Partially Measurable Sets in Measur Spaces

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1. Introduction. This article is concerned with any sets, whether measurable or not, in a general measure space (a general measure space is briefly defined in the article). A set S has an interior measure $m_i(S)$ and an exterior measure $m_e(S)$, where $0 \leq m_i(S) \leq m_e(S)$. Consider two disjoint sets S_1 and S_2, i.e. $S_1 \cap S_2 = \emptyset$, and their union set $S_1 \cup S_2$. There are the six non-negative quantities $m_i(S)$ and $m_e(S)$, for $S = S_1, S_2$, and $S_1 \cup S_2$. It is well-known that $m_e(S)$ is subadditive, i.e. $m_e(S_1 \cup S_2) \leq m_e(S_1) + m_e(S_2)$, and $m_i(S)$ is superadditive, i.e. $m_i(S_1 \cup S_2) \geq m_i(S_1) + m_i(S_2)$. The present article obtains more inequalities on the six quantities $m_i(S)$ and $m_e(S)$ for $S = S_1, S_2$, and $S_1 \cup S_2$; and indeed obtains a specific collection of six linear inequalities. One of these six inequalities states that the average measure $\frac{1}{2}(m_i(S) + m_e(S))$ is subadditive. Also, average measure is countably subadditive. Further, if the general measure space satisfies a certain two conditions, it is shown that these six inequalities form a *complete* collection of *independent* inequalities, valid for every pair of disjoint sets S_1, S_2. These two condition on the measure space do hold for the usual measure spaces.

Previous articles of the author considered the same matters for Lebesgue measure on the real number line or in Euclidean n-dimensional space. These are listed as [1] and [2] in the References at the end of the present article. The present article extends these results to a general measure space, subject to some limitations.

The inequalities on $m_e(S)$ and $m_i(S)$, for $S = S_1, S_2$, and $(S_1 \cup S_2)$, where $S_1 \cap S_2 = \emptyset$, are stated in Theorems 2 and 4 below, in §3. Furthermore, if the measure space satisfies two conditions, the partitionable condition and the basis condition, these inequalities are shown to be a *complete* collection of inequalities which are valid for every pair of disjoint sets contained in a measurable set M.

This is stated in Theorem 8 in §6. The partitionable condition on
a measure space is discussed in §4, and the basis condition in §5.
These conditions do hold for Lebesgue or Borel measure, and in the
usual measure spaces. The above results, without specifying the
two conditions, were obtained in the author's previous articles cited
above when the measure space is Lebesgue or Borel measure on the
real number line or in Euclidean n-dimensional space.

A set S for which $m_i(S) < m_e(S)$ has been called a non-measu-
rable set. In the introduction to my previous article [2] in Pa-
cific Journal of Mathematics, it is stated that a non-measurable set
S might be more appropriately called a partially measurable set
since one does have some information concerning the measure of S,
namely that its "measure" is something between $m_i(S)$ and $m_e(S)$
inclusive. In the present article, I am indeed calling such a set *par-
tially measurable*, as indicated in the title of the present article. In a
broad sense of measure, if one is thinking of applications, not only in
mathematics, an exact measurement might not be available for some
process or subject. But a lower value and an upper value might be
available, like interior measure and exterior measure. Or one might
be concerned with estimates rather than measurements. Finding
properties of the lower and upper values would be of interest.

Further results concerning partially measurable sets are obtained
in the present article, including some interesting partially measur-
able sets. These latter are obtained in §5 and §7, and in particular
in Theorems 7 and 9. Also, if one has two disjoint sets contained
in a partially measurable set S, instead of in a measurable set M, a
complete collection of inequalities is obtained in §3 and §7, stated
in Theorems 4 and 10. These hold as well for Lebesgue or Borel
measure on the real number line or in Euclidean space, and is a
supplement to the previous articles cited above. It should also be
stated that the present article, as well as the previous articles cited
above, makes use of transfinite induction and the axiom of choice
in set theory, but does not make use of the continuum hypothesis
of set theory (which is that $2^{\aleph_0} = \aleph_1$).

2. **Measure Spaces.** The definition and properties of set the-
ory and of measure spaces are fairly standard. For the sake of def-
initeness and generality, the definition and properties of a general

measure space, as used in the present article, will be stated briefly here.

A general measure space, written $(X, \mathcal{M}, m)$, is a non-empty set X of elements, which are called "points"; and a non-empty collection $\mathcal{M}$ of sets of points in X which form a *countable ring* of sets; and a measure function $m(M)$ defined for every set M in the collection $\mathcal{M}$, where $m(M)$ is a *non-negative real* number or ∞, and the measure function m is required to be *countably additive*, and $m(\emptyset) = 0$ where $\emptyset$ is the empty set.

The terms countable ring and countably additive are defined immediately below. A non-empty collection $\mathcal{M}$ of subsets of X is called a *ring* if, whenever M_1 and M_2 are both members of the collection $\mathcal{M}$, written $M_1 \in \mathcal{M}$ and $M_2 \in \mathcal{M}$, then $(M_1 \cup M_2) \in \mathcal{M}$ and $(M_1 - M_2) \in \mathcal{M}$, where $\cup$ is the symbol for the set-theoretic sum (or union) of sets, and $-$ in $M_1 - M_2$ is the symbol for the set-theoretic difference of sets. Note that the empty set $\emptyset$ is in M, since $\emptyset = M - M$ for any M in $\mathcal{M}$. It follows that $(M_1 \cap M_2) \in \mathcal{M}$, where $\cap$ is the symbol for set-theoretic product (or intersection) of sets, since $(M_1 \cap M_2) = M_1 - (M_1 - M_2)$, noting that $M_1 - M_2 = M_1 - (M_1 \cap M_2)$. Consequently, $\left(\bigcup_{j=1}^{n} M_j \right) \in \mathcal{M}$ and $\left(\bigcap_{j=1}^{n} M_j \right) \in \mathcal{M}$ if $M_j \in \mathcal{M}$ for every $j = 1, 2, \ldots, n$. A non-empty collection $\mathcal{M}$ of subsets of X is called a *countable ring* if $\mathcal{M}$ is a ring and if, whenever a *countable* infinity $M_j, j = 1, 2, \ldots, n, \ldots$, of sets are all in $\mathcal{M}$, then $\left(\bigcup_{j=1}^{\infty} M_j \right) \in \mathcal{M}$. It follows that $\left(\bigcap_{j=1}^{\infty} M_j \right) \in \mathcal{M}$, since

$$\left(\bigcap_{j=1}^{\infty} M_j \right) = M_1 - \bigcup_{j=1}^{\infty} (M_1 - M_j).$$

The *non-negative* set function $m(M)$, defined for all sets M in $\mathcal{M}$, is said to be *additive* if, whenever $M_1 \in \mathcal{M}$, $M_2 \in \mathcal{M}$, and $(M_1 \cap M_2) = \emptyset$ (which is described by the words "M_1 and M_2 are *disjoint* sets"), then $m(M_1 \cup M_2) = m(M_1) + m(M_2)$. The *non-negative* set function $m(M)$, defined over the countable ring $\mathcal{M}$, is said to be *countably additive* if it is *additive* and if, whenever M_j for $j = 1, 2, \ldots, n, \ldots$, are a *countable* infinity of *mutually disjoint* sets in $\mathcal{M}$, i.e., $(M_{j_1} \cap M_{j_2}) = \emptyset$ for every j_1 and j_2 with $j_1 \neq j_2$, then

$m\left(\bigcup\limits_{j=1}^{\infty} M_j\right) = \sum\limits_{j=1}^{\infty} m(M_j)$. Note that the values of $m(M)$ for all M
in $\mathcal{M}$ lie in the *extended* non-negative real number system, i.e., all
non-negative real numbers and ∞. Incidentally, note that if M_1, M_2
are both in $\mathcal{M}$ and $M_1 \subset M_2$ ($\subset$ is the symbol for set-theoretic
inclusion, and $\supset$ is the symbol for set-theoretic containment), then
$m(M_1) \leq m(M_2)$, since $M_2 = M_1 \cup (M_2 - M_1)$ and $M_1 \cap (M_2 - M_1) = \emptyset$, so that $m(M_2) = m(M_1) + m(M_2 - M_1) \geq m(M_1)$; and if
$m(M_1) < \infty$ then $m(M_2 - M_1) = m(M_2) - m(M_1)$. It also follows
that if there is a set $M \in \mathcal{M}$ with $m(M) < \infty$, then $m(\emptyset) = 0$ since
$\emptyset = M - M$ and $m(\emptyset) = m(M) - m(M) = 0$. Even if there is no
such $M \in \mathcal{M}$, it is supposed that $m(\emptyset) = 0$.

The above gives the definition of a measure space $(X, \mathcal{M}, m)$ as
used in this article. Note that the above definition of a general
measure space does not assume that the set X is in $\mathcal{M}$. If $X \notin \mathcal{M}$, the measure space $(X, \mathcal{M}, m)$ can be extended, if it is desired,
to a measure space $(X, \mathcal{M}^*, m^*)$ in which the countable ring $\mathcal{M}^*$
includes the set X as well as all of $\mathcal{M}$, and $m^*(M) = m(M)$ for
all $M \in \mathcal{M}$. This is discussed near the end of §6 herein. Also
note that the definition of a general measure space does not assume
the "countability condition for a measure space", which is stated
and discussed in the latter part of §6 herein, and this matter does
not enter prior to that, nor in §7. And the definition of a general
measure space does not assume that it is a complete measure space,
i.e. that every subset of a set M in $\mathcal{M}$ with $m(M) = 0$ also is in
$\mathcal{M}$ and has m-measure 0. The measure space can be extended, if
desired, by completing it.

Consider a general measure space $(X, \mathcal{M}, m)$. For *any* set S of
points in X, the interior measure $m_i(S)$ and exterior measure $m_e(S)$
can be defined (for $m_e(S)$, it is required that there be a set $M \in \mathcal{M}$
for which $S \subset M$). The *interior measure* of S (also called inner
measure of S), written $m_i(S)$, is defined as usual by

$$(2.1) \quad m_i(S) = \sup_{M \subset S} m(M), \text{ for all } M \in \mathcal{M} \text{ with } M \subset S,$$

where $\sup_{M \subset S} m(M)$ is the supremum, or least upper bound, of all
the numbers $m(M)$ with $M \in \mathcal{M}$ and $M \subset S$. There is at least one
such M, namely $\emptyset$ for which $m(\emptyset) = 0$. As is well-known, shown

from (2.1),

$$(2.2) \quad \begin{cases} m_i(S) \geq 0; \text{ and if } S_1 \subset S_2, \text{ then} \quad m_i(S_1) \leq m_i(S_2); \\ \text{and if } S \in \mathcal{M}, \text{ then} \quad m_i(S) = m(S); \\ \text{and if } S_j, \; j = 1, 2, \ldots \text{ to } n \text{ or} \\ \text{to } \infty, \text{ are } \textit{mutually disjoint}, \text{ then} \quad m_i\left(\bigcup_j S_j\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \geq \sum_j m_i(S_j). \end{cases}$$

The last in (2.2) states that the set function m is *superadditive*, and indeed is *countably superadditive*, for *disjoint* sets. Also,

$$(2.3) \qquad \text{there is a set } A \subset S \text{ for which}$$
$$A \in \mathcal{M} \text{ and } m(A) = m_i(S).$$

For the *exterior* measure $m_e(S)$ of a set S (also called outer measure of S), suppose that there is a set $M \in \mathcal{M}$ for which $S \subset M$. (If $X \in \mathcal{M}$, this is satisfied for every S.) The exterior measure $m_e(S)$ is defined as usual by

$$(2.4) \qquad m_e(S) = \inf_{M \supset S} m(M), \text{for all } M \in \mathcal{M} \text{ with } M \supset S,$$

where $\inf_{M \supset S}$ is the infimum, or greatest lower bound, of all the numbers $m(M)$ with $M \supset S$ and $M \in \mathcal{M}$. As is well-known, shown from (2.4)

$$(2.5) \quad \begin{cases} m_e(S) \geq m_i(S) \geq 0; \\ \quad \text{and if } S_1 \subset S_2, \text{ then} \qquad m_e(S_1) \leq m_e(S_2); \\ \quad \text{and if } S \in \mathcal{M}, \text{ then} \qquad m_e(S) = m(S); \\ \quad \text{and if } S_j, \; j = 1, 2, \ldots, \text{ to } n \\ \quad \text{or to } \infty, \text{ are a countable} \\ \quad \text{number of sets, then} \qquad m_e\left(\bigcup_j S_j\right) \leq \sum_j m_e(S_j). \end{cases}$$

The last in (2.5) states that the set function $m_e(S)$ is *subadditive*, and indeed is *countably subadditive*. Also,

$$(2.6) \qquad \text{there is a set } B \in \mathcal{M} \text{ for which } S \subset B \subset M \text{ and}$$
$$m(B) = m_e(S)$$

The above in (2.4), (2.5), (2.6) assumed that there was a set $M \in \mathcal{M}$ such that the sets S and S_1, S_2, and S_j are all contained in M.

