{1, (\mathscr{D}/\mathscr{C})^n\mathscr{C}^p, n \geq 0, p > 0\}$.*

As a consequence, $Z(g)$ is a free $\mathbb{C}[\mathscr{C}]$-module, with a basis $\{1, \mathscr{C}_n, n \geq 1\}$ such that $\mathscr{C}_n\mathscr{C}_p = \mathscr{C}\mathscr{C}_{n+p}$, $\forall n, p \geq 1$ (IV.5.1). Therefore, $Z(g)$ is not a noetherian algebra, and, a fortiori, not a finitely generated algebra (IV.5.2).

In §(V) we describe degenerate primitive quotients of $U(g)$, i.e. quotients by kernels of degenerate irreducible representations.

We introduce the algebra W_P, which is obtained when extending the Weyl algebra W by a parity P, in the following way:

Let σ be the automorphism of W defined by $(a)^\sigma = (-1)^{\deg a}a$, $a \in W$, then

$$W_P = \left\{ \begin{bmatrix} a & b \\ b^\sigma & a^\sigma \end{bmatrix}, a, b \in W \right\};$$

W is obviously contained in W_P, and $P = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Given any irreducible representation π of W in a space V, the subalgebra of $L(V)$, generated by $\pi(W)$ and the natural parity of V, is isomorphic to W_P (V.3.1), and W_P is a quasi-simple primitive algebra (V.3.3).

We prove (V.5.1):

THEOREM 5. *All but one primitive degenerate quotients of $U(g)$ are isomorphic to primitive quotients of $U(h)$, and the exceptional one is isomorphic to W_P.*

Let us recall the singularity which occurs in $\operatorname{Prim} U(h)$: primitive ideals of infinite codimension are always of type $U(h)/(C-c)$, except one, which is $U(h)/(L+1/4)$ (see **[16]** and (V.2.3)) and corresponds to the value $c = -1/16$ (metaplectic case); the corresponding quotient is the Weyl algebra W. The exceptional case of Theorem 5 is exactly this metaplectic case.

Conversely, if B is either a primitive quotient of $U(h)$ different from W, or if $B = W_P$, we define an explicit morphism from $U(g)$ onto B ((V.4.3)). As a consequence

THEOREM 6. *If π is any irreducible representation of h, there exists a degenerate representation $\hat{\pi}$ of g such that $\hat{\pi}|_h = \pi$.*

In §(VI), we give a classification of $\operatorname{Prim} U(g)$. We first distinguish between the degenerate and regular cases, and write

$$\operatorname{Prim} U(g) = \operatorname{Prim}_r U(g) \cup \operatorname{Prim}_d U(g).$$

Let $\mathscr{E} = \operatorname{Prim} U(h) \times Z_2$, $Z_2 = \{+, -\}$, with the identification:

$(I, +) = (I, -)$ if $I = U(h)(L + 1/4)$, or if I is the kernel of the trivial representation.

THEOREM 7. $\operatorname{Prim}_d U(g) = \mathscr{E}$.

We note that $\operatorname{Prim} U(g_{\bar{0}}) \simeq \operatorname{Prim} U(g_+) \simeq \operatorname{Prim} U(s) \times \mathbb{C}$ (VI.1.2) and define $\operatorname{Prim}_r U(g_{\bar{0}})$ in the following way: given $(I, \lambda_0) \in \operatorname{Prim} U(g_{\bar{0}}) = \operatorname{Prim} U(s) \times \mathbb{C}$, there exists $q \in \mathbb{C}$ such that $(Q - q) \in I$; then $(I, \lambda_0) \in \operatorname{Prim}_r U(g_{\bar{0}})$ if and only if $q - \lambda_0(\lambda_0 + 1) \neq 0$. Then we prove (VI.3.2):

THEOREM 8. $\operatorname{Prim}_r U(g) \simeq \operatorname{Prim}_r U(g_{\bar{0}})$.

Finally, the classification of $\operatorname{Prim} U(s)$ and $\operatorname{Prim} U(h)$ being well known (e.g., **[16]**) we obtain a classification of $\operatorname{Prim} U(g)$.

Though the distinction between degenerate irreducible representations, (which correspond to the vanishing of $\mathscr{C}$ and every $\mathscr{C}_n$ of $Z(g)$) and regular ones, seems quite natural, it is very interesting that the degenerate case can be interpreted in terms of irreducible representations of a simple subalgebra h, whence the regular case (via inducing techniques) is interpreted in terms of irreducible representations of $g_{\bar{0}}$.

It is proved in **[12]** that any element of $\operatorname{Prim} U(\omega)$ is the kernel of an irreducible Verma module, when ω is a classical simple Lie superalgebra (an extension of a classical result of Duflo). This gives a parametrization of $\operatorname{Prim} U(\omega)$, but unfortunately not one to one.

If ω is a semi-simple Lie algebra, minimal primitive ideals of $U(\omega)$ are well known: they are generated by maximal ideals of $Z(\omega)$ (**[4**, (8.4.4)**]**). For simple Lie superalgebras, the situation is more involved: for instance, the ideal $U(h)/(L+1/4)$, of $U(h)$, is minimal primitive, but not generated by its intersection with $Z(h)$ (**[16]**). Nevertheless, it is the only ideal of this type in $U(h)$; note that it comes from the metaplectic representation **[16]**. For $U(g)$, complexity is increasing, since:

THEOREM 9 (VI.5). *If $I \in \operatorname{Prim}_d U(g)$, and if* $\operatorname{codim} I = \infty$, *then I is minimal primitive, and I is not generated by its intersection with $Z(g)$.*

Finally, we prove two "structural" equations, holding in $U(g)$, and involving the (commuting) elements K, Q, $\mathscr{C}$ and $\mathscr{D}$. These equations give an explanation of the $g_{\bar{0}}$-reduction of regular irreducible g-modules. On the other hand, they do not give any information in the degenerate case. This stresses the difference between regular and degenerate primitive ideals.

(0) General conventions and notations.

(0.1) All vector spaces considered in this paper are vector spaces over the field of complex numbers $\mathbb{C}$. Accordingly, all (Lie, or super Lie, or associtive) algebras are algebras over $\mathbb{C}$. When (Lie or associative) Z_2-graded algebras are concerned, all considered objects are implicitly assumed (if the contrary is not mentioned) to be Z_2-graded: so, module (or representation) means Z_2-graded module, submodule means homogeneous submodule, ideal means homogeneous ideal, irreducibility means Z_2-irreducibility, primitive ideal means homogeneous primitive ideal, etc.

(0.2) Given an associative Z_2-graded algebra A, we define on A a Lie algebra, and a Lie superalgebra structure by:

$$a, b \in A, \quad [a, b]_L = ab - ba,$$
$$a \in A_{\deg(a)}, \quad b \in A_{\deg(b)}, \quad [a, b] = ab - (-1)^{\deg a \deg b} ba.$$

(0.3) Given a Lie algebra, or a Lie superalgebra ω, we denote by $\Pi(\omega)$ (resp. $\Pi^f(\omega)$) the set of (equivalence classes of) irreducible (resp. irreducible finite dimensional) representations of ω.

(0.4) We denote by $U(\omega)$ the enveloping algebra of ω, by $Z(\omega)$ the center of $U(\omega)$, and by $\operatorname{Prim} U(\omega)$ the set of primitive ideals of $U(\omega)$.

(0.5) We recall that a subset Π of $\Pi(\omega)$ is a *complete set of representations* of ω if:

$$u \in U(\omega), \quad \pi(u) = 0, \quad \forall \pi \in \Pi \Rightarrow u = 0.$$

(0.6) We mention that we use both terminology of ω-modules, or of representations of ω, as convenient.

(0.7) The *Weyl algebra* W is the associative Z_2-graded algebra generated by p and q and relations $[p, q] = 1$ (for details, and relations between W and $\operatorname{osp}(1, 2)$, see **[16]**).

(I) Notations.

(I.1) We denote by $g = g_{\bar{0}} \oplus g_{\bar{1}}$ the complex simple Lie super algebra sl(2, 1), or, equivalently W_2, in the notations of **[9]**. $g_{\bar{0}}$ is isomorphic to gl(2), so we write $g_{\bar{0}} = s \oplus \mathbb{C}\,\mathrm{Id}$, where $s \simeq \mathrm{sl}(2)$, and $g_{\bar{1}}$ reduces, under the adjoint representation of $g_{\bar{0}}$, into $g_{\bar{1}} = g_{-1} \oplus g_1$, $g_j \simeq D(1/2)$ under $\mathrm{ad}\, s$, and $\mathrm{ad}\,\mathrm{Id}|_{g_j} = -j\,\mathrm{Id}_{g_j}$, $j = -1, 1$.

We set $g_+ = g_{\bar{0}} \oplus g_1$. We introduce respective basis $\{Y, F, G, K\}$ of $g_{\bar{0}}$, $\{\Delta_\pm\}$ of g_1 and $\{E_\pm\}$ of g_{-1}, with (nonvanishing) brackets given by:

(I.1.1)
$$\begin{array}{lll}
[Y, F] = F, & [Y, G] = -G, & [F, G] = 2Y, \\
[Y, E_\pm] = \pm\frac{1}{2}E_\pm, & [Y, \Delta_\pm] = \pm\frac{1}{2}\Delta_\pm, & \\
[F, E_-] = -E_+, & [G, E_+] = -E_-, & \\
[F, \Delta_-] = -\Delta_+, & [G, \Delta_+] = -\Delta_-, & \\
[K, E_\pm] = \frac{1}{2}E_\pm, & [K, \Delta_\pm] = -\frac{1}{2}\Delta_\pm, & \\
[E_+, \Delta_+] = F, & [E_-, \Delta_-] = -G, & \\
[E_+, \Delta_-] = Y - K, & [E_-, \Delta_+] = Y + K. &
\end{array}$$

(I.1.2) We denote by $D_f(l)$, $l \in 1/2\mathbb{N}$, the irreducible representation of s of dimension $(2l+1)$, and by $D_f(l, \lambda_0)$, $l \in 1/2\mathbb{N}$, $\lambda_0 \in \mathbb{C}$, the irreducible representation $D_f(l)$ extended to $g_{\bar{0}}$ by setting $K \cdot v = \lambda_0 v$, $\forall v \in D_f(l)$. Using Schur's lemma, any finite dimensional representation of $g_{\bar{0}}$ is isomorphic to one (and only one) representation $D_f(l, \lambda_0)$.

(I.1.3) We introduce:

$$A_\pm = \frac{1}{\sqrt{2}}(E_\pm + \Delta_\pm), \quad \widetilde{A}_\pm = \frac{i}{\sqrt{2}}(-E_\pm + \Delta_\pm).$$

The subspaces h and h of g with respective basis $\{Y, F, G, A_\pm\}$ and $\{Y, F, G, \widetilde{A}_\pm\}$ are subalgebras both isomorphic to osp(1, 2), and the (nonvanishing) brackets are given by:

(I.1.4)
$$\begin{array}{lll}
[Y, A_\pm] = \pm 1/2 A_\pm, & [F, A_-] = -A_+, & [G, A_+] = -A_-, \\
[A_+, A_+] = F, & [A_-, A_-] = -G, & [A_+, A_-] = Y,
\end{array}$$

and similar brackets replacing $A_\pm$ by $\widetilde{A}_\pm$. Remaining (nonvanishing) brackets are given by:

$$[A_+, \widetilde{A}_-] = -iK, \quad [A_-, \widetilde{A}_+] = iK, \tag{I.1.5}$$
$$[K, A_\pm] = \frac{i}{2}\widetilde{A}_\pm, \quad [K, \widetilde{A}_\pm] = -\frac{i}{2}A_\pm.$$

(I.1.6) We denote by $\mathscr{D}_f(l)$, $l \in 1/2\mathbb{N}$, the irreducible representation of $\mathrm{osp}(1, 2)$ of dimension $(4l+1)$. As an $s = \mathrm{osp}(1, 2)_{\overline{0}}$-module, $\mathscr{D}_f(l)$ reduces into $D_f(l) \oplus D_f(l-1/2)$ (see a complete description of $\mathscr{D}_f(l)$ e.g. in **[3]**). As an h (or $\tilde{h}$)-module for the adjoint action, g reduces into $\mathscr{D}_f(1) \oplus \mathscr{D}_f(1/2)$.