Note that, in the general measure space $(X, \mathcal{M}, m)$, the interior measure $m_i(S)$ is defined for every set S in X, but the exterior measure $m_e(S)$ is defined for a set S only if there is a set M in $\mathcal{M}$ for which $S \subset M$. Whenever $m_e(S)$ is used in the present article it is assumed that there is a set $M \in \mathcal{M}$ with $S \subset M$. If $X \notin M$, there are sets S with no such M, for example the set X itself. If $X \notin \mathcal{M}$, the measure space $(X, \mathcal{M}, m)$ can be extended if desired to a measure space $(X, \mathcal{M}^*, m^*)$ in which $X \in \mathcal{M}^*$, as discussed near the end of §6 herein.

An example of a measure space $(X, \mathcal{M}, m)$ is Lebesgue measure on the real number line or in Euclidean n-dimensional space (or Borel measure). In this case, X is the set of points on the real number line or in Euclidean space, and $\mathcal{M}$ is the collection of Lebesgue (or Borel) measurable sets, and the measure function $m(M)$ is the Lebesgue (or Borel) measure of M. Then for any set S the interior measure $m_i(S)$ and exterior measure $m_e(S)$ are the usual Lebesgue interior and exterior measures of S.

Consider now a general measure space $(X, \mathcal{M}, m)$. There are some lemmas concerning the interior and exterior measures, $m_i(S)$ and $m_e(S)$.

LEMMA 1. *Suppose $M^j \in \mathcal{M}, j = 1, 2, \ldots$ to n or ∞, are a countable number of mutually disjoint sets in $\mathcal{M}$. Then for any set S,*

$$(2.7) \quad \begin{cases} m_i(S) &= \sum_j m_i(M^j \cap S) + m_i\left(\left(X - \bigcup_j M^j\right) \cap S\right), \\ m_e(S) &= \sum_j m_e(M^j \cap S) + m_e\left(\left(X - \bigcup_j M^j\right) \cap S\right), \end{cases}$$

where in the second line of (2.7), $S \subset M$ for some $M \in \mathcal{M}$.

Proof. Select a set $A \in \mathcal{M}$ with $A \subset S$ and $m(A) = m_i(S)$. Then $(M^j \cap A) \subset (M^j \cap S)$ so that $m(M^j \cap A) \leq m_i(M^j \cap S)$; and

$$\left(X - \bigcup_j M^j\right) \cap A = \left(A - \bigcup_j (M^j \cap A)\right) \in \mathcal{M}, \text{ and}$$

$$\left(\left(X - \bigcup_j M^j\right) \cap A\right) \subset \left(\left(X - \bigcup_j M^j\right) \cap S\right) \text{ so that}$$

$$m\left(A - \bigcup_j (M^j \cap A)\right) \leq m_i\left(\left(X - \bigcup_j M^j\right) \cap S\right). \text{ Since}$$

$$A = \left(\bigcup_j (M^j \cap A)\right) \cup \left(A - \bigcup_j (M^j \cap A)\right), \text{ one has}$$

$$m_i(S) = m(A) = \left(\sum_j m_i(M^j \cap A)\right) + m\left(A - \bigcup_j (M^j \cap A)\right)$$

$$\leq \left(\sum_j m_i(M^j \cap S)\right) + m_i\left(X - \bigcup_j (M^j \cap S)\right).$$

But interior measure m_i is countably superadditive for disjoint sets, as in (2.2), which is the reverse inequality for $m_i(S)$, so that the first equality of (2.7) is established.

Select a set $B \in \mathcal{M}$ with $S \subset B \subset M$ and $m(B) = m_e(S)$. Then $(M^j \cap S) \subset (M^j \cap B)$ so that $m(M^j \cap B) \geq m_e(M^j \cap S)$; and

$$\left(\left(X - \bigcup_j M^j\right) \cap S\right) \subset \left(\left(X - \bigcup_j M^j\right) \cap B\right) = \left(B - \bigcup_j (M^j \cap B)\right) \in \mathcal{M}$$

so that $m\left(B - \bigcup_j (M^j \cap B)\right) \geq m_e\left(\left(X - \bigcup_j M^j\right) \cap S\right)$. Since

$$B = \bigcup_j (M^j \cap B) \cup \left(B - \bigcup_j (M^j \cap B)\right) \text{ one has}$$

$$m_e(S) = m(B) = \left(\sum_j m(M^j \cap B)\right) + m\left(B - \bigcup_j (M^j \cap B)\right)$$

$$\geq \left(\sum_j m_e(M^j \cap S)\right) + m_e\left(\left(X - \bigcup_j M^j\right) \cap S\right).$$

But exterior measure m_e is countably subadditive, as in (2.5), which is the reverse inequality for $m_e(S)$, so that the second equality of (2.7) is established. Lemma 1 is proved.   □

Lemma 1 is an extension to a general measure space of Lemma 1 in the author's article, *Measure-theoretic Properties of Non-measurable Sets*, Pacific J. Math. **138**(1989), 357-389, which is listed in the Bibliography at the end of the present article as [2]. This lemma is on p.359, proof on p.360, of [2]; and Lemma 1 above, where

$S = \bigcup_{\nu} S_\nu$, is an extension to $(X, \mathcal{M}, m)$ since X need not be in $\mathcal{M}$. The article Lemma 1 concerns Lebesgue measure on the real number line or in Euclidean n-dimensional space.

Also, Lemmas 2 and 3 and their proofs, on pages 360, 361 of [2], hold in a general measure space $(X, \mathcal{M}, m)$. Just replace the word "measurable" and words "measurable set" by the words "in $\mathcal{M}$" and "set in $\mathcal{M}$" respectively. For the statements and proofs use only the countable ring and countable additivity properties of Lebesgue measure, and these hold in $(X, \mathcal{M}, m)$. Also, formulas (2.1) on the page 360 of [2] hold; just replace the sentence after formulas (2.1) by the sentence: This is by Lemma 1, since for the second formula of (2.1) S is contained in the set $(M - (L \cap M))$, which is in $\mathcal{M}$ and is disjoint from L, where $S \subset M \in \mathcal{M}$. Note again that throughout the present article, if $m_e(S)$ is used, then it is supposed that there is a set M in $\mathcal{M}$ such that $S \subset M$. Lemma 2 is stated below, as Lemma 2, since it is used frequently in later sections of the present article; its proof is on pages 360,361 of [2].

LEMMA 2. *Suppose that $(S_1 \cup S_2)$ is in $\mathcal{M}$, where S_1 and S_2 are disjoint sets, $S_1 \cap S_2 = \emptyset$. Then*

$$m(S_1 \cup S_2) = m_i(S_1) + m_e(S_2) = m_e(S_1) + m_i(S_2).$$

Theorem 4 on page 372 of [2] also holds in a general measure space $(X, \mathcal{M}, m)$. The inequalities for $m_e(S_1 \cup S_2)$ in that theorem are proved in the first paragraph of the proof on page 372 of [2], where L and B_1 are in $\mathcal{M}$. The inequalities for $m_i(S_1 \cup S_2)$ in that theorem can be proved similarly, instead of the proof that is given on pages 372, 373 of [2] (which involves the functionals d_i and d_e and $d_i \leq d_e$, to be taken up later in the present article), as follows. Pick a set $K \in \mathcal{M}$ with $K \subset (S_1 \cup S_2)$ and $m(K) = m_i(S_1 \cup S_2)$, and a set $A_2 \in \mathcal{M}$ with $A_2 \supset S_2$ and $m(A_2) = m_e(S_2)$. Then $(K - (K \cap A_2)) \subset S_1$, since a point in $(K - (K \cap A_2))$ is in K and not in A_2, and so not in S_2, and therefore is in S_1 since $K \subset (S_1 \cup S_2)$. Thus, $m(K - (K \cap A_2)) \leq m_i(S_1)$; also $m(K \cap A_2) \leq m(A_2) =$

$m_e(S_2)$. Now

$$K = (K - (K \cap A_2)) \cup (K \cap A_2) \quad \text{so that}$$
$$m_i(S_1 \cup S_2) = m(K) = m(K - (K \cap A_2)) + m(K \cap A_2)$$
$$\leq m_i(S_1) + m_e(S_2).$$

This is one of the inequalities for $m_i(S_1 \cup S_2)$ in theorem 4 of [2], and the other inequality is obtained by interchanging the roles of S_1 and S_2. Theorem 4 on page 372 of [2] is thus proved for a general measure space $(X, \mathcal{M}, m)$.

It will be important to consider those sets $\widetilde{M}$ in $\mathcal{M}$ for which

$$(2.8) \quad \begin{cases} \text{either} \quad m(\widetilde{M}) < \infty \\ \text{or} \quad m(\widetilde{M}) = \infty \text{ and } \widetilde{M} = \bigcup_{j=1}^{\infty} M^j \text{ where} \\ \quad M^j \in \mathcal{M} \text{ and } m(M^j) < \infty \text{ for all } j. \end{cases}$$

In (2.8), it suffices that $\widetilde{M} \subset \left(\bigcup_{j=1}^{\infty} M^j \right)$ since then $\widetilde{M} = \bigcup_{j=1}^{\infty} (\widetilde{M} \cap M^j)$.

In case $m(\widetilde{M}) = \infty$ and $\widetilde{M} = \bigcup_{j=1}^{\infty} M^j$ as in (2.8), then $\widetilde{M} = \bigcup_{j=1}^{\infty} (M^j - \bigcup_{k=1}^{j-1} M^k)$, where for $j = 1$, $\bigcup_{k=1}^{0} M^k$ means $\emptyset$. The sets $\widetilde{M}^j = M^j - \bigcup_{k=1}^{j-1} M^k$ for all $j = 1, 2, \ldots$ to ∞ are *mutually disjoint* since

$$\left(M^j - \bigcup_{k=1}^{j-1} M^k \right) \cap M^n = \emptyset \text{ for } n < j,$$

and $\widetilde{M}^j \in \mathcal{M}$ with $m(\widetilde{M}^j) \leq m(M^j) < \infty$, and unite all the $\widetilde{M}^j$ for which $m(\widetilde{M}^j) = 0$ together with an $\widetilde{M}^j$ for which $m(\widetilde{M}^j) > 0$; and renumber the resulting j's from 1 to ∞. Thus, a set $\widetilde{M}$ which satisfies (2.8) and has $m(\widetilde{M}) = \infty$ can be expressed as

$$(2.9) \quad \begin{cases} \widetilde{M} = \bigcup_{j=1}^{\infty} \widetilde{M}^j \quad \text{where } \widetilde{M}^j \in \mathcal{M} \text{ and} \\ \quad 0 < m(\widetilde{M}^j) < \infty \text{ for all } j, \text{ and} \\ \widetilde{M}^{j_1} \cap \widetilde{M}^{j_2} = \emptyset \quad \text{for all } j_1, j_2 \text{ with } j_1 \neq j_2. \end{cases}$$

The collection of all sets $\widetilde{M}$ in $\mathcal{M}$ which satisfy (2.8), will be designated by $\widetilde{\mathcal{M}}$:

$$(2.10) \qquad \widetilde{\mathcal{M}} = \text{ collection of all sets } \widetilde{M} \text{ in } \mathcal{M} \text{ which}$$

$$\text{satisfy (2.8), or the sentence below (2.8).}$$

Concerning the collection $\widetilde{\mathcal{M}}$, it is easily seen to be a countable ring. Also, $(M \cap \widetilde{M}) \in \widetilde{\mathcal{M}}$ for every $M \in \mathcal{M}, \widetilde{M} \in \widetilde{\mathcal{M}}$. Indeed, using the same measure $m(\widetilde{M})$ for all $\widetilde{M} \in \widetilde{\mathcal{M}}$ as in $(X, \mathcal{M}, m)$, one sees that $(X, \widetilde{\mathcal{M}}, m)$ is a measure space also. And note that if $S \subset \widetilde{M_1}$ for some $\widetilde{M_1} \in \widetilde{\mathcal{M}}$, then $m_i(S)$ and $m_e(S)$ obtained in $(X, \widetilde{\mathcal{M}}, m)$ have the same values as $m_i(S)$ and $m_e(S)$ obtained in $(X, \mathcal{M}, m)$. For, if $M \in \mathcal{M}$ and $M \subset S$ then $M \subset \widetilde{M_1}$ and $M = (M \cap \widetilde{M_1}) \in \widetilde{\mathcal{M}}$, so that $m_i(S)$ is the same. And if $M \in \mathcal{M}$ and $M \supset S$, then $(M \cap \widetilde{M}) \supset S$ for every $\widetilde{M} \in \widetilde{\mathcal{M}}$ with $\widetilde{M} \supset S$, and $m(M) \geq m(M \cap \widetilde{M})$ so that in taking the infimum of $m(M)$ one can limit oneself to sets $M \cap \widetilde{M}$, and these are in $\widetilde{\mathcal{M}}$; thus $m_e(S)$ is the same. Also, concerning $\widetilde{\mathcal{M}}$, there is Lemma 6 stated and proved in §7 of the present article. And in §6 of the present article, a condition is stated when $\widetilde{\mathcal{M}} = \mathcal{M}$.

3. Functionals and Inequalities for Two Disjoint Sets.

Consider a general measure space $(X, \mathcal{M}, m)$, and consider any subsets S of X. For two disjoint sets S_1 and S_2, several functionals of the pair S_1, S_2 will be defined. The notation and definitions of these functionals are the same as in the author's article, *Measure-theoretic Properties of Non-measurable Sets*, Pacific J. Math. 138(1989), 357-389, which is listed in the Bibliography at the end of the present article as [2]. Article [2] concerns Lebesgue measure on the real number line or in Euclidean n-dimensional space, and the results in [2] are being generalized to a general measure space $(X, \mathcal{M}, m)$ in the present article. The definitions will first be given immediately below for the case that

$$(3.1) \qquad m_e(S_1 \cup S_2) < \infty, \text{ where } (S_1 \cap S_2) = \emptyset.$$

The functionals are written as $d_i(S_1, S_2)$, $d_e(S_1, S_2)$, $g_1(S_1, S_2)$,

$g_2(S_1, S_2)$, and $h(S_1, S_2)$. Their definitions in the case (3.1) are:

$$
(3.2) \quad
\begin{cases}
d_i(S_1, S_2) & = m_i(S_1 \cup S_2) - m_i(S_1) - m_i(S_2), \\
d_e(S_1, S_2) & = m_e(S_1) + m_e(S_2) - m_e(S_1 \cup S_2), \\
g_1(S_1, S_2) & = m_e(S_1 \cup S_2) - m_i(S_1) - m_e(S_2), \\
g_2(S_1, S_2) & = m_e(S_1 \cup S_2) - m_i(S_2) - m_e(S_1), \\
h(S_1, S_2) & = m_i(S_1) + m_e(S_1) + m_i(S_2) + m_e(S_2) \\
& \quad - m_i(S_1 \cup S_2) - m_e(S_1 \cup S_2).
\end{cases}
$$

Another form for the last equation in (3.2) is

$$
(3.3) \qquad h(S_1, S_2) = d_e(S_1, S_2) - d_i(S_1, S_2),
$$

which follows from the first two equations in (3.2). These functionals will be shown to be *non-negative* in value, as in [2]. That the first two functionals in (3.2), i.e. $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$, are ≥ 0 is well-known, expressing the superadditivity of interior measure and subadditivity of exterior measure. Note that also $h(S_1, S_2)$ is stated to be ≥ 0, where $h(S_1, S_2)$ is given in (3.3).

Let $f(S_1, S_2)$ be any one of the functionals $d_i(S_1, S_2)$, $d_e(S_1, S_2)$, $g_1(S_1, S_2), g_2(S_1, S_2), h(S_1, S_2)$. There is the following theorem.

THEOREM 1. *Consider two disjoint sets S_1 and S_2 which are contained in $\widetilde{M}$ for some $\widetilde{M}$ in $\widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ is defined in (2.10):*

$$
(3.4) \quad (S_1 \cup S_2) \subset \widetilde{M} \text{ for some } \widetilde{M} \in \widetilde{\mathcal{M}}, \text{ where } (S_1 \cap S_2) = \emptyset.
$$

Let $f(S, S_2)$ be any one of the functionals $d_i(S_1, S_2), d_e(S_1, S_2)$, $g_1(S_1, S_2), g_2(S_1, S_2),$ $h(S_1, S_2)$. Suppose that L_k, for $k = 1, 2, \ldots$ to n or to ∞, are a countable number of mutually disjoint sets in $\mathcal{M}$. Then

$$
(3.5) \quad f(S_1, S_2) = \sum_k f(L_k \cap S_1, L_k \cap S_2)
$$

$$
+ f\left(\left(X - \bigcup_k L_k \right) \cap S_1, \left(X - \bigcup_k L_k \right) \cap S_2 \right).
$$

$\left(\text{Note that } \left(X - \bigcup_k L_k\right) \text{ in (3.5) can be replaced by } \left(\widetilde{M} - \bigcup_k L_k\right),\right.$

$\text{and also if } (S_1 \cup S_2) \subset \left(\bigcup_k L_k\right) \text{ then the last term in (3.5) is 0 since}$

$\left. f(\emptyset, \emptyset) = 0.\right)$

Proof. The theorem will first be proved when $m_e(S_1 \cup S_2) < \infty$. Given any two *disjoint* sets S_1 and S_2 such that $m_e(S_1 \cup S_2) < \infty$, there is a set M in $\mathcal{M}$ such that $(S_1 \cup S_2) \subset M$ and $m(M) = m_e(S_1 \cup S_2) < \infty$, so that $M \in \widetilde{\mathcal{M}}$ by (2.8), and (3.4) holds. Consider any particular case for $f(S_1, S_2)$, for example $f(S_1, S_2) = g_1(S_1, S_2)$. By (3.2) applied to the disjoint sets $L_k \cap S_1$, $L_k \cap S_2$ one has

$$g_1(L_k \cap S_1, L_k \cap S_2) = m_e(L_k \cap (S_1 \cup S_2)) - m_i(L_k \cap S_1) - m_e(L_k \cap S_2).$$

since $(L_k \cap S_1) \cup (L_k \cap S_2) = L_k \cap (S_1 \cup S_2)$, and summing for k going from 1 to n gives

$$(3.6) \quad \sum_{k=1}^{n} g_1(L_k \cap S_1, L_k \cap S_2) = \sum_{k=1}^{n} m_e(L_k \cap (S_1 \cup S_2))$$
$$- \sum_{k=1}^{n} m_i(L_k \cap S_1)$$
$$- \sum_{k=1}^{n} m_e(L_k \cap S_2).$$

If the index k in Theorem 1 runs from 1 to ∞, let $n \to \infty$ in (3.6), giving

$$(3.7) \quad \sum_{k=1}^{\infty} g_1(L_k \cap S_1, L_k \cap S_2) = \sum_{k=1}^{\infty} m_e(L_k \cap (S_1 \cup S_2))$$
$$- \sum_{k=1}^{\infty} m_i(L_k \cap S_1)$$
$$- \sum_{k=1}^{\infty} m_e(L_k \cap S_2),$$

since the quantities on the right-hand side are less than or equal to $m_e(S_1 \cup S_2)$, $m_i(S_1), m_e(S_2)$ respectively, by Lemma 1. There is also

$$(3.8) \quad g_1(Y \cap S_1, Y \cap S_2) = m_e(Y \cap (S_1 \cup S_2))$$
$$- m_i(Y \cap S_1) - m_e(Y \cap S_2),$$

where $Y = X - \bigcup_{k=1}^{n} L_k$ or $Y = X - \bigcup_{k=1}^{\infty} L_k$, according as k runs from 1 to n or 1 to ∞ in Theorem 1. Adding (3.8) to (3.6) or (3.7) accordingly yields, for the right-hand side of (3.5) when $f(S_1, S_2)$ is $g_1(S_1, S_2)$, the value

$$(3.9) \qquad m_e(S_1 \cup S_2) - m_i(S_1) - m_e(S_2),$$

by Lemma 1. The expression (3.9) is equal to $g_1(S_1, S_2)$ by (3.2). Thus, (3.5) is established when $f(S_1, S_2)$ is $g_1(S_1, S_2)$.

A similar proof holds for any of the other functionals $d_i(S_1, S_2)$, $d_e(S_1, S_2), g_2(S_1, S_2), \quad h(S_1, S_2)$ in (3.2). Theorem 1 is therefore proved for the case that $m_e(S_1 \cup S_2) < \infty$. $\qquad \square$

Incidentally, Theorem 1 above, for the case that $m_e(S_1 \cup S_2) < \infty$ and $f(S_1, S_2)$ is $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$, is stated and proved in [2], on page 362 of [2], as Lemma 4 on page 362, where $S_1 = \cup S_1^\nu, S_2 = \cup S_2^\nu$, and $(S_1^\nu \cup S_1^\nu) \subset L^\nu$, the L^ν being disjoint measurable sets. (There is a misprint on page 365 of [2], in formula (3.5) there; it should read $\sum_\nu$, not $\sum_\mu$.) It is stated in later pages of [2] that a similar proof holds for $g_1(S_1, S_2), g_2(S_1, S_2)$ and $h(S_1, S_2)$.

Still for the case that $m_e(S_1 \cup S_2) < \infty$, the quantities $d_i(S_1, S_2)$ and $d_e(S_1, S_2)$ are non-negative, since they state the superadditivity of interior measure respectively. And also $g_1(S_1, S_2) \geq 0$ and $g_2(S_1, S_2) \geq 0$, by Theorem 4 on page 372 of [2] (using the inequalities for $m_e(S_1 \cup S_2)$ in that Theorem 4), which holds in a general measure space $(X, \mathcal{M}, m)$, as stated and proved in the preceding section, §2 of the present article.

Also, the functional $h(S_1, S_2)$ is non-negative, stated in the form

$$(3.10) \qquad d_i(S_1, S_2) \leq d_e(S_1, S_2),$$

The inequality (3.10) is stated as a theorem on page 367 of [2], and proved on pages 367-369. The same proof holds in the general measure space $(X, \mathcal{M}, m)$, when $m_e(S_1 \cup S_2) < \infty$, replacing the words "measurable set" by "set in $\mathcal{M}$", and "(the entire space - B_1)" and "(the entire space - B_2)" by "$(M - B_1)$" and "$(M - B_2)$" respectively, where $M \in \mathcal{M}$ and $(S_1 \cup S_2) \subset M$ with $m(M) = m_e(S_1 \cup S_2) < \infty$; and also "bounded set" by "set with finite exterior

measure". Note that Lemma 4 in [2], referred to in the proof on page 367 of [2], is included in Theorem 1 of the present article, as stated two paragraphs above. Thus (3.10) holds, so that $h(S_1, S_2)$ in (3.3) and (3.2) is non-negative.

Thus, all the functionals $d_i(S_1, S_2), d_e(S_1, S_2), g_1(S_1, S_2), g_2(S_1, S_2)$ and $h(S_1, S_2)$, defined in (3.2) (and also (3.3)), for the case that $m_e(S_1 \cup S_2) < \infty$, are *non-negative*. Their definitions will now be given for two disjoint sets S_1, S_2, when

$$(3.11) \qquad m_e(S_1 \cup S_2) = \infty \text{ and}$$
$$(S_1 \cup S_2) \subset \widetilde{M} \text{ for some } \widetilde{M} \in \widetilde{\mathcal{M}}$$

where $\widetilde{\mathcal{M}}$ is defined in (2.10). Then $m(\widetilde{M}) = \infty$, and the set $\widetilde{M}$ can be expressed, by (2.9) as

$$(3.12) \quad \begin{cases} \widetilde{M} = \bigcup_{j=1}^{\infty} M^j, & \text{where } M^j \in \widetilde{M} \text{ and } m(M^j) < \infty \\ & \text{for all } j, \text{ and } M^{j_1} \cap M^{j_2} = \emptyset \\ & \text{for all } j_1, j_2 \text{ with } j_1 \neq j_2. \end{cases}$$

Let $f(S_1, S_2)$ be any one of the functionals $d_i(S_1, S_2), d_e(S_1, S_2),$ $g_1(S_1, S_2), g_2(S_1, S_2)$ or $h(S_1, S_2)$, and define $f(S_1, S_2)$ by

$$(3.13) \qquad f(S_1, S_2) = \sum_{j=1}^{\infty} f(M^j \cap S_1, M^j \cap S_2).$$

One has that $m_e((M^j \cap S_1 0 \cup (M^j \cap S_2)) \leq m(M^j) < \infty$, and the formula in (3.2) can be applied to obtain $f(M^j \cap S_1, M^j \cap S_2)$, which is finite and non-negative, and (3.13) yields $f(S_1, S_2)$. Note that ∞ is a possible value of $f(S_1, S_2)$. The values of $f(S_1, S_2)$ are in the extended non-negative real number system, including ∞ as well as non-negative real numbers.

Concerning $f(S_1, S_2)$ as defined in (3.12) and (3.13), if $\widetilde{M} = \bigcup_{j=1}^{\infty} N^j$, where $N^k \in \mathcal{M}$ and $m(N^k) < \infty$ and the N^k are mutually disjoint, is another expression for M as in (3.12), then

$$f(N^k \cap S_1, N^k \cap S_2) = \sum_{j=1}^{\infty} f(M^j \cap (N^k \cap S_1), M^j \cap (N^k \cap S_2))$$

by Theorem 1 already proved for the finite case since $m_e((N^k \cap S_1) \cup (N^k \cap S_1)) \leq m(N^k) < \infty$ and $((N^k \cap S_1) \cup (N^k \cap S_1)) \subset (S_1 \cup S_2) \subset \widetilde{M} = \bigcup_{j=1}^{\infty} M^j$. Summing for all k gives

$$\sum_{j=1}^{\infty} f(N^k \cap S_1, N^k \cap S_1) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} f(M^j \cap N^k \cap S_1, M^j \cap N^k \cap S_2)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f(N^k \cap (M^j \cap S_1), N^k \cap (M^j \cap S_2)) = \sum_{j=1}^{\infty} f(M^j \cap S_1, M^j \cap S_2)$$

by Theorem 1 since $m_e((M^j \cap S_1) \cup (M^j \cap S_2)) \leq m(M^j) < \infty$ and $((M^j \cap S_1) \cup (M^j \cap S_2)) \subset (S_1 \cup S_2) \subset \left(\bigcup_{k=1}^{\infty} N^k\right)$, the interchange of the order of summation being legitimate since all terms are non-negative. This shows that $\sum_{k=1}^{\infty} f(N^k \cap S_1, N^k \cap S_2)$ is equal to the right-hand side of (3.13), so that the definition (3.13) of $f(S_1, S_2)$ is independent of which division of $\widetilde{M}$ as in (3.12) is used.