(I.1.7) Given a Lie superalgebra ω, we denote by $U(\omega)$ its enveloping algebra, and by $Z(\omega)$ the center of $U(\omega)$. We shall use the following results:

(I.1.8) $Z(s)$ is the polynomial algebra $\mathbb{C}[Q]$, where $Q = \frac{1}{2}(FG + GF) + Y^2$, and $Z(g_{\overline{0}})$ is the polynomial algebra $\mathbb{C}[Q, K]$.

(I.1.9) $Z(h)$ (resp. $Z(\tilde{h})$) is the polynomial algebra $\mathbb{C}[C]$ (resp. $\mathbb{C}[\widetilde{C}]$), where $C = Q - 1/2[A_+, A_-]_L$ (resp. $\widetilde{C} = Q - 1/2[\widetilde{A}_+, \widetilde{A}_-]_L$) **[16]**. The Killing form of g is nondegenerate, so, by standard arguments, it provides an element $\mathscr{C} \in Z(g)$ (Casimir element) which is given by

$$\begin{aligned} \mathscr{C} &= Q - 1/2[A_+, A_-]_L - 1/2[\widetilde{A}_+, \widetilde{A}_-]_L - K^2 \\ &= Q - (\Delta_+ E_- - \Delta_- E_+) - K(K-1) \\ &= Q - (E_+ \Delta_- - E_- \Delta_+) - K(K+1). \end{aligned} \tag{I.1.10}$$

Introducing $L = [A_+, A_-]_L$, $\widetilde{L} = [\widetilde{A}_+, \widetilde{A}_-]_L$, one has **[16]**:

$$\begin{aligned} &Q = L(L+1) = \widetilde{L}(\widetilde{L}+1), \\ &C = L(L+1/2), \ \widetilde{C} = \widetilde{L}(\widetilde{L}+1/2), \\ &\mathscr{C} = 1/2L^2 + 1/2\widetilde{L}^2 - K^2. \end{aligned} \tag{I.1.11}$$

(II) Irreducible modules.

(II.1) Let V be an irreducible g-module. By Quillen's lemma (e.g., **[3]**), $Z(g)$ acts by scalars on V. We define $V_0 = \{v / \Delta_\pm v = 0\}$; then

(II.1.1) LEMMA. $V_0 \neq \{0\}$.

Proof. One has $\Delta_+\Delta_- V \subset V_0$. If $\Delta_+\Delta_- = 0$, then $\Delta_+ V + \Delta_- V \subset V_0$, so $V_0 = \{0\}$ implies $\Delta_+ = \Delta_- = 0$. We obtain a contradiction, unless $V = \{0\}$ which is usually not considered irreducible. □

It is easy to check that V_0 is a sub $g_{\bar{0}}$-module. Actually V_0 characterizes the g-module V:

(II.1.2) PROPOSITION. (1) *If* V *is an irreducible* g*-module, then* V_0 *is an irreducible* $g_{\bar{0}}$*-module.*

(2) *The mapping* $V \to V_0$ *from* $\Pi(g)$ *into* $\Pi(g_{\bar{0}})$ *is one-to-one and onto.*

Proof (*see* [**9**]). Let V be a nontrivial irreducible g-module. For any graded $g_{\bar{0}}$-invariant nonzero subspace $W \subset V_0$, one has $V = W + (E_+W + E_-W) + E_+E_-W$, since the written sum is g-stable. Let $W = V_0 \cap E_+E_-V_0$. If $W \neq \{0\}$, we find $V = W$, since $E_+^2 = E_-^2 = [E_+E_-] = 0$, and it follows that V must be trivial. So $V_0 \cap E_+E_-V_0 = \{0\}$.

Let $W = (V_0 \oplus E_+E_-V_0) \cap (E_+V_0 + E_-V_0)$; if $W \neq \{0\}$, then we find: $E_\pm W \subset W$, so W is g-stable, and therefore $W = V$. But then, we find $V = V_0 \oplus E_+EV_0 = E_+V_0 + E_-V_0$, so $E_+E_-V_0 = \{0\}$, and $V_0 = V$, from which follows that V must be trivial.

Finally, we obtain $V = V_0 \oplus (E_+V_0 + E_-V_0) \oplus E_+E_-V_0$. Given any graded sub-$g_{\bar{0}}$-module W of V_0, if $W \neq \{0\}$, we have $V = W \oplus (E_+W + E_-W) \oplus E_+E_-W$, so $W = V_0$ follows, and V_0 is an irreducible $g_{\bar{0}}$-module. Necessarily $V_0 \subset V_{\bar{0}}$, or $V_0 \subset V_{\bar{1}}$.

We assume given an irreducible $g_{\bar{0}}$-module W_0, that we consider as a $g_+ = (g_{\bar{0}} \oplus g_{-1})$ module by setting $\Delta_\pm v = 0$, $\forall v \in W_0$. We introduce the g-module $X = \mathrm{Ind}_{g_+ \uparrow g} W_0$.

Using the Poincaré-Birkhoff-Witt theorem, we have: $X = W_0 \oplus W_1 \oplus W_2$, where $W_1 = E_+W_0 \oplus E_-W_0$, $W_2 = E_+E_-W_0$ are $g_{\bar{0}}$-submodules.

From (I.1.1) and the irreducibility of W_0, there exists $\lambda_0 \in \mathbb{C}$ such that

$$K|_{W_0} = \lambda_0 \mathrm{Id}_{W_0}, \quad K|_{W_1} = (\lambda_0 + 1/2)\mathrm{Id}_{W_1},$$
$$K|_{W_2} = (\lambda_0 + 1)\mathrm{Id}_{W_2}.$$

Using (I.1.1), we then see that $\Delta_\pm$ map W_1 into W_0, and W_2 into W_1. Any K-stable subspace S reduces as $S = (S \cap W_0) \oplus (S \cap W_1) \oplus (S \cap W_2)$, so, if S is a g-submodule, one has $S \neq X$ if and only if $S \subset W_1 \oplus W_2$. It results that X has a biggest g-submodule $X_{\max} \neq X$, and one, and only one, irreducible quotient, namely $X/X_{\max}$.

Let $V = X/X_{\max}$, and $\mu \colon X \to V$ be the canonical projection. In order to show that $V_0 = W_0$, we consider $\mu^{-1}(V_0) = \{v | \Delta_\pm v \in X_{\max}\}$, and reduce $\mu^{-1}(V_0) = W_0 \oplus (\mu^{-1}(V_0) \cap W_1) \oplus \mu^{-1}(V_0) \cap W_2$. If $v \in \mu^{-1}(V_0) \cap W_i$, $i \neq 0$, then by the Poincaré-Birkhoff-Witt theorem one has $Uv \subset W_1 \oplus W_2$, so $U(g)v \neq X$, $v \in X_{\max}$, and $V_0 = W_0$, as claimed.

Actually, this proves that the map defined in (2) is onto. Now if an irreducible g-module V' satisfies $V'_0 = W_0$, then V' is an irreducible quotient of X, so $V' \simeq V$; this finishes the proof of (2). □

(II.1.3) PROPOSITION. *Let V be an irreducible g-module.*

(1) *As an s-module, V reduces into $V = V_0 \oplus V_1 \oplus V_2$, where V_0 is irreducible, $V_1 = E_+V_0 + E_-V_0$ is a quotient of $D(1/2) \otimes V_0$, and either $V_2 = E_+E_-V_0 = \{0\}$ or V_2 is isomorphic to V_0.*

(2) *K acts on V by*

$$Kv = \lambda_0 v\,,\ v \in V_0\,;\ Kv = (\lambda_0+1/2)v\,,\ v \in V_1\,;\ Kv = (\lambda_0+1)v\,,\ v \in V_2\,.$$

(3) *Let $V_{-1} = V_3 = \{0\}$; then $E_\pm$ map V_i into V_{i+1}, and $\Delta_\pm$ map V_i into V_{i-1}, for $i = 0, 1, 2$. Moreover $E_+E_-V_1 = \Delta_+\Delta_-V_1 = \{0\}$.*

(4) *Let $Q|_{V_0} = l(l+1)\mathrm{Id}_{V_0}$; then $\mathscr{C} = (l-\lambda_0)(l+\lambda_0+1)\mathrm{Id}_V$.*

Proof. If we look at the proof of (II.1.2), we see that V reduces as $V = V_0 \oplus V_1 \oplus V_2$, with V_0 $g_{\bar{0}}$-irreducible, and $V_1 = E_+V_0 + E_-V_0$, $V_2 = E_+E_-V_0$.

Moreover,

$$K|_{V_0} = \lambda_0 \mathrm{Id}_{V_0}\,, \quad K|_{V_1} = (\lambda_0 + 1/2)\mathrm{Id}_{V_1}\,, \quad K|_{V_2} = (\lambda_0+1)\mathrm{Id}_{V_2}\,.$$

The first assertion of (3) is an immediate consequence.

Since $E_\pm V_2 = \Delta_\pm V_0 = \{0\}$, one has $E_+E_-V_1 = \Delta_+\Delta_-V_1 = \{0\}$. From $[X, E_+E_-] = 0$, $\forall X \in s$, and the irreducibility of V_0, we deduce that $V_2 = E_+E_-V_0$ is either $\{0\}$, or isomorphic to V_0.

The mapping $X \otimes v \to Xv$ from $g_1 \otimes V_0$ onto V_1 is an s-morphism, and this achieves the proof of (1).

We can write $\mathscr{C} = Q - (E_+\Delta_- - E_-\Delta_+) - K - K^2$; V being irreducible, $\mathscr{C} = c\,\mathrm{Id}_V$, we compute $\mathscr{C}v = (l(l+1) - \lambda_0 - \lambda_0^2)v$, $v \in V_0$, and obtain $c = (l-\lambda_0)(l+\lambda_0+1)$. □

(II.1.4) REMARK. (1) (II.1.2) shows that the classification of irreducible g-modules is equivalent to the classification of irreducible $g_{\bar{0}}$-modules, which is not known explicitly. Nevertheless, (II.1.2) leads to a classification of series of irreducible g-modules, as will be shown in the following subsection, and we shall show that these series are complete.

(2) (II.1.3) shows that the s-content of an irreducible g-module is related to the reduction of tensor products $D_f(1/2) \otimes V_0$, V_0 an

irreducible s-module. Indecomposable $D_f(1/2) \otimes V_0$ can appear, we shall see examples in the next subsection.

(3) There is an analogue of (II.1.2) and (II.1.3) when replacing $\Delta_\pm$ by $E_\pm$, V_0 by $V_0' = \{v/E_\pm v = 0\}$, $V_1' = \Delta_+ V_0' + \Delta_- V_0'$, $V_2' = \Delta_+\Delta_- V_0'$.

We note that for an irreducible V, one has $V_1 \neq \{0\}$ unless V is trivial. Now, $V_2 = \{0\}$ does happen for nontrivial V, and examples will be given in next subsection. For the time being, we specify the parameter values for which it is the case:

(II.1.5) PROPOSITION. *We keep the assumptions and notations of* (II.1.3) *and* (II.1.4.(3)). *Then* $V_2 = \{0\}$ *(resp.* $V_2' = \{0\}$*) if and only if* $\mathscr{C} = 0$. *In this case, and if* V *is not trivial, one has* $V_0 = V_1'$, $V_1 = V_0'$, *so* V *splits into a direct sum of two irreducible* $g_{\bar{0}}$*-modules.* V *is an irreducible* h *(resp.* $\tilde{h}$*)-module, with Casimir values* $C = \widetilde{C} = l(l+1/2)$, *or* $C = \widetilde{C} = (l+1/2)(l+1)$.

Proof. Assuming $V_2 = \{0\}$, we take $v \neq 0$, $v \in V_0$, and compute $\Delta_+\Delta_-(E_+E_-v) = 0$, using (I.1.1).

We obtain $[Q - K(K+1)]v = (l-\lambda_0)(l+\lambda_0+1)v = 0$, so $\mathscr{C}v = 0$.

On the other hand, if $\mathscr{C}v = 0$ since $V_2 \subset V_0'$, which is $g_{\bar{0}}$-irreducible, and since $\Delta_+\Delta_- V_2 = \{0\}$, one has $V_2 = \{0\}$, or $V_2 = V_0'$ and $V_2' = \{0\}$. In the second case $V = V_2 \oplus V_1'$, and $V_1' \subset V_1$; then $V_0 = \{0\}$ and a contradiction, so $V_2 = \{0\}$. Then, $V_0' \cap V_0 = \{0\}$ if V is not trivial. But K is a scalar on V_0', so by (II.1.3), $V_0 \subset V_1$, $V_1' \subset V_0$ and $V_2' = \{0\}$. From (II.1.4.(3)), $V = V_0' \oplus V_1'$, so $V_0' = V_1$ is $g_{\bar{0}}$-irreducible.

As an s-module, V reduces into the irreducible submodules V_0 and V_1. If W is a nontrivial sub h-module, then V/W is an h-module which is s-irreducible, and this cannot happen unless V/W is trivial **[9]**. Then either V_0 or V_1 is a trivial s-module. If V_0 is trivial, then by (II.1.3), V_1 is not isomorphic to V_0, so $W = (W \cap V_0) \oplus (W \cap V_1)$. V_0 and V_1 being s-irreducible, one has $W = V_1$, so W is an h-module which is s-irreducible; therefore V_1 is trivial and there is a contradiction. Similar arguments using (II.1.4.(3)) give the same result if V_1 is trivial.

Finally C can be computed on V_0, and one finds, by (I.1.9), (I.1.1), $C = Q - 1/4(\Delta_+E_- - \Delta_-E_+)|_{V_0} = Q - 1/2K|_{V_0}$, so the announced values, and the same computation provides the same value for $\widetilde{C}$. □

(II.1.6) PROPOSITION. *Let* V_0 *be an irreducible* g_+*-module, with* $K = \lambda_0 \mathrm{Id}_{V_0}$, $\Delta_\pm = 0$, $Q = l(l+1)\mathrm{Id}_{V_0}$, *and let* $X = \mathrm{Ind}_{g_+ \uparrow g} V_0$. *Then*

$\mathscr{C} = (l - \lambda_0)(l + \lambda_0 + 1)\mathrm{Id}_X$. *If* $\mathscr{C} \neq 0$, *then* X *is irreducible, and, as s-modules,* $X_0 \simeq V_0$, $X_1 \simeq D_f(1/2) \otimes V_0$, $X_1 \simeq V_0$ (*see* (II.1.3) *for the notations*).

Proof. From the Poincaré-Birkhoff-Witt theorem, $X = V_0 \oplus V_1 \oplus V_2$, where $V_1 = E_+V_0 \oplus E_-V_0$, $V_2 = E_+E_-V_0$. From (I.1.1), K is diagonal on this reduction, with respective eigenvalues λ_0, $\lambda_0 + 1/2$, $\lambda_0 + 1$; moreover $V_1 \simeq D_f(1/2) \otimes V_0$, and $V_2 \simeq V_0$ as s-modules. Since $X = U(g)V_0$, we need only to compute $\mathscr{C}$ on V_0, but, using (I.1.10), $\mathscr{C}|_{V_0} = Q - K - K^2 = (l - \lambda_0)(l + \lambda_0 + 1)\mathrm{Id}_{V_0}$.

Next, from the proof of (II.1.2), X has a maximal g-submodule W, which is contained in $V_1 \oplus V_2$; moreover, X/W is irreducible and $(X/W)_0 = V_0$. W being K-stable, one has $W = (W \cap V_1) \oplus (W \cap V_2)$. Now, if $W \cap V_2 = V_2$, then $(X/W)_2 = \{0\}$, but then $\mathscr{C} = 0$, a contradiction with our assumption. Therefore $W \subset V_1$, but since $\Delta_\pm$ map V_1 into V_0, and $E_\pm$ map V_1 into V_2, one has $\Delta_\pm|_W = E_\pm|_W = 0$, and W is a trivial g-module, if $W \neq \{0\}$. Then $\mathscr{C}|_W = 0$, so a contradiction. Therefore $W = \{0\}$. □

(II.1.7) Corollary. *Let* V *be an irreducible* g*-module, such that* $\mathscr{C} \neq 0$, V_0 *is an irreducible* g_+*-module and* $V \simeq \mathrm{Ind}_{g_+\uparrow g} V_0$.

Proof. From the proof of (II.1.2), V is a quotient of $\mathrm{Ind}_{g_+\uparrow g} V_0$, then apply (II.1.6). □

(II.1.6) and (II.1.7) suggest the introduction of a partition of $\Pi(g)$:

(II.1.8) Definition. An irreducible g-module V is regular (resp. degenerate) if $\mathscr{C} \neq 0$ (resp. $\mathscr{C} = 0$) on V.

We denote by $\Pi_r(g)$ (resp. $\Pi_d(g)$) the set of (classes of) irreducible regular (resp. degenerate) g-modules.

(II.2) Let ω be a superalgebra, and k a subalgebra of $\omega_{\bar{0}}$, assumed reductive in ω. An ω-module V is a Harish-Chandra module if V is a semi-simple k-module, with finite dimensional isotypical components. Here, we shall consider the cases: $\omega = s$, $k = \mathbb{C}Y$; $\omega = g_{\bar{0}}$, $k = \mathbb{C}Y \oplus \mathbb{C}K$; $\omega = g$, $k = \mathbb{C}Y \oplus \mathbb{C}K$. In any of these cases, k is abelian, so the condition for V to be a Harish-Chandra module is that $V = \bigoplus_{\lambda \in k^*} V_\lambda$, where $V_\lambda = \{v \in V | X \cdot v = \lambda(X)v, \ \forall X \in k\}$, and $\dim V_\lambda < \infty$, $\forall \lambda \in k^*$.

Obviously, we can restrict (II.1.2) to the case of Harish-Chandra modules, and this leads to a classification of irreducible Harish-

Chandra g-modules, generalizing the classification of unitary irreducible g-modules given in **[6]**, [7].

(II.2.1) PROPOSITION. *Let V be an irreducible Harish-Chandra g-module; then V_0 is an irreducible Harish-Chandra $g_{\bar{0}}$-module, and characterizes V up to equivalence. The mapping $V \to V_0$ is one-to-one and onto.*

Now, we need some notations: We introduce the following H.C. s-modules: $D(l, m_0)$, $(l)\downarrow$, $(-l)\uparrow$, for complex l and m_0, $D_f(l)$, $l \in \frac{1}{2}\mathbb{N}$. $D(l, m_0)$ is irreducible when $(m_0 - l)$ and $(m_0 + l) \notin Z$, $(l)\downarrow$ and $(-l)\uparrow$ are irreducible when $l \notin 1/2\mathbb{N}$, $D_f(l)$ is irreducible and $\dim D_f(l) = 2l + 1$ (see e.g. **[3]**).

Defining $Kv = \lambda_0 v$, $\forall v$, we extend these s-modules to H.C. $g_{\bar{0}}$-modules that we denote by $D(l, m_0, \lambda_0)$, $(l, \lambda_0)\downarrow$, $(-l, \lambda_0)\uparrow$ and $D_f(l, \lambda_0)$.

Using (II.2.1) and e.g. **[3]**, we obtain a classification of irreducible H.C. g-modules:

(II.2.2) PROPOSITION. *Let V be an irreducible H.C. g-module, then V is one* (*and only one*) *of the modules of the following list*:

(1) $\mathscr{D}(l, m_0, \lambda_0)$, $l = -1/2 + \rho e^{i\theta}$, $0 \le \theta < \pi$, $\rho \in \mathbb{R}^+$, $0 \le \operatorname{Re} m_0 < 1$, $(m_0 - l)$ *and* $(m_0 + l) \notin Z$, $V_0 = D(l, m_0, \lambda_0)$,

(2) $[l, \lambda_0]\downarrow$, $l \neq h/2$, $h \in \mathbb{N}$, $V_0 = (l, \lambda_0)\downarrow$,

(3) $[-l, \lambda_0]\uparrow$, $l \neq h/2$, $h \in \mathbb{N}$, $V_0 = (-l, \lambda_0)\uparrow$,

(4) $\mathscr{D}_f(l, \lambda_0)$, $l \in 1/2\mathbb{N}$, $V_0 = D_f(l, \lambda_0)$, $\dim V_0 = 2l + 1$.

In any case, $Q|V_0 = l(l+1)$, *and* $\mathscr{C} = (l - \lambda_0)(l + \lambda_0 + 1)$.

Proof. It results from the well-known classification of irreducible H.C. s-modules. □

(II.2.2) gives a classification, and we now make s-reduction precise. We begin by degenerate Harish-Chandra modules:

(II.2.3) PROPOSITION. *Let V be an irreducible degenerate H.C. module. Parameters l and m_0 are specified as in* (II.2.2). *Then*:

(1) *If* $V = \mathscr{D}(l, m_0, l)$, (*resp.* $\mathscr{D}(l, m_0, -l-1)$, $l \neq -1/2$), $V_0 = D(l, m_0)$ *and* $V_1 = D(l-1/2, m_0+1/2)$ (*resp.* $D(l+1/2, m_0+1/2)$).

(2) *If* $V = [l, l]\downarrow$ (*resp.* $[l, -l-1]\downarrow$, $l \neq -1/2$), $V_0 = (l)\downarrow$ *and* $V_1 = (l - 1/2)\downarrow$ (*resp.* $(l + 1/2)\downarrow$).

(3) *If* $V = [-l, -l] \uparrow$ *(resp.* $[-l, l-1] \uparrow$, $l \neq -1/2$*)*, $V_0 = (-l) \uparrow$ *and* $V_1 = (1/2 - l) \uparrow$ *(resp.* $(-1/2 - l) \uparrow$*)*.

(4) *If* $V = \mathscr{D}_f(l, l)$, $l \neq 0$, *(resp.* $\mathscr{D}_f(l, -l-1)$*)*, $V_0 = D_f(l)$ *and* $V_1 = D_f(l - 1/2)$ *(resp.* $D_f(l + 1/2)$*)*.

(5) $V = \mathscr{D}_f(0, 0)$ *is the trivial representation.*

Proof. V is an irreducible h = osp(1, 2)-H.C.-module, and the s-reduction of such modules is known (e.g. **[3]**). □

We now study s-content of regular irreducible H.C. g-modules. Using (II.1.6), the problem is reduced to the s-content of $D_f(1/2) \otimes V_0$, when V_0 is an irreducible H.C. s-module. We need some notation:

There exists, up to equivalence, one and only one H.C. s-module which is a nontrivial extension of $(-1) \downarrow$ (resp. $(1) \uparrow$) by $D_f(0)$ (see e.g. **[10]**). We denote this indecomposable module by $E(0) \downarrow$ (resp. $E(0) \uparrow$). Moreover, there exists, up to equivalence, one and only one H.C. s-module which is a nontrivial extension of $D(0, m_0)$ by itself (see e.g. **[10]**).We denote this indecomposable module by $ED(0, m_0)$. Note that $E(0) \downarrow$ and $E(0) \uparrow$ are quasi-simple (actually $Q = 0$), whereas $ED(0, m_0)$ is not (one has $Q^2 = 0$, but $Q \neq 0$).

(II.2.4) LEMMA. *Let* V_0 *be an irreducible H.C. s-module, with* $Q = l(l+1)$. *Then*

∗ *first case: if* $l \neq -1/2$, *and*

- $V_0 \simeq D(l, m_0)$, *then* $D_f(1/2) \otimes V_0 \simeq D(l + 1/2, m_0 + 1/2) \oplus D(l - 1/2, m_0 - 1/2)$.
- $V_0 \simeq (l) \downarrow$, *then* $D_f(1/2) \otimes V_0 \simeq (l + 1/2) \downarrow \oplus (l - 1/2) \downarrow$.
- $V_0 \simeq (-l) \uparrow$, *then*

$$D_f(1/2) \otimes V_0 \simeq (-(l + 1/2)) \uparrow \oplus (-(l - 1/2)) \uparrow .$$

- $V_0 \simeq D_f(l)$, $l \neq 0$, *then*

$$D_f(1/2) \otimes V_0 \simeq D_f(l + 1/2) \oplus D_f(l - 1/2).$$

- $V_0 \simeq D_f(0)$, *then* $D_f(1/2) \otimes V_0 \simeq D_f(1/2)$.

∗ *second case: if* $l = -1/2$, *then* $D_f(1/2) \otimes V_0$ *is always indecomposable. One has*

- *if* $V_0 = D(-1/2, m_0)$, *then* $D_f(1/2) \otimes V_0 \simeq ED(0, m_0)$.
- *if* $V_0 \simeq (-1/2) \downarrow$ *(resp.* $(1/2) \uparrow$*) then* $D_f(1/2) \otimes V_0$ *is a nontrivial extension of* $E(0) \downarrow$ *(resp.* $E(0) \uparrow$*) by* $(-1) \downarrow$ *(resp.* $(1) \uparrow$*)*.