Also, the definition (3.13) of $f(S_1, S_2)$ is independent of which $\widetilde{M}$ as in (3.11) is used. If $m_e(S_1, S_2) = \infty$ and $(S_1 \cup S_2) \subset \widetilde{N}$ for some $\widetilde{N} \in \widetilde{\mathcal{M}}$, and $\widetilde{N} = \bigcup_{k=1}^{\infty} N^k$ with $N^k \in \mathcal{M}$ and $m(N^k) < \infty$ for all k and the N^k are mutually disjoint, then

$$\widetilde{M} = (\widetilde{M} - \widetilde{N}) \cup (\widetilde{M} \cap \widetilde{N}) = \left(\bigcup_{j=1}^{\infty} (M^j - \widetilde{N})\right) \cup \left(\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} (M^j \cap N^k)\right),$$

so that using $\widetilde{M}$ one has

$$f(S_1, S_2) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} f(M^j \cap N^k \cap S_1, M^j \cap N^k \cap S_2)$$

since $(M^j - \widetilde{N}) \cap S_1 = \emptyset, (M^j - \widetilde{N}) \cap S_2 = \emptyset$ and $f(\emptyset, \emptyset) = 0$.

Likewise, using $\widetilde{N}$ for the definition of $f(S_1, S_2)$, the same double sum expression is obtained for $f(S_1, S_2)$. This shows that the value of $f(S_1, S_2)$ is independent of which $\widetilde{M}$ is used in (3.11). Incidentally, note that if $(S_1 \cup S_2) \subset \widetilde{M}$ and $m_e(S_1 \cup S_2) < \infty$, then the

formulas (3.13) and (3.12) are correct formulas, by Theorem 1 already proved when $m_e(S_1 \cup S_2) < \infty$, since $\left(X - \bigcup\limits_{j=1}^{\infty} M^j\right) \cap S_1 = \emptyset$, $\left(X - \bigcup\limits_{j=1}^{\infty} M^j\right) \cap S_2 = \emptyset$, and $f(\emptyset, \emptyset) = 0$.

The following theorem has been proved above.

THEOREM 2. *Consider two disjoint sets S_1 and S_2 which are contained in $\widetilde{M}$ for some $\widetilde{M} \in \widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ is defined in (2.10) and (2.8). Then there are* non-negative *functionals of S_1, S_2, namely $d_i(S_1, S_2), d_e(S_1, S_2), g_1(S_1, S_2), g_2(S_1, S_2)$ and $h(S_1, S_2)$, i.e.,*

$$(3.14) \quad \begin{cases} f(S_1, S_2) \geq 0, & \text{where } f(S_1, S_2) \text{ is any one of} \\ & d_i(S_1, S_2), d_e(S_1, S_2), g_1(S_1, S_2), \\ & g_2(S_1, S_2), h(S_1, S_2). \end{cases}$$

These are defined in (3.2) when $m_e(S_1 \cup S_2) < \infty$, and in (3.12), (3.13) when $m_e(S_1 \cup S_2) = \infty$.

Returning to Theorem 1, it has been proved above, after its statement, when $m_e(S_1 \cup S_2) < \infty$. Consider now the case that $m_e(S_1 \cup S_2) = \infty$. Then $f(S_1, S_2)$ is given by (3.12), (3.13). Since $m_e((M^j \cap S_1) \cup (M^j \cap S_2)) \leq m(M^j) < \infty$, one has by (3.5) that

$$f(M^j \cap S_1, M^j \cap S_2) = \sum_k f(L_k \cap M^j \cap S_1, L_k \cap M^j \cap S_2)$$

$$+ f\left(\left(X - \bigcup_k L_k\right) \cap M^j \cap S_1, \left(X - \bigcup_k L_k\right) \cap M^j \cap S_2\right).$$

Summing for all j from 1 to ∞, one obtains

$$\sum_{j=1}^{\infty} f(M^j \cap S_1, M^j \cap S_2) = \sum_k \sum_{j=1}^{\infty} f(L_k \cap M^j \cap S_1, L_k \cap M^j \cap S_2)$$

$$+ \sum_{j=1}^{\infty} f\left(\left(X - \bigcup_k L_k\right) \cap M^j \cap S_1, \left(X - \bigcup_k L_k\right) \cap M^j \cap S_2\right),$$

having interchanged the order of summation, since all terms are non-negative. From (3.13), this is

$$f(S_1, S_2) = \sum_k f(L_k \cap S_1, L_k \cap S_2)$$

$$+ f\left(\left(X - \bigcup_k L_k\right) \cap S_1, \left(X - \bigcup_k L_k\right) \cap S_2\right)$$

which is (3.5). Theorem 1 is completely proved.

Incidentally, formulas (3.2) and (3.3) can be written in a form which is valid even if $m_e(S_1 \cup S_2) = \infty$. Namely, transpose all the negative terms in these formulas to the other side of the equations, resulting in

$$(3.15) \quad \begin{cases} d_i(S_1, S_2) + m_i(S_1) + m_i(S_2) = m_i(S_1 \cup S_2), \\ d_e(S_1, S_2) + m_e(S_1 \cup S_2) = m_e(S_1) + m_e(S_2), \\ g_1(S_1, S_2) + m_i(S_1) + m_e(S_2) = m_e(S_1 \cup S_2), \\ g_2(S_1, S_2) + m_i(S_2) + m_e(S_1) = m_e(S_1 \cup S_2), \\ h(S_1, S_2) + m_i(S_1 \cup S_2) + m_e(S_1 \cup S_2) \\ \quad = m_i(S_1) + m_e(S_1) + m_i(S_2) + m_e(S_2), \end{cases}$$

and

$$(3.16) \qquad h(S_1, S_2) + d_i(S_1, S_2) = d_e(S_1, S_2).$$

These are true when $m_e(S_1 \cup S_2) < \infty$, and also when $m_e(S_1 \cup S_2) = \infty$ and $(S_1 \cup S_2) \subset \widetilde{M} \in \widetilde{\mathcal{M}}$, by writing (3.15), (3.16) for the sets $M^j \cap S_1, M^j \cap S_2$ and then summing over all j from 1 to ∞, using (3.12), (3.13) and Lemma 1 for the sums involving m_i and m_e.

The inequality $d_i(S_1, S_2) \le d_e(S_1, S_2)$, or $h(S_1, S_2) \ge 0$, can be stated in an interesting form. For any set S define the *average measure* of S, written $m_a(S)$, by

$$(3.17) \qquad m_a(S) = \frac{1}{2}(m_i(S) + m_e(S)),$$

it being supposed that there is an $M \in \mathcal{M}$ with $S \subset M$. Note that $\frac{1}{2} m_e(S) \le m_a(S) \le m_e(S)$. Consider two disjoint sets S_1, S_2 for which $m_e(S_1) < \infty, m_a(S_2) < \infty$. The statement that $h(S_1, S_2) \ge 0$ can be written in the form

$$(3.18) \qquad m_a(S_1 \cup S_2) \le m_a(S_1) + m_a(S_2).$$

This follows from the formula for $h(S_1, S_2)$ in (3.2). If $m_e(S_j) = \infty$ for $j = 1$ or 2 or both, formula (3.18) is still true since then $m_a(S_j) = \infty$. Thus, $m_a(S)$ is subadditive; and it is also non-negative and monotone increasing. Incidentally, the subadditivity (3.18) also holds if S_1 and S_2 are not disjoint, since

$$m_a(S_1 \cup S_2) = m_a(S_1 \cup (S_2 - S_1)) \leq m_a(S_1) + m_a(S_2 - S_1)$$
$$\leq m_a(S_1) + m_a(S_2).$$

Indeed, $m_a(S)$ is countably subadditive, as in the following theorem.

THEOREM 3.   *The average measure* $m_a(S)$ *of a set* S, *defined in* (3.17), *is subadditive and indeed is countably subadditive. That is, if* S_j, *for* $j = 1, 2, \ldots$ *to* n *or to* ∞, *is a countable number of sets such that* $\left(\bigcup_j S_j\right) \subset M$ *for some* $M \in \mathcal{M}$, *then*

$$m_a\left(\bigcup_j S_j\right) \leq \sum_j m_a(S_j).$$

Proof. The subadditivity of $m_a(S)$ for two sets has been proved above, in (3.18). The proof for any countable number n or ∞ of sets S_j is given in [2], in the last paragraph on page 370 and on page 371 of [2]. $\Big($There is a misprint in the middle of page 371 of [2], in formula (5.1) there. The right-hand side of (5.1) should be $\sum_{\nu}^{\infty} m_a(S_\nu) + \epsilon.\Big)$ The same proof holds for the general measure space $(X, \mathcal{M}, m)$, replacing the words "measurable set" by "set in $\mathcal{M}$".   □

Also, the *complementation* and other properties of $m_a(S)$, as in the lower part of page 371 and the top paragraph of page 372 of [2], hold in the general measure space $(X, \mathcal{M}, m)$.

Returning to the non-negative functionals of two disjoint sets it will be shown in a later section of the present article that the four functionals $d_i(S_1, S_2), g_1(S_1, S_2),\ g_2(S_1, S_2)$, and $h(S_1, S_2)$ are independent functionals, while $d_e(S_1, S_2)$ is expressible in terms of them

by (3.16). To simplify the writing, place

$$(3.19) \quad \begin{cases} a_1 = m_i(S_1), & a_2 = m_i(S_2), a = m_i(S_1 \cup S_2), \\ b_1 = m_e(S_1), & b_2 = m_e(S_2), b = m_e(S_1 \cup S_2), \end{cases}$$

and

$$(3.20) \quad \begin{cases} d_i = d_i(S_1, S_2), & d_e = d_e(S_1, S_2), \\ g_1 = g_1(S_1, S_2), & g_2 = g_2(S_1, S_2), h = h(S_1, S_2), \end{cases}$$

all of which are non-negative real numbers or ∞. For the case that $m_e(S_1, S_2) < \infty$, the four formulas in (3.2) for d_i, g_1, g_2 and h (omitting the d_e) can be solved for the four quantities a, b_1, b_2, b with the result given in (3.23) shortly below, and also $d_e = d_i + h$. This is easily done, and is done on pages 374, 375 of [2], which use the same letters as above. Thus, the quantities b_1, b_2, a, b can be expressed in terms of the six quantities $a_1, a_2, d_i, h, g_1, g_2$ by (3.23), and so can a_1 and a_2 by $a_1 = a_1, a_2 = a_2$. The formulas (3.22) below follow from (3.23), since by (3.23) one has

$$a_1 + a_2 + d_i = a = m_i(S_1 \cup S_2) \leq m_i(S)$$

and

$$a_1 + a_2 + d_i + h + g_1 + g_2 = b = m_e(S_1 \cup S_2) \leq m_e(S).$$

The following theorem has been obtained above.

THEOREM 4. *Suppose that S_1 and S_2 are two disjoint sets, $S_1 \cap S_2 = \emptyset$, in a measure space $(X, \mathcal{M}, m)$, and that*

$$(S_1 \cup S_2) \subset S, \quad \text{where } S \subset \widetilde{M} \text{ for some } \widetilde{M} \subset \widetilde{\mathcal{M}},$$

where $\widetilde{\mathcal{M}}$ is defined in (2.8) and (2.10). Then there are 6 set functions, whose values are non-negative real numbers or ∞, namely

$$(3.21) \qquad m_i(S_1), m_i(S_2), d_i(S_1, S_2),$$

$$g_1(S_1, S_2), g_2(S_1, S_2), h(S_1, S_2),$$

all ≥ 0, which satisfy

$$(3.22) \quad \begin{cases} a_1 + a_2 + d_i \leq m_i(S), & \qquad and \\ a_1 + a_2 + d_i + g_1 + g_2 + h \leq m_e(S), \end{cases}$$

where the letters in (3.22) have the meaning given in (3.19) and (3.20). The 6 quantities $m_i(S)$ and $m_e(S)$, for $S = S_1, S_2$, and $(S_1 \cup S_2)$, can be expressed in terms of the six quantities in (3.21) by the formulas:

$$(3.23) \qquad \begin{cases} b_1 = a_1 + d_i + h + g_1, \\ b_2 = a_2 + d_i + h + g_2, \\ a = a_1 + a_2 + d_i, \\ b = a_1 + a_2 + d_i + h + g_1 + g_2, \end{cases}$$

and also $a_1 = a_1, a_2 = a_2$.

It will be shown in a later section of the present article that if the measure space $(X, \mathcal{M}, m)$ satisfies some conditions, which are satisfied in the usual measure spaces, then the 6 set functions $a_1, a_2, d_i, g_1, g_2, h$ are a complete collection of independent non-negative set functions.