Proof. Indication of proof: compute Q and try to diagonalize: this reduces to diagonalization of a series of 2×2 matrices, which turns out to be possible if $l \neq -1/2$. If $l = -1/2$, a long, but straightforward, computation using adapted basis, gives the results. □

(II.2.5) REMARK. (1) Using (II.1.3), (II.2.2), (II.2.3) and (II.2.4), we see that, in "most" cases, irreducible H.C. g-modules reduce as a direct sum of two, three or four irreducible H.C. s-modules. Nevertheless, from the second case of (II.2.4), there exists series of irreducible H.C. g-modules which are not semi-simple s-modules, but contain an indecomposable nonirreducible H.C. s-modules. Note that in cases $V_0 = (-1/2)\downarrow$ or $V_0 = (1/2)\uparrow$, one has $\text{length}_s(V) = 5$, in all other cases, $\text{length}_s(V) = 2, 3$ or 4.

(2) The question of h (or $\tilde{h}$)-content of an irreducible H.C. g-module V is natural. Actually, it can be given a complete answer: as seen in (II.1.5) for a degenerate V, V is h (or $\tilde{h}$)-irreducible; for a regular V, there are two cases: assuming that $Q|_{V_0} = l(l+1)\text{Id}_{V_0}$, then if $l = -1/2$, V is an indecomposable nonirreducible h or $\tilde{h}$ H.C.-module; if $l \neq -1/2$, V is a direct sum of two irreducible h or $\tilde{h}$ H.C.-modules; note that the underlying subspaces of these h, or $\tilde{h}$ reductions need not coincide; actually, there is coincidence only if $\lambda_0 = -1/2$. The proof of this last claim needs a complete computation of explicit action of generators of g on a suitable basis of V and will not be given here.

(III) Complete sets of representations.

(III.1) PROPOSITION. *Let L be a subset of $1/2\mathbb{N}$, such that $(l+1) \in L$ if $l \in L$, and let Λ be an infinite subset of $\mathbb{C}$; We assume that $(l-\lambda_0)(l+\lambda_0+1) \neq 0$, $\forall (l, \lambda_0) \in L \times \Lambda$. Then $\{\mathscr{D}_f(l, \lambda_0), (l, \lambda_0) \in L \times \Lambda\}$ is a complete set of representations of g.*

COROLLARY 1. $\Pi_r^f(g)$ *is a complete set of representations of* g.

COROLLARY 2. $\Pi^f(g)$ *is a complete set of representations of* g.

Proof. To prove (III.1), we have to introduce some notation:

(III.2) Given a space V on which s acts by $D_f(l)$, a *standard basis* of V is a basis $\{\varphi_{-l}, \varphi_{-l+1}, \dots, \varphi_l\}$ such that the action is given by:

$Y\varphi_n = n\varphi_n$, $F\varphi_n = -(n-l)\varphi_{n+1}$, $G\varphi_n = (n+l)\varphi_{n-1}$, where, once and for all, undefined vectors have to be interpreted as 0.

Now, we introduce a suitable basis of $\mathscr{D}_f(l, \lambda_0)$, when $\mathscr{C} \neq 0$, and $l \neq 0$:

We start with a standard basis $\{\varphi_n, n = -l, -l+1, \dots, l\}$ of V_0, and introduce:

$$v_n = (n + l + 1/2)E_+\varphi_{n-1/2} + (n - l - 1/2)E_-\varphi_{n+1/2},$$
$$n = l - 1/2, -l + 1/2, \dots, l + 1/2,$$

and

$$\omega_n = E_+\varphi_{n-1/2} + E_-\varphi_{n+1/2}, \quad n = -l + 1/2, -l + 3/2, \dots, l - 1/2$$

(where, once more, undefined φ-factors have to be interpreted as 0). Moreover, we introduce $z_n = E_+E_-\varphi_n$, $n = -l, -l+1, \dots, l$. Then it is an easy computation to check that the reduction $V_1 \simeq D_f(l+1/2) \oplus D_f(l - 1/2)$ of (II.2.4), is actually realized on the subspaces V_1^+ and V_1^- generated respectively by $\{v_n\}$ and $\{\omega_n\}$, and that $\{v_n\}$ and $\{\omega_n\}$ are standard basis; moreover $\{z_n\}$ is a standard basis of V_2. Complete computation of the action of g on these basis gives:

(III.2.1)
$$E_\pm\varphi_n = \frac{\pm 1}{2l+1}\{v_{n\pm 1/2} - (n \mp l)\omega_{n\pm 1/2}\},$$
$$E_\pm v_n = \pm(n \mp (l + 1/2))z_{n\pm 1/2}, \quad E_\pm\omega_n = \pm z_{n\pm 1/2},$$

(III.2.2)
$$\Delta_\pm v_n = \pm(\lambda_0 - l)(n \mp (l + 1/2))\varphi_{n\pm 1/2},$$
$$\Delta_\pm \omega_n = \pm(\lambda_0 + l + 1)\varphi_{n\pm 1/2},$$
$$\Delta_\pm z_n = \frac{\pm 1}{2l+1}\{-(\lambda_0 + l + 1)v_{n\pm 1/2} + (\lambda_0 - l)(n \mp l)\omega_{n\pm 1/2}\}.$$

(III.3) Keeping the notation of (III.2), we prove (III.1):

We first note that, since $g_{\bar{0}}$ is a central extension of s by $\mathbb{C}K$, any set of representations of $g_{\bar{0}}$ of type $\{D_f(l, \lambda_0), l \in \mathscr{L}$ infinite $\subset 1/2\mathbb{N}, \lambda_0 \in \Lambda$ infinite$\}$ is a complete set of representations of $g_{\bar{0}}$.

Given $u \in U(g)$, we write

$$u = \sum_{\substack{\alpha, \beta = 0, 1 \\ \alpha', \beta' = 0, 1}} u_{\alpha\beta\alpha'\beta'} E_+^\alpha E_-^\beta \Delta_+^{\alpha'} \Delta_-^{\beta'},$$

with $u_{\alpha\beta\alpha'\beta'} \in U(g_{\bar{0}})$.

We assume that u vanishes in any irreducible finite dimensional representation specified in (III.1).

We start with $\mathscr{D}_f(l, \lambda_0)$, with $l \neq 0$, and computing $u \cdot \varphi_n = 0$, we obtain:

$$u_{1100} z_n = 0 \quad \text{(component on } V_2\text{)},$$
$$u_{1000} v_{n+1/2} - u_{0100} v_{n-1/2} = 0 \quad \text{(component on } D(l+1/2)\text{)},$$
$$(n-l) u_{1000} \omega_{n+1/2} - (n+l) u_{0100} \omega_{n-1/2} = 0 \qquad \text{(component on } D(l-1/2)\text{)}.$$

From the first equation, and our preliminary remark, we deduce that $u_{1100} = 0$. We then note that changing l into $(l+1)$, and writing the third equation, which will be the component on $D(l+1/2)$, we obtain, identifying $V_1^{l,+} \simeq V_1^{(l+1),-}$:

$$(n-l-1) u_{1000} v_{n+1/2} - (n+l+1) u_{0100} v_{n-1/2} = 0.$$

The system satisfied by $u_{1000} v_{n+1/2}$ and $u_{0100} v_{n-1/2}$, has determinant $-2(l+1)$, has leads to $u_{1000} v_{n+1/2} = u_{0100} v_{n-1/2} = 0$ and then $u_{1000} = u_{0100} = 0$.

Similar arguments, using $u \cdot z_n = 0$, will lead to $u_{0011} = u_{0010} = u_{0001} = 0$.

We next compute $u \cdot v = 0$, $v \in V_1$, and note that this will split into a component on V_2, and a component on V_1. The first one gives: $(u_{1110} E_+ E_- \Delta_+ + u_{1101} E_+ E_- \Delta_-) v = 0$, taking $v = v_n$, $v = \omega_n$, we deduce:

$$(n - (l+1/2)) u_{1110} z_{n+1/2} - (n+l+1/2) u_{1101} z_{n-1/2} = 0,$$
$$u_{1110} z_{n+1/2} - u_{1101} z_{n-1/2} = 0,$$

from which we deduce that $u_{1110} = u_{1101} = 0$.

Now, the second component, taking $v = v_n$, $v = \omega_n$, will lead to:

$$u_{1010} v_{n+1} - u_{1001} v_n - u_{0110} v_n + u_{0101} v_{n-1} = 0,$$
$$(n-l+1/2) u_{1010} \omega_{n+1} - (n-l-1/2) u_{1001} \omega_n - (n+l+1/2) u_{0110} \omega_n + (n+l-1/2) u_{0101} \omega_{n-1} = 0,$$
$$(n-l-1/2) u_{1010} v_{n+1} - (n+l+1/2) u_{1001} v_n - (n-l-1/2) u_{0110} v_n + (n+l+1/2) u_{0101} v_{n-1} = 0,$$
$$\begin{aligned}(n-l-1/2)(n-l+1/2) u_{1010} \omega_{n+1} &- (n+l+1/2)(n-l-1/2) u_{1001} \omega_n \\ &- (n-l-1/2)(n+l+1/2) u_{0110} \omega_n \\ &+ (n+l+1/2)(n+l-1/2) u_{0101} \omega_{n-1} = 0.\end{aligned}$$

Once more, we change l into $(l+1)$, and deduce the following system:

$$\begin{aligned}
&u_{1010}v_{n+1} - u_{1001}v_n - u_{0110}v_n + u_{0101}v_{n-1} = 0\,,\\
&(n-l-1/2)u_{1010}v_{n+1} - (n-l-3/2)u_{1001}v_n - (n+l+3/2)u_{0110}v_n\\
&\qquad + (n+l+1/2)u_{0101}v_{n-1} = 0\,,\\
&(n-l-1/2)u_{1010}v_{n+1} - (n+l+1/2)u_{1001}v_n - (n-l-1/2)u_{0110}v_n\\
&\qquad + (n+l+1/2)u_{0101}v_{n-1} = 0\,,\\
&(n-l-3/2)(n-l-1/2)u_{1010}v_{n+1} - (n+l+3/2)(n-l-3/2)u_{1001}v_n\\
&\qquad - (n-l-3/2)(n+l+3/2)u_{0110}v_n\\
&\qquad + (n+l+3/2)(n+l+1/2)u_{0101}v_{n-1} = 0\,.
\end{aligned}$$

The determinant of this system is $-2(l+1)(2l+1)^2(2l+3)$, so we conclude that:

$$u_{1010}v_{n+1} = u_{1001}v_n = u_{0110}v_n = u_{0101}v_{n-1} = 0\,,$$

and then

$$u_{1010} = u_{1001} = u_{0110} = u_{0101} = 0\,.$$

Now, we get $u = u_{1111}E_+E_-\Delta_+\Delta_-$, and we compute $uz_n = 0$, to obtain

$$(\lambda_0 - l)(\lambda_0 + l + 1)u_{1111}z_n = 0\,, \quad \text{so} \quad u_{1111} = u = 0\,. \qquad \square$$

(III.4) Remark. The proof of (III.1) can easily be adapted to construct other complete sets of representations of g. For instance, the irreducible H.C.-modules $\mathscr{D}(l, m_0, \lambda_0)$ of (II.2.2) provide a complete set, and so do the irreducible H.C.-modules of type $[l, \lambda_0]\downarrow$, or $[-l, \lambda_0]\uparrow$. (III.5) (III.1) reveals to be useful to prove structural identities in $U(g)$: actually, one only has to verify that the wanted identity is valid in any finite dimensional irreducible regular representation, and it will hold in $U(g)$. We give an example:

Let us recall the "structural" identity $4Q^2-(8C-1)Q+2C(2C-1) = 0$, which holds in $U(h)$ **[16]** (roughly speaking, this identity contains the $U(s)$ reduction of $U(h)$). We now establish the corresponding identity for $U(g)$; it involves the commuting elements Q, K and $\mathscr{C}$.

(III.5.1) Proposition.

$$\begin{aligned}
&[\mathscr{C} - Q + K(K-1)][\mathscr{C} - Q + K(K+1)]\\
&\qquad \times [\mathscr{C} - (\mathscr{C} - Q + K(K-1))(\mathscr{C} - Q + K(K+1))] = 0\,.
\end{aligned}$$

Proof. Using formula given in (III.2), and (II.1.3), the identity holds in any $\pi \in \Pi_r^f(g)$, so we conclude using (III.1). □

In physics terminology, (III.5.1) expresses a relation between the isospin numbers which are the possible values of Q, and the baryonic numbers, which are the possible values of K, in an irreducible finite dimensional representation (a multiplet).