Incidentally, concerning the set functions $f(S_1, S_2)$, a consequence of Theorem 1 is the following. For given disjoint sets S_1, S_2 with $(S_1 \cup S_2) \subset \widetilde{M} \in \widetilde{\mathcal{M}}$, consider the functional $f(M \cap S_1, M \cap S_2)$ as a functional of the set $M \in \mathcal{M}$. If $M = \bigcup_k M_k$, and $M_k \in \mathcal{M}$ for all k and the M_k are mutually disjoint, then

$$(3.24) \qquad f(M \cap S_1, M \cap S_2) = \sum_k f(M_k \cap S_1, M_k \cap S_2).$$

For, in Theorem 1, replace S_1, S_2 by $M \cap S_1, M \cap S_2$ respectively, and note that $M_k \cap M = M_k$ and $\left(X - \bigcup_k M_k\right) \cap M = \emptyset$. Formula (3.24) and Theorem 1 state that the non-negative functional $f(M \cap S_1, M \cap S_2)$, when considered as a functional of the set $M \in \mathcal{M}$, is countably additive for disjoint sets $M \in \mathcal{M}$. This is like $m(M)$, and also $m_i(M \cap S)$ and $m_e(M \cap S)$ by Lemma 1. And also note that, if A and B are two sets $\in \mathcal{M}$ with $A \subset B$, then

$$f(A \cap S_1, A \cap S_2) \leq f(B \cap S_1, B \cap S_2),$$

since

$$f(B \cap S_1, B \cap S_2) = f(A \cap S_1, A \cap S_2) + f((B-A) \cap S_1, (B-A) \cap S_2)$$

by (3.24) for two sets M_k.

4. Partitionable Sets. Consider a general measure space $(X, \mathcal{M}, m)$, and a set $M \in \mathcal{M}$ with $m(M) > 0$. The set M will be called a *partitionable* set if, for each set B in $\mathcal{M}$ with $B \subset M$ and $m(B) > 0$, there is a set $A \in \mathcal{M}$, where A depends on B, with

$$(4.1) \qquad A \subset B \quad \text{and} \quad 0 < m(A) < m(B).$$

Note that if M is partitionable, then every set $B \subset M$ with $B \in \mathcal{M}$ and $m(B) > 0$ is also partitionable. A measure space $(X, \mathcal{M}, m)$ for which every set $M \in \mathcal{M}$, with $m(M) > 0$, is partitionable will be called a *partitionable measure space*. The following lemma will first be proved.

LEMMA 3. *Suppose that $M \in \mathcal{M}$ is a partitionable set with $m(M) > 0$, in a measure space $(X, \mathcal{M}, m)$. Then for any positive real number r, there is a set*

$$(4.2) \qquad A \in \mathcal{M}, A \subset M, \text{ and } 0 < m(A) \leq r.$$

Proof. Consider first the case that $m(M) < \infty$. There is a set $B \in \mathcal{M}, B \subset M$ with $0 < m(B) < m(M)$, since M is a partitionable set. Then $m(M) = m(B) + m(M - B)$, and at least one of the sets B or $M - B$ has its m-measure $\leq \frac{1}{2}m(M)$. If $r \geq \frac{1}{2}m(M)$, then this set can be taken for the set A. For any positive r, let n be a positive integer such that $\frac{1}{2^n}m(M) \leq r$. There is a set $A_1 \in \mathcal{M}, A_2 \subset A_1$, with $0 < m(A_2) \leq \frac{1}{2}m(A_1) \leq \frac{1}{2^2}m(M)$; etc., etc.; at the n-th step, there is a set $A_n \in \mathcal{M}, A_n \in M$, with $0 < m(A_n) \leq \frac{1}{2^n}m(M) \leq r$. The set A_n is a set A of the lemma. If $m(M) = \infty$, there is a set $B \subset M, B \in \mathcal{M}$ with $0 < m(B) < \infty$, by the partitionability of M. Pick a set $A \subset B, A \in \mathcal{M}$ with $0 < m(A) \leq r$. Lemma 3 is proved. $\qquad\square$

THEOREM 5. *Suppose that $M \in \mathcal{M}$ is a partitionable set, with $m(M) > 0$, in a measure space $(X, \mathcal{M}, m)$. Then, if c is any positive real number or ∞ with $c \leq m(M)$, there is a set $C \in \widetilde{\mathcal{M}}, C \subset M$, with $m(C) = c$. ($\widetilde{\mathcal{M}}$ is defined in (2.10) and (2.8)). If $m(M) = \infty$*

and $c = \infty$, *there is a set* $C \in \widetilde{\mathcal{M}}, C \subset M$ *with* $m(C) = \infty$ *and* $m(M - C) = \infty$.

Proof. (If $c = 0$, the set $\emptyset$ will do, and consider $c > 0$.) There is by Lemma 3, a set $C_1 \in \widetilde{\mathcal{M}}, C_1 \subset M$ with $0 < m(C_1) \le \min(c, 1)$, where $\min(c, 1)$ is the minimum of c and 1. Then $m(M - C_1) = m(M) - m(C_1) \ge c - m(C_1) \ge 0$. If $c - m(C_1) = 0$, pick $C_2 = \emptyset$. If $c - m(C_1) > 0$, there is by Lemma 2 a set $C_2 \in \widetilde{\mathcal{M}}, C_2 \subset (M - C_1)$ with $0 < m(C_2) \le \min(c - m(C_1), 1)$; then

$$m(M - (C_1 \cup C_2)) = m((M - C_1) - C_2)$$
$$= m(M - C_1) - m(C_2)$$
$$\ge c - m(C_1) - m(C_2) \ge 0.$$

If $c - m(C_1) - m(C_2) = 0$, pick $C_3 = \emptyset$; if $c - m(C - 1) - m(C_2) > 0$, pick $C_3 \in \widetilde{\mathcal{M}}, C_3 \subset (M - (C_1 \cup C_2))$ with $0 < m(C_3) \le \min(c - m(C_1) - m(C_2), 1)$. Continue by transfinite induction over countable ordinal numbers. For any countable ordinal number γ, suppose that sets $C_\alpha \in \widetilde{\mathcal{M}}, C_\alpha \subset M$, have been selected for all ordinals $\alpha < \gamma$, such that for every ordinal $\alpha < \gamma$:

$$(4.3) \quad \begin{cases} C_\alpha \subset (M - \bigcup_{\beta < \alpha} C_\beta), \\[1em] \sum_{\beta < \alpha} m(C_\beta) \le c, \\[1em] \text{and if } \sum_{\beta < \alpha} m(C_\beta) = c \quad \text{then } C_\alpha = \emptyset, \\[1em] \text{while if } \sum_{\beta < \alpha} m(C_\beta) < c \quad \text{then } 0 < m(C_\alpha) \\[1em] \qquad\qquad\qquad\qquad \le \min\left(c - \sum_{\beta < \alpha} m(C_\beta), 1\right). \end{cases}$$

The sets C_1, C_2, C_3 have been obtained above, and satisfy (4.3) for $\gamma = 1, 2, 3$. If $\sum_{\alpha < \gamma} m(C_\alpha) = c$, select $C_\gamma = \emptyset$. If $\sum_{\alpha < \gamma} m(C_\alpha) < c$, then $\left(\bigcup_{\alpha < \gamma} C_\alpha\right) \in \widetilde{\mathcal{M}}$ since γ is a countable ordinal number, and

$$m\left(M - \bigcup_{\alpha < \gamma} C_\alpha\right) = m(M) - m\left(\bigcup_{\alpha < \gamma} C_\alpha\right) \ge c - m\left(\bigcup_{\alpha < \gamma} C_\alpha\right) > 0, \text{ and}$$

select $C_{\underline{\gamma}} \in \widetilde{\mathcal{M}}, C_{\underline{\gamma}} \subset \left(M - \bigcup_{\underline{\alpha} < \underline{\gamma}} C_{\underline{\alpha}}\right)$ with $0 < m(C_{\underline{\gamma}}) \le \min\Big(c -$

$m\Big(\bigcup_{\underline{\alpha} < \underline{\gamma}} C_{\underline{\alpha}}\Big), 1\Big)$. Then (4.3) holds for $\alpha = \gamma$ also.

By the principle of transfinite induction, the mutually disjoint sets $C_{\underline{\alpha}} \in \widetilde{\mathcal{M}}$ and $C_{\underline{\alpha}} \subset M$ have been selected for all countable ordinal numbers, with (4.3) being satisfied for all these ordinals $\underline{\alpha}$. And for every countable ordinal $\underline{\alpha}$, either $C_{\underline{\alpha}} = \emptyset$ or $0 < m(C_{\underline{\alpha}}) \le 1$. For any given positive integer n, there are only a countable number of $C_{\underline{\alpha}}$ with $m\Big(C_{\underline{\alpha}} \ge \dfrac{1}{2^n}\Big)$. For, if there are an infinity of them, as $C_{\underline{\alpha}_1}, C_{\underline{\alpha}_2}, \ldots, C_{\underline{\alpha}_j}, \ldots$ where $\underline{\alpha}_1 < \underline{\alpha}_2 < \ldots < \underline{\alpha}_j < \underline{\alpha}_{j+1} < \ldots$ for all integral j from 1 to ∞. Then $m\Big(\bigcup_{j=1}^{\infty} C_{\underline{\alpha}_j}\Big) = \sum_{j=1}^{\infty} m(C_{\underline{\alpha}_j}) = \infty$ since $m(C_{\underline{\alpha}_j}) \ge \dfrac{1}{2^n}$ for every j. If $\underline{\gamma}$ is any countable ordinal $> \underline{\alpha}_j$ for all j, i.e. $\underline{\gamma} \ge \lim_{j \to \infty} \underline{\alpha}_j$, then $m\Big(\bigcup_{\underline{\alpha} < \underline{\gamma}} C_{\underline{\alpha}}\Big) \ge m\Big(\bigcup_{j=1}^{\infty} C_{\underline{\alpha}_j}\Big) = \infty$, and $c = \infty$, so that $C_{\underline{\gamma}} = \emptyset$ by (4.3). In any case, there are at most a countable infinity of $C_{\underline{\alpha}}$ for which $m(C_{\underline{\alpha}}) > 0$. Since the number of countable ordinals is uncountably infinite, there are countable ordinals larger than all these. Let $\underline{\gamma}$ be the smallest countable ordinal such that $m(C_{\underline{\gamma}}) = 0$, so that $\sum_{\underline{\beta} < \underline{\gamma}} m(C_{\underline{\beta}}) = c$ by (4.3).

Place $C = \bigcup_{\underline{\beta} < \underline{\gamma}} C_{\underline{\beta}}$, which $\in \widetilde{\mathcal{M}}$ by (2.8) and (2.10) since $\underline{\gamma}$ is countable ordinal number. Theorem 5, up to the last sentence, is proved. For the last sentence of Theorem 5, which is the case $m(M) = \infty$ and $c = \infty$, one has $C \in \widetilde{\mathcal{M}}, C \subset M$, with $m(C) = \infty = m(M)$. Now, by (2.9), $C = \bigcup_{j=1}^{\infty} M^j$, where the M^j are mutually disjoint and $m(M^j) < \infty$ for all j. By the first part of Theorem 5 already proved, there is $A^j \in M, A^j \subset M^j$ with $m(A^j) = \frac{1}{2}m(M^j)$, and therefore $m(M^j - A^j) = \frac{1}{2}m(M^j)$. Then, the set $A = \bigcup_{j=1}^{\infty} A^j \in$ $\widetilde{\mathcal{M}}$ and $A \subset C \subset M$ and has $m(A) = \sum_{j=1}^{\infty} m(A^j) = \frac{1}{2} \sum_{j=1}^{\infty} m(M^j) =$

$\frac{1}{2}m(C) = \infty$. Also, $(C - A) = \bigcup_{j=1}^{\infty}(M^j - A^j)$ and $m(C - A) =$

$\sum_{j=1}^{\infty}m(M^j - A^j) = \sum_{j=1}^{\infty}\frac{1}{2}m(M^j) = \infty$. Theorem 5 is proved, with the set A as the C of the last sentence of the theorem. $\quad\square$

One easily shows from Theorem 5 that, for a partitionable set $M \in \mathcal{M}$ with $m(M) > 0$, if $c_j \geq 0, j = 1, 2, \ldots$ to n or to ∞, are a countable number of non-negative real numbers or ∞ such that $\sum_j c_j \leq m(M)$, then there are *mutually disjoint* sets $C_j \in \widetilde{\mathcal{M}}$ with $C_j \subset M$ and $m(C_j) = c_j$, for all $j = 1, 2, \ldots$ to n or to ∞. A proof is given on page 384 of the Pacific Journal of Mahtematics article [2], Lemma 8.

There is also the following lemma.

LEMMA 4. *A partitionable set $M \in \widetilde{\mathcal{M}}$ with $m(M) > 0$ contains an uncountable infinity of points. If the partitionable set $M \in \widetilde{\mathcal{M}}$, then any point x in M is contained in a set $M_0 \in \mathcal{M}, M_0 \subset M$, with $m(M_0) = 0$.*

Proof. Considering the second sentence of Lemma 4, suppose first that $0 < m(M) < \infty$. (If $m(M) = 0$, then M_0 can be taken as M.) There is by Theorem 5 a set $C \in \widetilde{\mathcal{M}}$ with $C \subset M$ and $m(C) = \frac{1}{2}m(M)$, and then $m(M - C) = \frac{1}{2}m(M)$ also. One of the sets C or $M - C$ contains the point x, and designating this one by A_1 one has $A_1 \in \widetilde{\mathcal{M}}, A_1 \subset M, x \in A_1$, and $m(A_1) = \frac{1}{2}m(M)$. Then there is $A_2 \in \widetilde{\mathcal{M}}, A_2 \subset A_1, x \in A_2$, and $m(A_2) = \frac{1}{2}m(A_1) = \frac{1}{2^2}m(M)$; and there is $A_3 \in \widetilde{\mathcal{M}}, A_3 \subset A_2, x \in A_3$, and $m(A_3) = \frac{1}{2}m(A_2) = \frac{1}{2^3}m(M)$. Continuing for all possible integers n, there are $A_n \in \widetilde{\mathcal{M}}, A_n \subset A_{n-1}, x \in A_n$, and $m(A_n) = \frac{1}{2^n}m(M)$. Placing $M_0 = \left(\bigcap_{n=1}^{\infty} A_n\right) \subset M$, then $M_0 \in \mathcal{M}, x \in M_0$, and $m(M_0) = 0$, which is a desired M_0. If $m(M) = \infty$, then since $M \in \widetilde{\mathcal{M}}$ one has $M = \bigcup_{j=1}^{\infty} M^j$ by (2.9), where the M^j are mutually disjoint sets

$\in \mathcal{M}$ with $0 < m(M^j) < \infty$, so that $x \in M^{j_1}$ for some j_1. By the preceding there is $M_0 \in \mathcal{M}$ with $M_0 \subset M^{j_1} \subset M, x \in M_0$, and $m(M_0) = 0$. The second sentence of Lemma 4 is proved.