(IV) Center of $U(g)$.

(IV.1) Given a Lie algebra, or a Lie superalgebra ω, we recall that $Z(\omega)$ denotes the center of $U(\omega)$. Letting V be an ω-module, we denote by V^ω the submodule of ω-invariant vectors; for instance, when $V = U(\omega)$ with the adjoint action, one has $U(\omega)^\omega = Z(\omega)$.

When ω is a semi-simple Lie algebra, then $Z(\omega)$ is a polynomial algebra $\mathbb{C}[Q_1, \ldots, Q_r]$, where $r = \operatorname{rank}\omega$. For simple Lie superalgebras, the situation is not so simple, as will be shown by the description of $Z(g)$, $g = \mathrm{sl}(2, 1)$, that we shall now give.

(IV.2) We introduce the elements $u_1 = 1$, $u_2 = E_+\Delta_- - E_-\Delta_+$, $u_3 = FE_-\Delta_- - GE_+\Delta_+ - Y(E_-\Delta_+ + E_+\Delta_-)$, $u_4 = E_+E_-\Delta_+\Delta_-$, of $U(g)$. It is easily seen that $u_i \in U(g)^{g_{\bar{0}}}$, $\forall i$.

(IV.2.1) Lemma. *$U(g)^{g_{\bar{0}}}$ is an abelian algebra, and a free $\mathbb{C}[Q, K]$-module with basis $\{u_1, u_2, u_3, u_4\}$.*

Proof. Let W be the subspace of $U(g)$ with basis $\{E_+^\alpha E_-^\beta \Delta_+^{\alpha'} \Delta_-^{\beta'}$, $\alpha, \beta, \alpha', \beta' = 0, 1\}$. As a $g_{\bar{0}}$-module (for the adjoint action), $U(g) \simeq U(g_{\bar{0}}) \otimes W$. It is known (e.g. **[1]**) that, as an s-module, $U(s) \simeq \mathbb{C}[Q] \otimes H$, with $H = \sum_{n \in \mathbb{N}} H_n$, and $H_n \simeq D_f(n)$; therefore, as a $g_{\bar{0}}$-module, $U(g_{\bar{0}}) \simeq \mathbb{C}[Q, K] \otimes H$, and $H_n \simeq D_f(n, 0)$. Moreover, as a $g_{\bar{0}}$-module,

$$\begin{aligned} W &\simeq \mathrm{Ext}(D_f(1/2, -1/2)) \otimes \mathrm{Ext}(D_f(1/2, 1/2)) \\ &\simeq D_f(0, 0) \oplus [D_f(1/2, 1/2) \oplus D_f(1/2, -1/2)] \\ &\quad \oplus [D_f(0, 1) \oplus D_f(0, -1) \oplus D_f(1, 0) \oplus D_f(0, 0)] \\ &\quad \oplus [D_f(1/2, 1/2) \oplus D_f(1/2, -1/2)] \oplus D_f(0, 0), \end{aligned}$$

where the reduction is written according to increasing degree from 0 to 4, the 0-degree $D_f(0, 0)$ being $\mathbb{C}$, the 2-degree $D_f(0, 0)$ being $\mathbb{C}u_2$, the 4-degree $D_f(0, 0)$ being $\mathbb{C}u_4$. In the reduction of the $g_{\bar{0}}$-module $H \otimes W$, there will appear one more invariant, coming from $H_1 \otimes D_f(1, 0)$, and only one, which is exactly u_3, so $(H \otimes W)^{g_{\bar{0}}} =$

$\mathbb{C} \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_3 \oplus \mathbb{C}u_4$. Following the notations of (III.2), let $V = V_0 \oplus V_1^+ \oplus V_1^- \oplus V_2$ be a g-module of type $\mathscr{D}_f(l, \lambda_0)$, with $\mathscr{C} \neq 0$; let $u, v \in U(g)^{g_{\bar{0}}}$, then u and v act by scalars on V_0, V_1^+, V_1^- and V_2, so $[u, v]_L = 0$ on V, and by (III.1), $[u, v]_L = 0$ in $U(g)$. □

(IV.3) In order to give an explicit basis of $Z(g)$, we have to introduce a second Casimir operator by

$$\mathscr{D} = 2K\mathscr{C} + (\Delta_+\Delta_-E_+E_- - E_+E_-\Delta_+\Delta_-).$$

(IV.3.1) LEMMA. $\mathscr{D} \in Z(g)$.

Proof. This can be seen by direct computation, or, preferably, as follows:

Using (I.1.1) and (IV.2), one has:

(IV.3.2) $\Delta_+\Delta_-E_+E_- = Q - K(K+1) - Ku_2 + u_3 + E_+E_-\Delta_+\Delta_-$.

Therefore, if V is any irreducible g-module, with $Q|_{V_0} = l(l+1)$ and $K|_{V_0} = \lambda_0$, one has $\Delta_+\Delta_-E_+E_-|_{V_1} = \Delta_+\Delta_-E_+E_-|_{V_2} = 0$ and $\Delta_+\Delta_-E_+E_-|_{V_0} = (l-\lambda_0)(l+\lambda_0+1)$. By similar computations, $E_+E_-\Delta_+\Delta_-|_{V_0} = E_+E_-\Delta_+\Delta_-|_{V_1} = 0$, and, since $V_2 = E_+E_-V_0$, $E_+E_-\Delta_+\Delta_-|_{V_2} = (l-\lambda_0)(l+\lambda_0+1)$. It follows that $\mathscr{D}|_{V_0} = \mathscr{D}|_{V_1} = \mathscr{D}|_{V_2} = (2\lambda_0+1)(l-\lambda_0)(l+\lambda_0+1)$. Therefore, for any $u \in U(g)$, one has $[\mathscr{D}, u] = 0$, in any irreducible g-module, so, using (III.1), $\mathscr{D} \in Z(g)$. □

(IV.3.3) REMARK. (1) We note that the value of $\mathscr{D}$ in an irreducible g-module V such that $Q|_{V_0} = l(l+1)$, $K|_{V_0} = \lambda_0$, is $(2\lambda_0+1)(l-\lambda_0)(l+\lambda_0+1)$. Actually, if $\mathscr{C} = 0$ on V, then $\mathscr{D} = 0$ on V. Special cases corresponding to $\mathscr{D} = 0$, coming from $\lambda_0 = -1/2$, are discussed in (II.2.5.(2)).

(2) From (I.1.10) and (IV.3.2), one has:

(IV.3.4) $\mathscr{D} = (2K+1)(Q - K(K+1)) - 3Ku_2 + u_3$.

(IV.4) Let A be an associative algebra, and Z an element of the center of A. Assuming that Z is not a zero-divisor in A, we can define the fraction algebra A_Z, generated by A and Z^{-1} (see e.g. [**4**, (3.6)]). Using (III.1) it is clear that $\mathscr{C}$ is not a zero-divisor in U, so can we introduce $\widetilde{U} = U(g)_{\mathscr{C}}$. Let $\widetilde{Z}(g)$ be the center of $\widetilde{U}$; obviously $\widetilde{Z}(g) = Z(g)_{\mathscr{C}}$.

(IV.4.1) PROPOSITION. *Let* $\Lambda = \mathscr{D}\mathscr{C}^{-1}$. *Then*

(1) $\widetilde{Z}(g) = \mathbb{C}[\Lambda, \mathscr{C}]_{\mathscr{C}}$.

(2) *$Z(g)$ is the subalgebra of* $\mathbb{C}[\Lambda, \mathscr{C}]_{\mathscr{C}}$, *with basis* $\{1, \Lambda^n\mathscr{C}^p, n \geq 0, p > 0\}$.

Proof. Given $z \in Z(g)$, using (IV.2.1), we write $z = P_0(z) + P_2(z)u_2 + P_3(z)u_3 + P_4(z)u_4$, $P_i \in \mathbb{C}[Q, K]$.

We consider the mapping $P_0 \colon Z(g) \to \mathbb{C}[Q, K]$. It is clear that P_0 is linear. Moreover, if $z, z' \in Z(g)$,

$$\begin{aligned} zz' = {} & P_0(z)P_0(z') + (P_2(z)u_2 + P_3(z)u_3 + P_4(z)u_4)P_0(z') \\ & + (P_0(z) + P_2(z)u_2 + P_3(z)u_3 + P_4(z)u_4) \\ & \times (P_2(z')u_2 + P_3(z')u_3 + P_4(z')u_4). \end{aligned}$$

Therefore, in any g-module V of type $\mathscr{D}_f(l, \lambda_0)$, with $\mathscr{C} \neq 0$, one has: $zz'|_{V_0} = P_0(zz')|_{V_0} = P_0(z)P_0(z')|_{V_0}$, from which we deduce that $P_0(zz') = P_0(z)P_0(z')$ (note that $u_2|_{V_0} = u_3|_{V_0} = u_4|_{V_0} = 0$).

If we assume that $P_0(z) = 0$, then, since $uz = zu$, $\forall u \in U(g)$, and $V = U(g)V_0$, using (III.1), we deduce that $z = 0$.

We have an injective morphism P_0 from $Z(g)$ into $\mathbb{C}[Q, K]$, such that $P_0(\mathscr{C}) = Q - K(K+1)$, so we can extend P_0 to an injective morphism $\widetilde{P}_0$ from $\widetilde{Z}(g) = Z(g)_{\mathscr{C}}$ into $\mathbb{C}[Q, K]_{Q-K(K+1)}$. Then $\widetilde{P}_0(\mathscr{C}^{-h}z) = (Q - K(K+1))^{-h}P_0(z)$, so $\widetilde{P}_0(\Lambda) = (2K+1)$, and $K = \widetilde{P}_0(1/2(\Lambda - 1))$; moreover $\widetilde{P}_0(\mathscr{C} + 1/4(\Lambda^2 - 1)) = P_0(\mathscr{C}) + K(K+1) = Q$, so $\widetilde{P}_0$ is onto, and (1) is proved.

To prove (2), we first compute the action of u_2, u_3 and u_4, in a g-module V of type $D_f(l, \lambda_0)$, with $\mathscr{C} \neq 0$, and $l \neq 0$. We follow the notations of (III.2), and use the relations (I.1.10): $\mathscr{C} = Q - K(K+1) - u_2$, (IV.3.2): $\Delta_+\Delta_-E_+E_- - E_+E_-\Delta_+\Delta_- = Q - K(K+1) - Ku_2 + u_3$, and the proof of (IV.3.1), to obtain:

	u_2	u_3	u_4
V_0	0	0	0
V_1^+	$l - \lambda_0$	$-(l + 3/2)(l - \lambda_0)$	0
V_1^-	$-(l + \lambda_0 + 1)$	$-(l - 1/2)(l + \lambda_0 + 1)$	0
V_2	$-2(\lambda_0 + 1)$	$-2l(l+1)$	$(l - \lambda_0)(l + \lambda_0 + 1)$

Now, let us assume that $\Lambda^n \in Z(g)$, $n > 0$. Then, since $P_0(\Lambda^n) = (2K+1)^n$, one has $\Lambda^n = (2\lambda_0 + 1)^n$ in V. We write $\Lambda^n = P_0 + P_2u_2 + P_3u_3 + P_4u_4$, $P_i \in \mathbb{C}[Q, K]$, and compute $\Lambda^n|_{V_1^+}$ and $\Lambda^n|_{V_1^-}$ respectively for

$$V = \mathscr{D}_f(l - 1/2, \lambda_0 - 1/2) \quad \text{and} \quad V = \mathscr{D}_f(l + 1/2, \lambda_0 - 1/2).$$

We obtain a system between $p_2 = P_2(l(l+1), \lambda_0)$ and $p_3 = P_3(l(l+1), \lambda_0)$:

$$\begin{cases} (l-\lambda_0)p_2 - (l-\lambda_0)(l+1)p_3 = 2^n[\lambda_0^n - (\lambda_0+1/2)^n], \\ (l+\lambda_0+1)p_2 - l(l+\lambda_0+1)p_3 = 2^n[\lambda_0^n - (\lambda_0+1/2)^n]. \end{cases}$$

It follows that

$$p_2 = \frac{\lambda_0 2^n[\lambda_0^n - (\lambda_0+1/2)^n]}{(l-\lambda_0)(l+\lambda_0+1)}, \quad \text{and} \quad p_3 = -\frac{2^n[\lambda_0^n - (\lambda_0+1/2)^n]}{(l-\lambda_0)(l+\lambda_0+1)},$$

and this is a contradiction, since p_2 and p_3 are not polynomials of l and λ_0, as they should be. So we conclude that $\Lambda^n \notin Z(g)$, if $n > 0$. Now, we prove that $\Lambda^n\mathscr{C} \in Z(g)$, $\forall n \geq 0$. We write $\Lambda^n\mathscr{C} = P_0 + P_2u_2+P_3u_3+P_4u_4$, $P_i \in \mathbb{C}[Q, K]$ and compute $\Lambda^n\mathscr{C}$ on V_1^+, V_1^- and V_2, respectively in $\mathscr{D}_f(l-1/2, \lambda_0-1/2)$, $\mathscr{D}_f(l+1/2, \lambda_0-1/2)$ and $\mathscr{D}_f(l, \lambda_0-1)$. Denoting $p_2 = P_2(l(l+1), \lambda_0)$, $p_3 = P_3(l(l+1), \lambda_0)$ and $p_4 = P_4(l(l+1), \lambda_0)$, we obtain:

$$\begin{aligned} &p_2 - (l+1)p_3 = (l+\lambda_0)(2\lambda_0)^n - (l+\lambda_0+1)(2\lambda_0+1)^n, \\ &p_2 + lp_3 = (l-\lambda_0)(2\lambda_0+1)^n - (l-\lambda_0+1)(2\lambda_0)^n \\ &\qquad - 2\lambda_0 p_2 - 2l(l+1)p_3 + (l-\lambda_0+1)(l+\lambda_0)p_4 \\ &\quad = (l-\lambda_0+1)(l+\lambda_0)(2\lambda_0-1)^n - (2\lambda_0+1)^n(l-\lambda_0)(l+\lambda_0+1) \end{aligned}$$

from which we deduce $p_2 = (\lambda_0-1)(2\lambda_0)^n - \lambda_0(2\lambda_0+1)^n$, $p_3 = (2\lambda_0+1)^n - (2\lambda_0)^n$, and $p_4 = (2\lambda_0-1)^n - 2(2\lambda_0)^n + (2\lambda_0+1)^n$. Let $P_2 = 2^n\{(K-1)K^n - K(K+1/2)^n\}$, $P_3 = 2^n\{(K+1/2)^n - K^n\}$, and $P_4 = 2^n\{(K-1/2)^n - 2K^n + (K+1/2)^n\}$, we obtain that $\Lambda^n\mathscr{C} = P_0+P_2u_2+P_3u_3+P_4u_4$, in any $\mathscr{D}_f(l, \lambda_0)$, with $\mathscr{C} \neq 0$, and $l \geq 1/2$, and therefore in $U(g)$ by (III.1). This proves (2). □

(IV.5) Let us set $\mathscr{C}_n = \Lambda^n\mathscr{C}$, $n \geq 1$; then, from (IV.4.1), $Z(g)$ is a free $\mathbb{C}[\mathscr{C}]$-module with basis $\{1, \mathscr{C}_n, n \geq 1\}$ and one has:

(IV.5.1) $\mathscr{D} = \mathscr{C}_1$, $\mathscr{C}_n\mathscr{C}_p = \mathscr{C}\,\mathscr{C}_{n+p}$, $\forall n, p \geq 1$.

This last relation has several consequences: First, given an irreducible g-module π such that $\pi(Q)|_{V_0} = l(l+1)$ and $\pi(K)|_{V_0} = \lambda_0$, then, using (IV.3.3), one has:

$$\pi(\mathscr{C}_n) = (2\lambda_0+1)^n(l-\lambda_0)(l+\lambda_0+1).$$

On the other hand, given complex numbers c and k, there exists a character ξ_{ck} of $Z(g)$ such that $\xi_{ck}(\mathscr{C}) = c$ and $\xi_{ck}(\mathscr{D}) = kc$; ξ is defined by $\xi_{ck}(\mathscr{C}_n) = k^n c$. Note that, when $c = 0$, then ξ_{0k} is the trivial character ε for any k (defined by $\varepsilon(\mathscr{C}) = \varepsilon(\mathscr{C}_n) = 0$, $\forall n \geq 1$).

From (IV.4.1), $\{\xi_{ck}, \ c, k \in \mathbb{C}\}$ exhausts the characters of $Z(g)$. Obviously, any character ξ of $Z(g)$ is the infinitesimal character of an irreducible g-module (see e.g. §II and (IV.3)). Secondly, one has:

(IV.5.1) PROPOSITION. *$Z(g)$ is not a noetherian algebra.*

COROLLARY. *$Z(g)$ is not a finitely generated algebra.*

Proof. Let $I = Z(g)\mathscr{C}$, and $A = Z(g)/I$; given $z \in Z(g)$, let us denote by $\tilde{z}$ its class in A. Then $\{1, \widetilde{\mathscr{C}_n}, \ n \geq 1\}$ is a basis of the vector space A, and one has

$$\widetilde{\mathscr{C}_n} \cdot \widetilde{\mathscr{C}_p} = 0, \quad \forall n, p \geq 1.$$

It follows that any subspace contained in the subspace general by $\{\widetilde{\mathscr{C}_n}, n \geq 1\}$, is an ideal, so A is not noetherian. A fortiori (since A is a quotient of $Z(g)$), the same holds for $Z(g)$. □

(V) Degenerate irreducible representations and corresponding primitive quotients. In this section, we use representation notation (and not module notation) to avoid confusions.

(V.1) We recall that an irreducible representation π of g in V is *degenerate* (resp. regular) if $\pi(\mathscr{C}) = 0$ (resp. $\pi(\mathscr{C}) \neq 0$); an ideal I of $U(g)$ is a *degenerate primitive* (*resp. regular primitive*) if $I = \operatorname{Ker} \pi$, for some irreducible degenerate (resp. regular) π. In the degenerate case, with the notations of (IV.5), $I \cap Z(g) = \operatorname{Ker} \varepsilon$, so $\mathscr{C}_n = I$, $\forall n \geq 1$. Quotients $U(g)/I$, where I is degenerate (resp. regular) primitive will be called *degenerate* (*resp. regular*) *primitive quotients* of $U(g)$.

(V.2) We develop structural results about $U(h)$, which will be needed later. We introduce $L = [A_+, A_-]_L$ and note that

(V.2.1) $$Q = L(L+1), \quad C = L(L+1/2), \quad [X, L] = 0, \quad \forall X \in s \ [16].$$

Using [9, (1.7.1) and (1.7.2)], we obtain:

(V.2.2) $$A_\pm(L + 1/4) = -(L + 1/4)A_\pm.$$

Given an irreducible representation π of h, one has $\pi(C) = c\,\mathrm{Id}$, so $\pi(L)^2 + 1/2\pi(L) - c = 0$; therefore $\pi(L)$ can be diagonalized, with eigenvalues l_1 and $l_2 = -1/2 - l_1$. Comparing with the reduction $V = V_{\bar{0}} \oplus V_{\bar{1}}$ into two irreducible representations of s [16], and using (V.2.1), we deduce that $V_{\bar{0}}$ and $V_{\bar{1}}$ are eigenspaces of $\pi(L)$.

We say that a complex number l is *admissible*, if $l = -1/4 + re^{i\theta}$, with $r \geq 0$ and $0 \leq \theta < \pi$. Up to isomorphism, we can assume that $\pi(L)|_{V_{\bar{0}}} = l\,\mathrm{Id}_{V_{\bar{0}}}$, with l admissible, and then $\pi(L)|_{V_{\bar{1}}} = (-1/2-l)\mathrm{Id}_{V_{\bar{1}}}$.

We denote by $P_{\bar{0}}$ and $P_{\bar{1}}$ the projections onto $V_{\bar{0}}$ and $V_{\bar{1}}$. If $l \neq -1/4$, it is easily seen that

$$P_{\bar{0}} = 1/2\frac{L+l+1/2}{l+1/4}, \quad P_{\bar{1}} = -1/2\frac{L-l}{l+1/4}. \tag{V.2.3}$$

In the singular case $l = -1/4$ (which is the case e.g. of the metaplectic representation), one has

(V.2.4) LEMMA. *If $l = -1/4$, $M = \operatorname{Ker}\pi = U(h)(L+1/4) = (L+1/4)U(h)$ and $U(h)/\operatorname{Ker}\pi$ is the Weyl algebra W (for definition of W, see* (0.7)).

Proof. From (V.2.1), $(L+1/4) \in \operatorname{Ker}\pi$, and from (V.2.2), using the fact that $U(h)$ is generated by A_+ and A_-, we deduce that $U(h)(L+1/4) \subset \operatorname{Ker}\pi$. Now $[\pi(A_+), \pi(A_-)]_L = -1/4$, so $\pi(U(h))$ is a quotient of the Weyl algebra W, which is known to be quasi-simple, therefore $U(h)/\operatorname{Ker}\pi$ is the Weyl algebra. It is proved in **[16]**, that $\operatorname{Ker}\pi = (Q-3C)U(h) + (C+1/4)U(h)$, but $Q - 3C = -2L(L+1/4)$ and $C + 1/16 = (L+1/4)^2$. □

(V.3) The Weyl algebra W is a domain, so, in the case $l = -1/4$, there cannot exist u and v in $U(h)$ such that $\pi(u) = P_{\bar{0}}$ and $\pi(v) = P_{\bar{1}}$. Therefore, we have to introduce the algebra $W_{P_{\bar{0}},P_{\bar{1}}}$ generated by $\pi(U(h))$, $P_{\bar{0}}$ and $P_{\bar{1}}$, or, equivalently, the algebra W_P generated by $\pi(U(h))$ and the parity operator P, defined by $Pv = (-1)^i v$, $v \in V_i$. It is not obvious that W_P' does not depend on the choice of π, so we prove it:

(V.3.1) PROPOSITION. *Let π be an irreducible representation of the Weyl algebra W in a space V, and let P be the parity operator of V. Let W_P be the subalgebra of $L(V)$ generated by $\pi(W)$ and P; then $W_P = \pi(W) \oplus P\pi(W)$.*

COROLLARY. *Let π and π' be two irreducible representations of W; then the corresponding W_P and $W_{P'}$, are isomorphic.*

Proof. From $P^2 = 1$, it is clear that $W_P = \pi(W) + P\pi(W)$. Let us set $V = V_{\bar{0}} \oplus V_{\bar{1}}$, $u = u_{\bar{0}} + u_{\bar{1}}$, if $u \in L(V)$, with

$$u_{\bar{0}} = \begin{bmatrix} u_{\bar{0}\bar{0}} & 0 \\ 0 & u_{\bar{1}\bar{1}} \end{bmatrix} \quad \text{and} \quad u_{\bar{1}} = \begin{bmatrix} 0 & u_{\bar{0}\bar{1}} \\ u_{\bar{1}\bar{0}} & 0 \end{bmatrix}.$$

Given $a \in W$, we set

$$\tilde{a} = \pi(a) = \begin{bmatrix} \tilde{a}_{\overline{00}} & \tilde{a}_{\overline{01}} \\ \tilde{a}_{\overline{10}} & \tilde{a}_{\overline{11}} \end{bmatrix}.$$

If we assume that $\tilde{a}_{\overline{00}} = 0$, then $(\tilde{a}_{\bar{0}}\tilde{p})^2 = 0$, and, recalling that W is a quasi-simple domain, we deduce $a_{\bar{0}} = 0$; similarly $\tilde{a}_{\overline{11}} = 0$ implies $a_{\bar{0}} = 0$. If we assume that $\tilde{a}_{\overline{10}} = 0$, then $\tilde{a}_{\bar{1}}^2 = 0$, so $a_{\bar{1}} = 0$; similarly $\tilde{a}_{\overline{01}} = 0$ implies $a_{\bar{1}} = 0$.

Now let $\tilde{a} + P\tilde{b} = 0$, with $a, b \in W$. Then

$$\begin{bmatrix} \tilde{a}_{\overline{00}} + \tilde{b}_{\overline{00}} & \tilde{a}_{\overline{01}} + \tilde{b}_{\overline{01}} \\ \tilde{a}_{\overline{10}} - \tilde{b}_{\overline{10}} & \tilde{a}_{\overline{11}} - \tilde{b}_{\overline{11}} \end{bmatrix} = 0.$$

It follows that $(a\tilde{+}b)_{\overline{00}} = 0$, so $(a\tilde{+}b)_{\overline{11}} = \tilde{a}_{\overline{11}} + \tilde{b}_{\overline{11}} = 0$, and then $\tilde{a}_{\overline{11}} = \tilde{b}_{\overline{11}} = 0$. Therfore, $a_{\bar{0}} = b_{\bar{0}} = 0$. Similarly, $(a\tilde{+}b)_{\overline{01}} = 0$ implies $(a\tilde{+}b)_{\overline{10}} = 0$, and then $\tilde{a}_{\overline{10}} = \tilde{b}_{\overline{10}} = 0$; therefore $a_{\bar{1}} = b_{\bar{1}} = 0$. Finally $a = b = 0$.