For the first sentence of Lemma 4, there is a set $B \in \mathcal{M}, B \subset M$ with $0 < m(B) < m(M)$. The set $B \in \widetilde{\mathcal{M}}$ since $m(B) < \infty$, and each point x in B is contained in a set $M_0 \in \mathcal{M}, M_0 \subset B$, with $x \in M_0$ and $m(M_0) = 0$. If there were only a countable number of points x in B, then B would be a union of a countable number of sets M_0 with $m(M_0) = 0$, and one would have $m(B) = 0$ contrary to $m(B) > 0$. Thus B, and therefore M, contains an uncountable infinity of points. Lemma 4 is proved. $\qquad\square$

Lemma 4 is stated since one could have a two-valued measure space $(X, \mathcal{M}, m)$ which doesn't contain an infinity of points. Two-valued measure spaces are discussed in §6.

5. A Basis Condition for a Measure Space.

Consider a general measure space $(X, \mathcal{M}, m)$. A *measure basis* of $(X, \mathcal{M}, m)$ will be defined as a collection $\mathcal{C}$ of sets C_α in $\mathcal{M}$ with $m(C_\alpha) > 0$, such that every set M in $\mathcal{M}$ with $m(M) > 0$ contains a set C_α in $\mathcal{C}$, $C_\alpha \subset M$ (where C_α depends on M). There is at least one such collection $\mathcal{C}$, namely all sets M in $\mathcal{M}$ with $m(M) > 0$. Consider the cardinal number of all the different sets C_α in such collection $\mathcal{C}$, and let $\chi^{(f)}$ be the least cardinal number for all such collections $\mathcal{C}$. Or, it suffices to let $\chi^{(f)}$ be the cardinal number of one such collection $\mathcal{C}$. Let $w^{(f)}$ be the least ordinal number with the cardinality $\chi^{(f)}$. There is a collection $\mathcal{C}^{(f)}$ of cardinality $\chi^{(f)}$, called a *basis* collection, and let C_α denote all the sets C_α with $m(C_\alpha) > 0$ in this collection $\mathcal{C}^{(f)}$, where $\underline{\alpha}$ runs over all the non-negative ordinal numbers $\underline{\alpha} < w^{(f)}$.

Consider the cardinal number of points in a set $M \in \mathcal{M}$ with $m(M) > 0$, and let $\chi^{(p)}$ denote the least cardinal number of points in such sets M, among all sets $M \in \mathcal{M}$ with $m(M) > 0$. Thus, every set $M \in \mathcal{M}$ with $m(M) > 0$ has at least $\chi^{(p)}$ points.

The measure space $(X, \mathcal{M}, m)$ will be said to satisfy the *basis condition* if

$$(5.1) \qquad\qquad \chi^{(p)} \geq \chi^{(f)},$$

i.e. if the number of points in every set $M \in \mathcal{M}$ with $m(M) > 0$ is at least $\chi^{(f)}$.

For the usual measure spaces $(X, \mathcal{M}, m)$, the cardinal numbers $\chi^{(p)}$ and $\chi^{(f)}$ are both $2^{\aleph_0}$, the cardinal number of the continuum, so that the usual measure spaces do satisfy the basis condition. For, consider Borel or Lebesgue measure on the real number line or in Euclidean space. There is a measure basis consisting of all bounded closed sets of positive measure. Here $\chi^{(f)} = 2^{\aleph_0}$ (since closed sets are complements of open sets, and the number of these is $2^{\aleph_0}$). And every measurable set M of positive measure contains a bounded closed set of positive measure, and thus contains a continuum number $2^{\aleph_0}$ of points. So, $\chi^{(f)} = \chi^{(p)} = 2^{\aleph_0}$, and the basis condition (5.1) is satisfied.

Consider a general measure space $(X, \mathcal{M}, m)$ and suppose that:

$$(5.2) \qquad \begin{cases} \text{there is at least one set } M \in \mathcal{M} \\ \text{with } m(M) > 0; \text{ and every set } M \in \mathcal{M} \\ \text{with } m(M) > 0 \text{ has an infinity of points.} \end{cases}$$

This is stated since one could have a two-valued measure space with a finite number of points. Two-valued measure spaces are discussed in §6. The following theorem will be proved.

THEOREM 6. *Suppose that $(X, \mathcal{M}, m)$ is a measure space satisfying the basis condition (5.1), and also (5.2). Then there are $\chi^{(p)}$ mutually disjoint sets $Z_{\underline{\beta}} \subset X$, where $\underline{\beta}$ varies over all ordinal numbers of cardinality $< \chi^{(p)}$, such that*

$$(5.3) \qquad X = \bigcup_{\underline{\beta}} Z_{\underline{\beta}}, \text{ where } Z_{\underline{\beta}_1} \cap Z_{\underline{\beta}_2} = \emptyset \text{ for all } \underline{\beta}_1 \neq \underline{\beta}_2,$$

and

$$(5.4) \qquad m_i(Z_{\underline{\beta}}) = 0, \quad m_i(X - Z_{\underline{\beta}}) = 0 \text{ for all } \underline{\beta}.$$

Proof. Let $\underline{w}^{(p)}$ be the smallest ordinal number of cardinality $\chi^{(p)}$ and $\underline{w}^{(p)}$ the smallest ordinal number of cardinality $\chi^{(f)}$, where $\underline{w}^{(f)} \leq \underline{w}^{(p)}$ by (5.1). Since $\chi^{(f)} \chi^{(p)} = \chi^{(p)}$, the set of all ordered

pairs $(\underline{\alpha}, \underline{\beta})$, for all non-negative ordinal numbers $\underline{\alpha}$ and $\underline{\beta}$ for which $\underline{\alpha} < \underline{w}^{(f)}$ and $\underline{\beta} < \underline{w}^{(p)}$, has the cardinality $\chi^{(p)}$ and so can be put into one-to-one correspondence with the set of all non-negative ordinal numbers $\underline{\gamma} < \underline{w}^{(p)}$. For any given $\underline{\gamma} < \underline{w}^{(p)}$, designate the $(\underline{\alpha}, \underline{\beta})$ of the associated pair $(\underline{\alpha}, \underline{\beta})$ in this one-to-one correspondence by $\underline{\alpha}(\underline{\gamma}), \underline{\beta}(\underline{\gamma})$. Thus, for any $\underline{\alpha} < \underline{w}^{(f)}$ and any $\underline{\beta} < \underline{w}^{(p)}$, there is one and only one $\underline{\gamma} < \underline{w}^{(p)}$ for which $\underline{\alpha}(\underline{\gamma}) = \underline{\alpha}$, $\underline{\beta}(\underline{\gamma}_1) = \underline{\beta}$; and if $\underline{\gamma}_1 \neq \underline{\gamma}_2$, then either $\underline{\alpha}(\underline{\gamma}_1) \neq \underline{\alpha}(\underline{\gamma}_2)$ or $\underline{\beta}(\underline{\gamma}_1) \neq \underline{\beta}(\underline{\gamma}_2)$ or both. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Let $\mathcal{C}^{(f)}$, consisting of sets $C_{\underline{\alpha}} \in \mathcal{M}$ with $m(C_{\underline{\alpha}}) > 0$ where $\underline{\alpha}$ varies over all ordinal numbers $\underline{\alpha} < \underline{w}^{(f)}$, be a measure basis of the measure space $(X, \mathcal{M}, m)$. Continuing with the proof of Theorem 6, the following lemma will first be established.

LEMMA 5. *There are points $y_{\underline{\alpha}}$ and $x_{(\underline{\alpha}, \underline{\beta})}$ in X, for all ordinal numbers $\underline{\alpha} < \underline{w}^{(f)}$ and all ordinal numbers $\underline{\beta} < \underline{w}^{(p)}$, which are all different points, and*

(5.5) $\quad y_{\underline{\alpha}} \in C_{\underline{\alpha}}, x_{(\underline{\alpha}, \underline{\beta})} \in C_{\underline{\alpha}}$ *for all $\underline{\alpha} < \underline{w}^{(f)}$ and all $\underline{\beta} < \underline{w}^{(p)}$.*

Proof. For $\underline{\gamma} = 0$, pick two different points $y_{\underline{\alpha}(0)}$ and $x_{(\underline{\alpha}(0), \underline{\beta}(0))}$ in $C_{\underline{\alpha}(0)}$. For $\underline{\gamma} = 1$, pick two different points $y_{\underline{\alpha}(1)}$ and $x_{(\underline{\alpha}(1), \underline{\beta}(1))}$ in $C_{\underline{\alpha}(1)} - \{y_{\underline{\alpha}(0)}, x_{(\underline{\alpha}(0), \underline{\beta}(0))}\}$. Continue by transfinite induction. Suppose, for an ordinal number $\underline{\gamma} < \underline{w}^{(p)}$, that $y_{\underline{\alpha}(\underline{\delta})}$ and $x_{(\underline{\alpha}(\underline{\delta}), \underline{\beta}(\underline{\delta}))}$ have been selected for all ordinal numbers $\underline{\delta} < \underline{\gamma}$ in such a way that they are all *different* points, and

(5.6) $\qquad y_{\underline{\alpha}(\underline{\delta})} \in C_{\underline{\alpha}(\underline{\delta})}, x_{(\underline{\alpha}(\underline{\delta}), \underline{\beta}(\underline{\delta}))} \in C_{\underline{\alpha}(\underline{\delta})}$ for all $\underline{\delta} < \underline{\gamma}$.

Since $\underline{\gamma} < \underline{w}^{(p)}$, the total number of points $y_{\underline{\alpha}(\underline{\delta})}$ and $x_{(\underline{\alpha}(\underline{\delta}), \underline{\beta}(\underline{\delta}))}$ for all $\underline{\delta} < \underline{\gamma}$ has cardinality $< \chi^{(p)}$, and so the set

$$C_{\underline{\alpha}(\underline{\gamma})} - \bigcup_{\underline{\delta} < \underline{\gamma}} \{y_{\underline{\alpha}(\underline{\delta})}, x_{(\underline{\alpha}(\underline{\delta}), \underline{\beta}(\underline{\delta}))}\}$$

has cardinality $\geq \chi^{(p)}$ and is therefore an infinite set. Select two different points $y_{\underline{\alpha}(\gamma)}$ and $x_{(\underline{\alpha}(\gamma),\underline{\beta}(\gamma))}$ in this set:

$$(5.7) \qquad y_{\underline{\alpha}(\gamma)} \text{ and } x_{(\underline{\alpha}(\gamma),\underline{\beta}(\gamma))} \text{ both}$$

$$\in \left(C_{\underline{\alpha}(\underline{\delta})} - \bigcup_{\underline{\delta} < \underline{\gamma}} \{y_{\underline{\alpha}(\underline{\delta})}, x_{(\underline{\alpha}(\underline{\delta}),\underline{\beta}(\underline{\delta}))}\} \right),$$

where $y_{\underline{\alpha}(\gamma)} \neq x_{(\underline{\alpha}(\gamma),\underline{\beta}(\gamma))}$. Thus, $y_{\underline{\alpha}(\underline{\delta})}$ and $x_{(\underline{\alpha}(\underline{\delta}),\underline{\beta}(\underline{\delta}))}$ have been selected for all $\underline{\delta} \leq \underline{\gamma}$, and are all different points, and satisfy (5.6) for all $\underline{\delta} \leq \underline{\gamma}$. This can be done for all $\underline{\gamma} < \underline{w}^{(p)}$, and so by the principle of transfinite induction, one has selected $y_{\underline{\alpha}(\gamma)}$ and $x_{(\underline{\alpha}(\gamma),\underline{\beta}(\gamma))}$ for all $\underline{\gamma} < \underline{w}^{(p)}$ in such a way that they are all *different* points in X, and

$$(5.8) \qquad y_{\underline{\alpha}(\gamma)} \in C_{\underline{\alpha}(\gamma)}; x_{(\underline{\alpha}(\gamma),\underline{\beta}(\gamma))} \in C_{\underline{\alpha}(\gamma)} \text{ for all } \underline{\gamma} < \underline{w}^{(p)}.$$