For the corollary, let P' be the parity of V' and $\pi'(a) = \hat{a}$, $a \in W$. We define ϕ: $W_P \to W_{P'}$ by $\phi(\tilde{a} + P\tilde{b}) = \hat{a} + P'\hat{b}$; ϕ is clearly an isomorphism. □

From (V.3.1) and its corollary, using the quasi-simplicity of W, we can give the following intrinsic definition of W_P: W_P is the algebra generated by p, q and P with relations:

(V.3.2) $\quad [p, q]_L = 1, \quad pP = -Pp, \quad qP = -Pq \quad \text{and} \quad P^2 = 1.$

This algebra has a realization as matrix with coefficient in W, which is as follows: let σ be the automorphism of W defined by $a^\sigma = (-1)^{\deg a} a$, $a \in W_{\deg a}$, then, in W_P, one has $Pa = a^\sigma P$, $\forall a \in W$. Therefore

$$W_P \simeq \left\{ \begin{bmatrix} a & b \\ b^\sigma & a^\sigma \end{bmatrix}, a, b \in W \right\};$$

W is realized by

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & a^\sigma \end{bmatrix}, a \in W \right\} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(V.3.3) PROPOSITION. *W_P is a quasi-simple primitive algebra.*

Proof. W_P is primitive from its initial definition. Let J be a two-sided ideal in W_P; assuming $J \neq W_P$, we can fix a maximal left ideal J' such that $J \subset J'$. Let π be the irreducible representation of W_P

on $V = W_P/J'$. Writing $V = V_{\overline{0}} \oplus V_{\overline{1}}$, from $\deg(P) = \overline{0}$ and $P^2 = 1$, we deduce that $\pi(P)$ is diagonal in $V_{\overline{0}}$ and $V_{\overline{1}}$, with eigenvalues ± 1. Denote by $V_{\overline{0},\pm 1}$ and $V_{\overline{1},\pm 1}$ the corresponding eigenspaces. Assuming, for instance, that $V_{\overline{0}1} \neq \{0\}$, and using (V.3.2), we find that $V_{\overline{0},1} \oplus V_{\overline{1},-1}$ is stable, and therefore $V = V_{\overline{0},1} \oplus V_{\overline{1},-1}$. This proves that $\pi(P)$ is exactly the parity operator of V. Up to an isomorphism of V exchanging the grading, the same holds if $V_{\overline{0},-1} \neq \{0\}$. If V' is $\pi(W)$-stable, from $V' = V'_{\overline{0}} \oplus V'_{\overline{1}}$, we deduce that V' is $\pi(W_P)$-stable, so $\pi|_W$ is irreducible. Let $u = a + Pb \in \operatorname{Ker}\pi$, $a, b \in W$, using (V.3.1), we see that $\pi(a) = \pi(b) = 0$, and since W is quasi-simple, it results that $a = b = 0$. So $\operatorname{Ker}\pi = \{0\}$, but $J \subset \operatorname{Ker}\pi$ implies $J = \{0\}$. □

(V.4) Given any complex $l \neq -1/4$, let $c = l(l + 1/2)$, $I_l = (C - c)U(h)$ and $B_l = U(h)/I_l$; if $l = -1/4$, we set $B_{-1/4} = W_P$. For any value of l, we have a morphism from $U(h)$ into B_l (recall that $W = U(h)/(L + 1/4)U(h))$; given $u \in U(h)$ we denote by $\overline{u}$ its image in B_l. We now define an element of P_l of B_l in the following way: if $l = -1/4$, we set $P_{-1/4} = P$; if $l \neq -1/4$, we set

$$P_l = \frac{\overline{L} + 1/4}{l + 1/4}. \tag{V.2.1}$$

As a consequence of (V.2.1) and (V.2.2), one has $P_l^2 = 1$,

$$[X, P_l] = 0, \quad \forall X \in s \quad \text{and} \quad \overline{A}_\pm P_l = -P_l \overline{A}_\pm .$$

We now define elements of B_l by

(V.4.2). $Y_l = \overline{Y}$, $F_l = \overline{F}$, $G_l = \overline{G}$, $(A_{l\pm}) = \overline{A}_\pm$, $(K_l) = P_l\overline{L}$ and $\widetilde{A}_{l\pm} = iP_l\overline{A}_\pm$.

(V.4.3) Proposition. *The mapping $X \in g \to X_l$ defined by* (V.4.2) *is an isomorphism from g onto a subsuperalgebra of B_l, and it extends to a morphism ϕ_l from $U(g)$ onto B_l, which satisfies $\phi_l(\mathscr{C}) = 0$.*

Corollary 1. *The algebras B_l are primitive quotients of $U(g)$; any irreducible representation of B_l can be extended to a degenerate representation of $U(g)$.*

Corollary 2. *For any irreducible representation π of h, there exists a degenerate representation $\hat{\pi}$ of g such that $\hat{\pi}|_h = \pi$.*

Proof. For (V.4.3), one has to prove that the elements defined in (V.4.2) do satisfy the commutation rules (I.1.4), (I.1.5); it is a straightforward computation, using essentially (V.2.1) and (V.2.2). Then, by the universal property of $U(g)$, we can extend to a morphism ϕ_l from $U(g)$ into B_l. If $l \neq -1/4$, it is clear that ϕ_l is onto. If $l = -1/4$, recalling that $\overline{L} + 1/4 = 0$, we get $K_{-1/4} + 1/4P_{-1/4} = P_{-1/4}(\overline{L} + 1/4) = 0$, so $P = P_{-1/4} = -4K_{-1/4}$ and therefore $B_{-1/4} = W_P = \phi_{-1/4}[U(g)]$. Finally,

$$\begin{aligned}\phi_l(\mathscr{C}) &= \overline{Q} - \tfrac{1}{2}([A_{l+}, A_{l-}]_L + [\widetilde{\overline{A}}_{l+}, \widetilde{\overline{A}}_{l-}]_L) - K_l^2 \\ &= \overline{L}(\overline{L}+1) - \tfrac{1}{2}(\overline{L}+\overline{L}) - \overline{L}^2 = 0.\end{aligned}$$

Corollaries 1 and 2 are immediate consequences of (V.4.3). □

(V.5) Let us now start with an irreducible degenerate representation π of g in V, and introduce the subspaces V_0 and V_1 of (II.1.5), such that $V = V_0 \oplus V_1$, $\pi(\Delta_\pm)|_{V_0} = 0$, $\pi(E_\pm)|_{V_1} = 0$. Note that $\pi(E_+E_-) = \pi(\Delta_+\Delta_-) = 0$. Using (II.1.5), π is an irreducible representation of h, so $\pi(C) = c\,\mathrm{Id}_V$.

By (I.1.9), one has $\pi(L) = 2(\pi(Q) - \pi(C))$; since $V = V_0 \oplus V_1$ is a reduction of $\pi|_s$ into irreducibles ((II.1.5)), we must have $\pi(L)|_{V_0} = l_0$ and $\pi(L)|_{V_1} = l_1$. But $\pi(C) = \pi(L)(\pi(L) + 1/2)$ by (V.2.1), so $l_1 = l_0$ or $l_1 = -l_0 - 1/2$. From (I.1.3), $\pi(A_\pm)V_0 = \pi(E_\pm)V_0$, so, by (II.1.3), $V_1 = \pi(A_+)V_0 + \pi(A_-)V_0$. Using (V.2.2), $\pi(L)\pi(A_\pm)v = (-l_0 - 1/2)\pi(A_\pm)v$, if $v \in V_0$, so we conclude that $l_1 = -l_0 - 1/2$. We set $l = l_0$, and use (V.2.1) to obtain $\pi(C) = l(l+1/2)$, $\pi(Q)|_{V_0} = l(l+1)$ and $\pi(Q)|_{V_1} = (l - 1/2)(l + 1/2)$. So, if we assume that V is infinite dimensional, we have $\pi(U(h)) \simeq B_l$, if $l \neq -1/4$, and $\pi(U(h)) \simeq W$, if $l = -1/4$ **[16]**.

Now $L = [A_+, A_-]_L = 1/2[E_+ + \Delta_+, E_- + \Delta_-]_L$, so, using (I.1.1), and $\Delta_\pm|_{V_0} = 0$, $E_\pm|_{V_1} = 0$, $\Delta_+\Delta_- = E_+E_- = 0$ on V, we deduce

$$\begin{cases} \pi(L)|_{V_0} = \pi(K)|_{V_0}, \\ \pi(L)|_{V_1} = -\pi(K)|_{V_1}. \end{cases}$$

It follows that if P' is the parity operator defined by $P'v = (-1)^i v$, $v \in V_i$, one has $\pi(K) = P'\pi(L)$, and by similar arguments $\pi(\widetilde{A}_\pm) = iP'\pi(A_\pm)$.

If $l = -1/4$, then $\pi(L) = -1/4\mathrm{Id}_V$, so $\pi(K) = -1/4P'$, and if we introduce $B_{-1/4} \simeq \pi(U(h)) \oplus P'\pi(U(h))$, we deduce that $\pi(U(g)) \simeq B_{-1/4}$.

If $l \neq -1/4$, then $\pi(P_l)|_{V_0} = 1$, $\pi(P_l)|_{V_1} = -1$, so $\pi(P_l) = P'$, and therefore $\pi(U(g)) = \pi(U(h)) \simeq B_l$. So we have proved

(V.5.1) PROPOSITION. *Let V be a degenerate infinite dimensional primitive quotient of $U(g)$; then there exists l such that $V \simeq B_l$.*

(V.5.2) REMARK. Note that $B_l = B_{l'}$ if $l' = -l - 1/2$, so we can restrict to admissible values of l.

(V.6) We now treat the case of finite dimensional degenerate primitive quotients. The discussion is very similar to the preceding case, so we give fewer details.

First, we note that, any irreducible finite dimensional degenerate representation π of g being actually an irreducible representation of h, since Burnside's theorem holds for finite dimensional irreducible representations of h [**16**], one has $\pi(U(g)) = \pi(U(h)) = L(V)$, if V is the space of π. Moreover, from [**3**], there exists $n \in 1/2\mathbb{N}$ such that $\dim V = 4n$, $V = V_{\bar{0}} \oplus V_{\bar{1}}$, with $\dim V_{\bar{0}} = 2n+1$, $\dim V_{\bar{1}} = 2n-1$, and $\pi(C) = n(n+1/2)$. Actually π induces an isomorphism $\overline{\pi}$ from B_{n/J_n} onto $L(V)$, where J_n is the unique nontrivial two-sided ideal of B_n [**16**]. Let us note $V = V_n$, $\pi = \pi_n$, and introduce $\overline{\phi}_n = \overline{\pi}_n \circ \phi_n$; we get a surjective morphism from $U(g)$ onto $L(V_n)$.

(V.6.1) Summarizing:

PROPOSITION. *Any primitive degenerate nontrivial finite dimensional quotient of $U(g)$ is an algebra $L(V_n)$, $n \in 1/2\mathbb{N}$, $\dim V_n = 4n$, $V_n = V_{n\bar{0}} \oplus V_{n\bar{1}}$, with $\dim V_{n\bar{0}} = 2n+1$, $\dim V_{n\bar{1}} = 2n-1$.*

(VI) A classification of primitive ideals of $U(g)$.

(VI.1) Primitive ideals of $U(s)$, and $U(h)$ are well known (see e.g. [**16**]). It will turn out that classification of primitive ideals of $U(g)$ is related to both classifications, according to the fact that one has to distinguish between degenerate and regular cases. We introduce the following notation: we denote by $\mathrm{Prim}_d U(g)$ the set of degenerate primitive, and by $\mathrm{Prim}_r U(g)$ the set of regular primitive, so we have a partition $\mathrm{Prim}\, U(g) = \mathrm{Prim}_d U(g) \cup \mathrm{Prim}_r U(g)$. Now, we need a description of $\mathrm{Prim}\, U(g_{\bar{0}})$ and of $\mathrm{Prim}\, U(g_+)$ (where $g_+ = g_{\bar{0}} \oplus g_{\bar{1}}$) which is achieved by the following lemma:

(VI.1.1) LEMMA. *Given a primitive ideal I of $U(s)$ and a complex λ_0, let*

$$\varphi(I, \lambda_0) = I \oplus (K - \lambda_0)U(s),$$
$$\psi(I, \lambda_0) = \varphi(I, \lambda_0) \oplus U(g_{\bar{0}})\Delta_+ \oplus U(g_{\bar{0}})\Delta_-.$$

Then φ *and* ψ *are one-to-one mappings from* $\operatorname{Prim} U(s)$ *onto* $\operatorname{Prim} U(g_{\overline{0}})$ *and* $\operatorname{Prim} U(g_+)$.