For any $\underline{\alpha} < \underline{w}^{(f)}$ and $\underline{\beta} < \underline{w}^{(p)}$, there is a single $\underline{\gamma} < \underline{w}^{(p)}$ such that $\underline{\alpha}(\underline{\gamma}) = \underline{\alpha}$ and $\underline{\beta}(\underline{\gamma}) = \underline{\beta}$, and (5.8) is (5.5). Lemma 5 is proved. □

Returning to the proof of Theorem 6, define the set $Z_{\underline{\beta}}$ by

$$(5.9) \quad \begin{cases} Z_{\underline{\beta}} = \{x_{(\underline{\alpha},\underline{\beta})}; \text{ for all } \underline{\alpha} < \underline{w}^{(f)}\}, & \text{for } 0 < \underline{\beta} < \underline{w}^{(p)}, \\ Z_0 = X - \bigcup\limits_{1 \leq \underline{\beta} < \underline{w}^{(p)}} Z_{\underline{\beta}}. \end{cases}$$

Then (5.3) is satisfied, and note that $Z_0 \supset \{y_{\underline{\alpha}}; \text{ for all } \underline{\alpha} < \underline{w}^{(f)}\}$. Let M be a set $\in \mathcal{M}$ with $M \subset Z_{\underline{\beta}}$. If $m(M) > 0$, then there is a $C_{\underline{\alpha}} \in \mathcal{C}^{(f)}$ with $M \supset C_{\underline{\alpha}}$, so that $C_{\underline{\alpha}} \subset Z_{\underline{\beta}}$. But for $\underline{\beta} > 0, y_{\underline{\alpha}} \in C_{\underline{\alpha}}$ and $y_{\underline{\alpha}} \neq Z_{\underline{\beta}}$, so that $C_{\underline{\alpha}} \not\subset Z_{\underline{\beta}}$; and for $\underline{\beta} = 0, x_{(\underline{\alpha},1)} \in C_{\underline{\alpha}}$ and $x_{(\underline{\alpha},1)} \notin Z_0$, so that $C_{\underline{\alpha}} \not\subset Z_0$. These contradictions show that $m(M) = 0$, and therefore $m_i(Z_{\underline{\beta}}) = 0$. Concerning the complementary set $X - Z_{\underline{\beta}}$, let M be a set $\in \mathcal{M}$ with $M \subset (X - Z_{\underline{\beta}})$. If $m(M) > 0$, there is a set $C_{\underline{\alpha}} \in \mathcal{C}^{(f)}$ with $M \supset C_{\underline{\alpha}}$, so that $C_{\underline{\alpha}} \subset (X - Z_{\underline{\beta}})$. But for $\underline{\beta} > 0, x_{(\underline{\alpha},\underline{\beta})} \in C_{\underline{\alpha}}$ and $x_{(\underline{\alpha},\underline{\beta})} \notin (X - Z_{\underline{\beta}})$ so that $C_{\underline{\alpha}} \not\subset (X - Z_{\underline{\beta}})$; and for $\underline{\beta} = 0, y_{\underline{\alpha}} \in C_{\underline{\alpha}}$ and $y_{\underline{\alpha}} \notin (X - Z_0)$ so that $C_{\underline{\alpha}} \not\subset (X - Z_0)$. These contradictions show that $m(M) = 0$, and therefore $m_i(X - Z_{\underline{\beta}}) = 0$. Thus, (5.4) holds, and Theorem 6 is proved. (Incidentally, the different points $y_{(\underline{\alpha}}$ and $x_{(\underline{\alpha},\underline{\beta})}$ in the above proof can be replaced by mutually disjoint point sets of less than $\chi^{(p)}$ points each.)

For Lebesgue or Borel measure on the real number line or in Euclidean space, in which case $\chi^{(p)} = \chi^{(f)} = 2^{\aleph_0}$, Theorem 6 shows that there are a continuum number of mutually disjoint sets Z_β as in Theorem 6. This is an addition to a similar theorem in the author's articles [1], [2], and in particular an addition to Theorem 7 on page 381 of [2]. Likewise for Theorem 7 below.

An immediate consequence of Theorem 6 is Theorem 7.

THEOREM 7. *Suppose that $(X, \mathcal{M}, m)$ is a measure space satisfying the basis condition (5.1), and also (5.2). Then, for any integer $N \geq 2$, and also for any $\chi \leq \chi^{(p)}$, there are N and also χ, mutually disjoint sets Z_γ for $\underline{\gamma} = 0, 1, \ldots, N-1$, or for $\underline{\gamma} = $ all ordinal numbers of cardinality $< \chi$, such that $X = \bigcup_\gamma Z_{\underline{\gamma}}, m_i(Z_{\underline{\gamma}}) = 0$ and $m_i(X - Z_{\underline{\gamma}}) = 0$ for all $\underline{\gamma}$. For any $M \in \mathcal{M}$ with $m(M) > 0$, one has $M = \bigcup_\gamma (M \cap Z_{\underline{\gamma}})$ and*

$$(5.10) \qquad m_i(M \cap Z_{\underline{\gamma}}) = 0 \quad \text{and}$$
$$m_i(M - (M \cap Z_{\underline{\gamma}})) = 0 \text{ for all } \underline{\gamma}$$

and

$$(5.11) \qquad m_e(M \cap Z_{\underline{\gamma}}) = m(M),$$
$$m_e(M - (M \cap Z_{\underline{\gamma}})) = m(M) \text{ for all } \underline{\gamma}.$$

Proof. The first two sentences of Theorem 7 for the case that $\chi = \chi^{(p)}$ is Theorem 6. For the case that $\chi < \chi^{(p)}$, replace Z_0 of Theorem 6 by $\widehat{Z}_0 = Z_0 \cup \left(\bigcup_{\beta \geq N \text{ or } \underline{w}} Z_{\underline{\beta}} \right)$, where $\underline{w}$ is the smallest ordinal number of cardinality χ. Now, $\widehat{Z}_0 \cap Z_1 = \emptyset$ so that $\widehat{Z}_0 \subset (X - Z_1)$ and $m_i(Z_0) = 0$ since $m_i(X - Z_1) = 0$; and $(X - \widehat{Z}_0) \subset (X - Z_0)$ so that $m_i(X - \widehat{Z}_0) = 0$ since $m_i(X - Z_1) = 0$. This establishes the first two sentences of Theorem 7.

Equation (5.10) is a consequence of $(M \cap Z_{\underline{\gamma}}) \subset Z_{\underline{\gamma}}$ and $(M - (M \cap Z_{\underline{\gamma}})) \subset (X - Z_{\underline{\gamma}})$. Also, by Lemma 2,

$$m_e(M \cap Z_{\underline{\gamma}}) + m_i(M - (M \cap Z_{\underline{\gamma}})) = m(M),$$

and
$$m_i(M \cap Z_{\underline{\gamma}}) + m_e(M - (M \cap Z_{\underline{\gamma}})) = m(M),$$
so that (5.11) follows from (5.10). Theorem 7 is proved.   □

(Incidentally, §5 also holds if $\chi^{(p)}$ is a finite number, which would require that the measure space be a direct sum of a finite number of 2-valued measure spaces and of a zero-measure space; and also if the second line of (5.2) is not satisfied, so that $\chi^{(p)}$ can be a finite number ≥ 2, as can be shown. Two-valued measure spaces are discussed briefly in the latter part of §6.)

6. A Complete Set of Independent Inequalities.
The following main theorem will now be proved.

THEOREM 8. *Suppose that $(X, \mathcal{M}, m)$ is a partitionable measure space satisfying the basis condition (5.1). For any set $\widetilde{M}$ in $\widetilde{\mathcal{M}}$ with $m(\widetilde{M}) > 0$, where $\widetilde{\mathcal{M}}$ is defined in (2.10) and (2.8), and any six non-negative real numbers or ∞, namely $a_1, a_2, d_i, h, g_1, g_2$, satisfying*

$$a_1 + a_2 + d_i + h + g_1 + g_2 \leq m(\widetilde{M}),$$

there are two disjoint sets S_1, S_2 in $\widetilde{M}$, $S_1 \subset \widetilde{M}, S_2 \subset \widetilde{M}, S_1 \cap S_2 = \emptyset$, such that (3.20) holds and

$$(6.1) \quad \begin{cases} m_i(S_1) = a_1, \\ m_e(S_1) = a_1 + d_i + h + g_1 & (\text{ which } = b_1), \\ m_i(S_2) = a_2, \\ m_e(S_2) = a_2 + d_i + h + g_2 & (\text{ which } = b_2), \\ m_i(S_1 \cup S_2) = a_1 + a_2 + d_i & (\text{ which } = a), \\ m_e(S_1 \cup S_2) \\ \quad = a_1 + a_2 + d_i + h + g_1 + g_2 & (\text{ which } = b). \end{cases}$$

Proof. (A partitionable measure space is defined in the first paragraph of §4.) The theorem and proof are exactly the same as in [2]. The theorem and proof appears on pages 382-385 of that article, with the symbol X in the article replaced by $\widetilde{M}$ for Theorem 8

above, and the words "measurable set" replaced by "set in $\widetilde{\mathcal{M}}$". Lemma 7 in that proof appears in the present article as Theorem 5, and Lemma 1 is the same as Lemma 1 in the present article; and the sets Z_1 and Z_2 in that proof are the sets Z_1 and Z_2 in Theorem 7 for $N = 3$, since $Z_1 \cup Z_2 = X - Z_0$ so that $m_i(Z_1 \cup Z_2) = 0$ and $m_i(X - (Z_1 \cup Z_2)) = 0$. A misprint appears in the cited article [2] on page 385, line 3; it should read: $Z_1 \cap Z_2 = \emptyset$, not $Z_1 \cap Z_2 \neq \emptyset$. Theorem 8 is proved. $\qquad\qquad\square$

A proof of Theorem 8 is also contained in the proof of the more general Theorem 10 in §7 of the present article. Theorem 10 concerns two disjoint sets contained in any set S, where $S \subset \widetilde{M}$ for some $\widetilde{M}$ in $\widetilde{\mathcal{M}}$. Theorem 8 is the case of Theorem 10 when $S = \widetilde{M}$. Then $m_i(S) = m_e(S) = m(\widetilde{M})$, and the second line of (7.11) is a consequence of the first line of (7.11), and the sets K and L appearing in the proof of Theorem 10 are also $= \widetilde{M}$.

Theorems 2 and 8 are the main theorems concerning the quantities $m_i(S)$ and $m_e(S)$ for $S = S_1, S_2$, and $S_1 \cup S_2$, where S_1 and S_2 are disjoint sets contained in a set $\widetilde{M}$ in $\widetilde{\mathcal{M}}$ with $m(\widetilde{M}) > 0$. They state that the six quantities a_1, b_1, a_2, b_2, a, b defined in (3.19) are subject to six independent inequalities, that $a_1, a_2, d_i, h, g_1, g_2$ are each ≥ 0, and the inequality $a_1 + a_2 + d_i + h + g_1 + g_2 \leq m(M)$, where b_1, b_2, a, b and a_1, a_2 are expressible in terms of them by (6.1). These are valid for every pair of disjoint sets S_1, S_2 contained in $\widetilde{M}$, and any other numerical relation involving a_1, b_1, a_2, b_2, a, b and $m(\widetilde{M})$ which is valid for *every* pair of disjoint sets $\subset \widetilde{M}$ is a consequence of these. For, Theorem 8 states that the six non-negative quantities $a_1, a_2, d_i, h, g_1, g_2$ can have any values independently subject merely to their sum being $\leq m(\widetilde{M})$. In any particular case or cases of two disjoint sets S_1, S_2 in a set $M \in \widetilde{\mathcal{M}}$, there may be more information applicable to the particular case or cases.

Without the use of the symbols d_i, h, g_1, g_2, the inequalities are

$$(6.2) \quad \begin{cases} a_1 \geq 0, \quad a_2 \geq 0, \quad a \geq a_1 + a_2, \\ a + b \geq a_1 + b_1 + a_2 + b_2, \\ b \geq a_1 + b_2, \quad b \geq a_2 + b_1, \\ b \leq m(\widetilde{M}). \end{cases}$$

In other words, the non-negativeness of interior measure, and the

superadditivity of interior measure for disjoint sets and subadditivity of *average* measure for disjoint sets, and the inequalities on the third line of (6.2), and the monotone increasing property of exterior measure, form a complete set of conditions on the quantities $m_i(S)$ and $m_e(S)$ for $S = S_1, S_2,$ and $S_1 \cup S_2$, valid for every pair of disjoint sets S_1, S_2 contained in $\widetilde{M} \in \mathcal{M}$.

In Theorem 8 above, consideration was given to partitionable measure spaces. If the measure space $(X, \mathcal{M}, m)$ is not partitionable, then from §4 there would be a set $M \in \mathcal{M}$ with $m(M) > 0$ which is not a partitionable set, which means there is a set $B \subset M$, with $B \in \mathcal{M}$ and $m(B) > 0$, for which there is no set $A \in \mathcal{M}$ satisfying (4.1). Thus, for this set B, every set $A \in \mathcal{M}$ with $A \subset B$ has $m(A) = m(B)$ or $m(A) = 0$. The subset of M consisting of all sets $A \in \mathcal{M}$ for which $A \subset B$, which could be designated by $\mathcal{M} \cap B$ and includes the set B, forms a countable ring, and the measure m defined for $\mathcal{M} \cap B$ is countably additive. Thus $(B, \mathcal{M} \cap B, m)$ is also a measure space which is a 2-valued measure space, having only the values 0 and $m(B)$ for the measures $m(A)$ of all sets $A \in (\mathcal{M} \cap B)$. This measure space is a part of $(X, \mathcal{M}, m)$.