Proof. Actually, this comes from the fact that, starting from an irreducible representation π of s, one can extend to $g_{\overline{0}}$ by $\pi(K) = \lambda_0$, and then to g_+ by $\pi(\Delta_+) = \pi(\Delta_-) = 0$, and, conversely, any irreducible representation of $g_{\overline{0}}$, or g_+ is obtained by this way as is easily seen from the commutation rules (I.1.1). □

(VI.1.2) Corollary. $\operatorname{Prim} U(g_{\overline{0}}) = \operatorname{Prim} U(g_+) = \operatorname{Prim} U(s) \times \mathbb{C}$.

(VI.2). We classify $\operatorname{Prim}_d U(g)$. Let $\mathscr{E}$ be the set obtained from $\operatorname{Prim} U(h) \times Z_2$, $Z_2 = \{+, -\}$, by the identification $(I, +) = (I, -)$, if $I = U(h)(L + 1/4)$, or if $\operatorname{codim} I = 1$. Let $I \in \operatorname{Prim} U(h)$ ([**16**]),

• if $I = (C - c)U(h)$, with $c = l(l + 1/2)$, l admissible, and $l \neq -1/4$, we define:

$E_+(I) = E(I, +) = \operatorname{Ker} \phi_l$, $E_-(I) = E(I, -) = \operatorname{Ker} \phi_{-(l+1/2)}$.

• if $I = (L + 1/4)U(h)$, we define:

$E_+(I) = E(I, +) = E_-(I) = E(I, -) = \operatorname{Ker} \phi_{-1/4}$.

• if $I = \operatorname{Ker} \mathscr{D}_f(n)$, $n \in 1/2\mathbb{N}$, setting $\pi_n = \mathscr{D}_f(n)$, we define:

$$E_+(I) = E(I, +) = \operatorname{Ker}(\pi_n \circ \phi_n),$$
$$E_-(I) = E(I, -) = \operatorname{Ker}(\pi_n \circ \phi_{-n-1/2}),$$

if $n \neq 0$, and $E_\pm(\operatorname{Ker} \mathscr{D}_f(0)) = \operatorname{Ker} \mathscr{D}_f(0, 0)$.

So we get a mapping $E : \mathscr{E} \to \operatorname{Prim}_d U(g)$.

(VI.2.1) Proposition. *E is a one-to-one mapping from $\mathscr{E}$ onto* $\operatorname{Prim}_d U(g)$.

Proof. E is onto by (V.4), (V.5), and (V.6). Moreover, $E_\pm(I) \cap U(h) = I$; assume that $E_+(I) = E_-(I) = J$, $I \neq (L + 1/4)U(h)$. Let l be admissible such that $(C - l(l + 1/2))U(h) \subset I$. Then by (V.2.1)

$$(K_l - l)(K_l - (l + 1/2)) = \frac{1}{(l + 1/4)^2}[(\overline{L} - l)(\overline{L} + l + 1/4)(\overline{L} - l - 1/4)(\overline{L} + l + 1/2)] = 0,$$

and since $K_{-l-1/2} = -K_l$, the same holds when replacing l by $-(l + 1/2)$. Therefore $(K - l)(K - (l + 1/2)) \in E_+(I)$, and $(K + l) \times (K + (l + 1/2)) \in E_-(I)$. So $K \in J$. From (I.1.5), we deduce $h \subseteq J$, so $h \subseteq I$, and $I = \operatorname{Ker} \mathscr{D}_f(0)$. This proves that E is one-to-one. □

(VI.3) We now classify $\operatorname{Prim}_r U(g)$.

Given $(I, \lambda_0) \in \operatorname{Prim} U(g_+)$, with $I \in \operatorname{Prim} U(s)$, let us fix any irreducible representation ρ of g_+ such that $\operatorname{Ker} \rho = (I, \lambda_0)$, and introduce $J = \operatorname{Ker}[\operatorname{Ind}_{g_+ \uparrow g}(\rho)]$, which is a two-sided ideal of $U(g)$. Actually, J is not necessarily primitive, so we have to restrict to $\operatorname{Prim}_r U(g_+)$, which is defined as follows:

Let $(I, \lambda_0) \in \operatorname{Prim} U(g_+)$; then, from the classification of primitive ideals of $U(s)$ (see e.g. **[16]**), there exists a (unique) $q \in \mathbb{C}$ such that $(Q - q)U(s) \subset I$. We then compute $q - \lambda_0(\lambda_0 + 1)$, and define $\operatorname{Prim}_r U(g_+)$ as the subset of (I, λ_0) satisfying $q - \lambda_0(\lambda_0 + 1) \neq 0$.

So we assume $(I, \lambda_0) \in \operatorname{Prim}_r U(g_+)$, and then, by (II.1.6) $\pi = \operatorname{Ind}_{g_+ \uparrow g}(\rho)$ is irreducible, so $J = \operatorname{Ker} \pi$ is primitive. Given ρ' such that $\operatorname{Ker} \rho' = (I, \lambda_0)$, $\pi' = \operatorname{Ind}_{g_+ \uparrow g}(\rho')$ and $J' = \operatorname{Ker} \pi'$, one has

(VI.3.1) LEMMA. $J' = J$.

Proof. By **[4**, (5.1)], J (resp. J') is the biggest two-sided ideal of $U(g)$ contained in $U(g)(I, \lambda_0)$.

From (VI.3.1), we can now define a mapping E from $\operatorname{Prim}_r U(g_+)$ into $\operatorname{Prim}_r U(g)$ by $E(I) = J$.

(VI.3.2) PROPOSITION. *E is one-to-one mapping from* $\operatorname{Prim}_r U(g_+)$ *onto* $\operatorname{Prim}_r U(g)$.

Proof. Let $J = E(I, \lambda_0) = E(I', \lambda_0')$. Let $\hat{\pi}$ be any irreducible representation of g, acting on $\widehat{V}$, and such that $\operatorname{Ker} \hat{\pi} = J$. With the notations of (I), let $\hat{\rho}$ be the corresponding irreducible representation of g_+ on V_0; then by (II.1.7) $\hat{\pi} = \operatorname{Ind}_{g_+ \uparrow g} \hat{\rho}$. Now let $\hat{\rho}(K) = \hat{\lambda}_0$, and $\hat{\rho}(Q) = \hat{q}$; the minimal polynomial of $\widehat{K} = \pi(K)$ is

$$(\widehat{K} - \hat{\lambda}_0)(\widehat{K} - (\hat{\lambda}_0 + 1/2))(\widehat{K} - (\hat{\lambda} + 1)) = m(\widehat{K});$$

therefore $J \cap \mathbb{C}[K] = m(K)\mathbb{C}[K]$, and this proves (since $\hat{\lambda}_0$ is the eigenvalue with smallest real part) that $\hat{\lambda}_0$, which was a priori dependent of $\hat{\rho}$, is actually not. It follows that $\lambda_0 = \lambda_0' = \hat{\lambda}_0$. Secondly, there exists a unique $c \in \mathbb{C}$ such that $\mathscr{C} - c \in J$, and one has $c = \hat{q} - \lambda_0(\lambda_0 + 1)$ (see (II.1.6)). So $\hat{q}$ is also independent of $\hat{\rho}$, and therefore $q = q' = \hat{q}$.

If J is of infinite codimension, then I and I' have to be also of infinite codimension, so, from the classification of $\operatorname{Prim} U(s)$ (see e.g. **[16]**), $I = (Q - q)U(s)$, $I' = (Q - q')U(s)$ and then $I = I'$. If J is of finite codimension, then I and I' are two primitive ideals of

$U(s)$, both of finite codimension, and both containing $(Q-q)U(s)$; therefore $I = I'$. So we have proved that E is one-to-one.

Now given $J \in \mathrm{Prim}_r U(g)$, using (II.1.7), there exists $(I, \lambda_0) \in \mathrm{Prim}_r U(g_+)$ such that $E(I, \lambda_0) = J$; so E is onto. □

(VI.4) $\mathrm{Prim}\, U(s)$ and $\mathrm{Prim}\, U(h)$ are very well known and classified (e.g. **[16]**), so (VI.3.2), together with (VI.2.1) and (VI.1.1), gives a complete classification of $\mathrm{Prim}\, U(g)$.

(VI.5) PROPOSITION. *If I is a degenerate primitive ideal of $U(g)$, and if* $\operatorname{codim} I = \infty$, *then I is minimal primitive, and I is not generated by its intersection with $Z(g)$.*

Proof. Assuming I degenerate primitive, and $\operatorname{codim} I = \infty$, we first prove that I is minimal primitive:

If J is primitive, and $J \subset I$, then J is degenerate; we set $J = E(J', \alpha)$, $I = E(I', \beta)$, with $J', I' \in \mathrm{Prim}\, U(h)$ and $\alpha, \beta \in Z_2$. Then $J' = U(h) \cap J$, $I' = U(h) \cap I$, $\operatorname{codim} J' = \operatorname{codim} I' = \infty$, and $J' \subset I'$, so, using the results of **[9]**, $J' = I'$. Then $E(I', \alpha) \subset E(I', \beta)$; If $\alpha \neq \beta$, arguments similar to the proof of (VI.2.1) give $K \in I = E(I', \beta)$, therefore, by (I.1.5), $I = \operatorname{Ker} D_f(0, 0)$ (a contradiction), so $I = J$.

Note that $I \cap Z(g) = \operatorname{Ker}\varepsilon$ ((IV.5)), $U(g)\operatorname{Ker}\varepsilon \subset I$. By (VI.2.1), there exist infinitely many degenerate primitive I with $\operatorname{codim} I = \infty$, which are all minimal primitive from beginning of the proof, so, necessarily, they all satisfy $I \neq U(g)\operatorname{Ker}\varepsilon$. □

(VI.6) As pointed out in **[16]**, the $h_{\bar{0}}$-reduction of irreducible h-modules is contained in the following "structural" equation:

$$4Q^2 - (8C-1)Q + 2C(2C-1) = 0$$

which holds in $U(h)$ between the respective Casimir elements of $h_{\bar{0}}$ and h. We now give the corresponding "structural" equations in $U(g)$, which involve Q, K, $\mathscr{C}$ and $\mathscr{D}$. They contain the $g_{\bar{0}}$-reduction of regular irreducible g-modules:

(VI.6.1) PROPOSITION.

$$[(2K+1)\mathscr{C} - \mathscr{D}][2K\mathscr{C} - \mathscr{D}][(2K-1)\mathscr{C} - \mathscr{D}] = 0,$$

$$\begin{aligned}&[\mathscr{C}^2 Q - (\mathscr{C}^3 + 1/4(\mathscr{D}^2 - \mathscr{C}^2))]\\ &\quad \times [(\mathscr{C}^2 Q - (\mathscr{C}^3 + 1/4\mathscr{D}^2))^2 - \mathscr{C}^2(\mathscr{C}^3 + 1/4\mathscr{D}^2)] = 0.\end{aligned}$$

Proof. These formulae can be checked directly (e.g. with the help of a computer!), or preferably as follows:

Replacing both, $\mathscr{C}$ and $\mathscr{D}$ respectively by $(l-\lambda_0)(l+\lambda_0+1)$ and $(2\lambda_0+1)(l-\lambda_0)(l+\lambda_0+1)$, they reduce to

$$\begin{aligned}&(l-\lambda_0)^3(l+\lambda_0+1)^3((2K+1)-(2\lambda_0+1))\\ &\qquad \times(2K-(2\lambda_0+1))((2K-1)-(2\lambda_0+1))=0,\\ &(l-\lambda_0)^6(l+\lambda_0+1)^6(Q-l(l+1))\\ &\qquad \times(Q-(l+1/2)(l-1/2))(Q-(l+1/2)(l+3/2))=0,\end{aligned}$$

and these last two relations are true in any representations $\mathscr{D}_f(l, \lambda_0)$ ((II.2.2), (II.2.4)). So using (III.1), we obtain (VI.6.1). □

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