If one considers all the sets $M \in \mathcal{M}$ for which $M \subset (X - B)$, then these form a countable ring $\mathcal{M} \cap (X - B)$ with a countably additive set function m, and so one has a measure space $(X - B, \mathcal{M} \cap (X - B), m)$, which is also part of $(X, \mathcal{M}, m)$. And the original measure space $(X, \mathcal{M}, m)$ is a direct sum of the two measure spaces $(B, \mathcal{M} \cap B, m)$ and $(X - B, \mathcal{M} \cap (X - B), m)$. Consideration can now be given to the measure space $(X - B, \mathcal{M} \cap (X - B), m)$, breaking off another 2-valued measure space, if this is not a partitionable measure space, etc.. Suppose now that the original measure $(X, \mathcal{M}, m)$ satisfies the following *countability condition for a measure space*:

If $M_\alpha \in \mathcal{M}$ with $m(M_\alpha) > 0$, are a collection of *mutually disjoint* sets $\in \mathcal{M}$ with *positive* measure, then there are at most a countable infinity of M_α.

If $(X, \mathcal{M}, m)$ satisfies this countability condition for a measure space, then one easily shows that $(X, \mathcal{M}, m)$ is a *direct sum* of a countable number of mutually disjoint 2-valued measure spaces and of a partitionable measure space. Indicating this countable number of disjoint 2-valued measure spaces by the index j, where

$j = 1, 2, \ldots,$ to n or to ∞, one has that

$$(6.3) \qquad (X, \mathcal{M}, m) = \bigoplus_j (B_j, \mathcal{M} \cap B_j, m)$$

$$\oplus \left(X - \bigcup_j B_j, \mathcal{M} \cap \left(X - \bigcup_j B_j \right), m \right),$$

where $B_j \in \mathcal{M}$ and $m(B_j) > 0$, and $(B_j, \mathcal{M} \cap B_j, m)$ is a 2-valued measure space, and $B_{j_1} \cap B_{j_2} = \emptyset$ for all j_1, j_2 with $j_1 \neq j_2$, and $\left(X - \bigcup_j B_j, \mathcal{M} \cap \left(X - \bigcup_j B_j \right), m \right)$ is a partitionable measure space

(which may be a 0-measure space in which every M in $\left(\mathcal{M} \cap \left(X - \bigcup_j B_j \right) \right)$ has $m(M) = 0$). For any set S in X, one has $S = \bigcup_j (S \cap B_j) \cup \left(S \cap \left(X - \bigcup_j B_j \right) \right)$, and by Lemma 1,

$$m_i(S) = \sum_j m_i(S \cap B_j) + m_i \left(S \cap \left(X - \bigcup_j B_j \right) \right)$$

and

$$m_e(S) = \sum_j m_e(S \cap B_j) + m_e \left(S \cap \left(X - \bigcup_j B_j \right) \right).$$

For $m_i(S \cap B_j)$ and $m_e(S \cap B_j)$ one has only the two possible values 0 and $m(B_j)$, since $(B_j, \mathcal{M} \cap B_j, m)$ is 2-valued measure space.

Thus, in considering $m_i(S)$ and $m_e(S)$ for any set S, it is best to decompose the measure space as in (6.3), and treat the first part $\bigoplus_j (B_j, \mathcal{M} \cap B_j, m)$ separately from the second part $\left(X - \bigcup_j B_j, \mathcal{M} \cap \left(X - \bigcup_j B_j \right), m \right)$ which is a partitionable measure space. This latter partitionable measure space also satisfies the countability condition for a measure space. Furthermore, it is easily shown that: If $(X, \mathcal{M}, m)$ is a partitionable measure space and satisfies the *countability condition for a measure space*, than every set M which $\in \mathcal{M}$ also $\in \widetilde{\mathcal{M}}$, so that $\widetilde{\mathcal{M}} = \mathcal{M}$.

An example of a 2-valued measure space $(X, \mathcal{M}, m)$ is one in which $\mathcal{M}$ consists of the two sets $\emptyset$ and X, and $m(\emptyset) = 0$ and $m(X) > 0$. Then for any set S in X, other than $\emptyset$ and X, one has $m_i(S) = 0$ and $m_e(S) = m(X)$. The set X could have any number of points, including a finite number. Another example of a two-valued measure space $(X, \mathcal{M}, m)$ is one in which X consists of an uncountable infinity of points, and $\mathcal{M}$ consists of all subsets of X, and $m(M) = \infty$ if M contains an uncountable infinity of points while $m(M) = 0$ if the number of points in M is countable. For this 2-valued measure space, every set is measurable. Still another example of a two-valued measure space is to take any measure space $(X, \mathcal{M}, m)$ and form a new measure space $(X, \mathcal{M}, \hat{m})$ with the same X and $\mathcal{M}$ but with a different $\hat{m}$, where $\hat{m}(M) = 0$ if $m(M) = 0$ and $\hat{m}(M) = \infty$ if $m(M) > 0$.

Incidentally, in connection with Theorem 8 for disjoint sets in a *given* $\widetilde{M} \in \widetilde{\mathcal{M}}$, one need only have the basis condition (5.1) with the $\chi^{(f)}$ and $\chi^{(p)}$ for the measure space $(\widetilde{M}, \mathcal{M} \cap \widetilde{M}, m)$ instead of the containing measure space $(X, \mathcal{M}, m)$, if these are different. Note that the measure space $(\widetilde{M}, \mathcal{M} \cap \widetilde{M}, m)$ in which S_1 and S_2 lie does satisfy the countability condition for a measure space.

In the definition of a general measure space $(X, \mathcal{M}, m)$, it was not assumed that $X \in \mathcal{M}$. If $X \notin \mathcal{M}$, one can extend the measure space $(X, \mathcal{M}, m)$, if it is desired, to a measure space $(X, \mathcal{M}^*, m^*)$ in which the extended countable ring $\mathcal{M}^*$ includes the set X as well as all of $\mathcal{M}$, and the extended measure function m^* coincides with m over all of $\mathcal{M}$. The smallest such countable ring $\mathcal{M}^*$ is:

(6.4) $\quad \mathcal{M}^* = \mathcal{M}$ and the sets $\quad X - M \quad$ for all $\quad M \in \mathcal{M}$.

Indeed, one easily shows that $\mathcal{M}^*$ in (6.4) is a countable ring. Note that the two parts of $\mathcal{M}^*$ in (6.4) have no sets in common, because $X - M_1 \neq M_2$ for all $M_1 \in \mathcal{M}, M_2 \in \mathcal{M}$ since $X \notin \mathcal{M}$. Define the measure function m^* by:

(6.5) $\quad \begin{cases} m^*(M) = m(M) & \text{for all} \quad M \in \mathcal{M}, \\ m^*(X - M) = m_i(X - M) & \text{for all} \quad M \in \mathcal{M}. \end{cases}$

One can show that m^* is countably additive for disjoint sets in $\mathcal{M}^*$. This uses $(X - M_1) \cap (X - M_2) = X - (M_1 \cup M_2) \neq \emptyset$, so that two

sets of the form $X - M$ are not disjoint. Thus, $(X, \mathcal{M}^*, m^*)$ is a measure space, an extension of $(X, \mathcal{M}, m)$.

In the extended measure space $(X, \mathcal{M}^*, m^*)$, any set S has an interior measure $m_i^*(S)$ and an exterior measure $m_e^*(S)$. One can prove that:

$$(6.6) \quad \begin{cases} m_i^*(S) = m_i(S) & \text{for all } S, \text{ and }, \\ m_e^*(S) = m_e(S) & \text{for all } S \text{ such that } \quad S \subset M \in \mathcal{M}. \end{cases}$$

The measure space $(X, \mathcal{M}^*, m^*)$ is thus not only an extension of $(X, \mathcal{M}, m)$ but it also gives the same interior measure for every set S in X and it is an extension of the exterior measure applicable now to every set S in X.

Concerning the countable ring $\widetilde{\mathcal{M}}$ defined in (2.10) and (2.8), one obtains a similar countable ring $\widetilde{\mathcal{M}}^*$ in the measure space $(X, \widetilde{\mathcal{M}}^*, m^*)$, and this includes $\widetilde{\mathcal{M}}$. Also, it is easy to show that if $(X, \mathcal{M}, m)$ is a partitionable measure space, so is $(X, \mathcal{M}^*, m^*)$ a partitionable measure space. And the measure spaces $(X, \mathcal{M}, m)$ and $(X, \mathcal{M}^*, m^*)$ have the same value for $\chi^{(f)}$, and the same $\chi^{(p)}$. And if $(X, \mathcal{M}, m)$ satisfies the countability condition for a measure space, so does $(X, \mathcal{M}^*, m^*)$.

It might be desirable to have a complete measure space, i.e. one in which every subset of a set of measure 0 is also measurable and has measure 0. The measure space $(X, \mathcal{M}^*, m^*)$, if it isn't complete, can be extended to a complete measure space by the usual process of completion. The above extensions can be made beforehand, if it is desired, and have X as measurable also. $\bigl($Incidentally, if there are points of X not contained in any $M \in \mathcal{M}$, one can limit consideration to the point set $\bigcup\limits_{M \in \mathcal{M}} M$, and replace X in the above by $\bigcup\limits_{M \in \mathcal{M}} M$, or consider $\bigcup\limits_{M \in \mathcal{M}} M$ as X.$\bigr)$

If one has any extension $(X, \mathcal{M}^*, \hat{m})$ of $(X, \mathcal{M}, m)$ to the countable ring $\mathcal{M}^*$, one can prove that $\hat{m}$ is given by

$$(6.7) \qquad \hat{m} = m(M), \quad \hat{m}(X - M) = m_i(X - M) + \hat{c}$$
$$\text{for all} \quad M \in \mathcal{M},$$

where $\hat{c}$ is a non-negative constant, $\hat{c} \geq 0$. Indeed, for any $\hat{c} \geq 0$, (6.7) does furnish a countably additive measure function $\hat{m}$. If there

is a set $M_1 \in \mathcal{M}$ for which $m_i(X - M_1) < \infty$, in which case there is a set $M_2 \in \mathcal{M}$ for which $m_i(X - M_2) = 0$, then for $\hat{c} > 0$ the extension $(X, \mathcal{M}^*, \hat{m})$ is different from $(X, \mathcal{M}^*, m^*)$. It might be desired to have such an extension, to represent some unknown measure. But note that $\hat{m}_i(X - M_1) = \hat{m}(X - M_1) = m_i(X - M) + \hat{c} > m_i(X - M_1)$, and $\hat{m}_i$ does not agree with m_i for all sets S, as in (6.6). Thus, the extension $(X, \mathcal{M}^*, m^*)$ is the only extension of $(X, \mathcal{M}, m)$ to the countably ring $\mathcal{M}^*$ which satisfies (6.6).

It should be stated, in closing §6, that in all of the present article, and also in the author's articles [1] and [2], no hypothesis is made concerning the continuum $2^{\aleph_0}$; the continuum hypothesis of set theory, which is that $2^{\aleph_0} = \chi_1$, is not used. Use has been made of transfinite induction and of the axiom of choice.

7. Disjoint Sets in a Set S.

In all of the preceding, consideration was given to disjoint sets S_1, S_2 contained in a set $\widetilde{M} \in \widetilde{\mathcal{M}}$. Now, consider disjoint sets S_1, S_2 contained in *any* set S. First, for any set S contained in $M \in \mathcal{M}$, there is $m_i(S)$ and $m_e(S)$ for which $0 \leq m_i(S) \leq m_e(S)$, and the sets A and B in (2.3) and (2.6) will be chosen more carefully in the following lemma.

LEMMA 6. *Suppose that S is a set in a measure space $(X, \mathcal{M}, m)$ and $S \subset \widetilde{M} \in \widetilde{\mathcal{M}}$, where $\widetilde{M}$ is defined in (2.10) and (2.8). Then there are sets $K \in \widetilde{\mathcal{M}}$ and $L \in \widetilde{\mathcal{M}}$ such that*

(7.1)
$$K \subset S \subset L \subset \widetilde{M}, \quad with$$
$$m(K) = m_i(S), \quad m(L) = m_e(S),$$

(7.2)
$$\begin{cases} m_i(S - K) = 0, & m_i(L - S) = 0, \\ m_e(S - K) = m(L - K), & m_e(L - S) = m(L - K). \end{cases}$$

Further, considering $M \cap S$ for any $M \in \mathcal{M}$, and using K, L satisfying (7.1) and (7.2), then (7.1) and (7.2) also hold with $M \cap K, M \cap S, M \cap L$ in place of K, S, L respectively.

Proof. Suppose first that $m_e(S) < \infty$. Then sets A and B in (2.3) and (2.6) can be selected as the sets K and L of Lemma 6, and

(7.1) is satisfied. Let C_1 by any set $\in \mathcal{M}$ with $C_1 \subset (S - K)$. Then $C_1 \cap K = \emptyset$ and $(C_1 \cup K) \subset S$, so that $m(C_1) + m(K) = m(C_1 \cup K) \leq m_i(S) = m(K)$. Transposing $m(K)$ since $m(K) < \infty$ shows that $m(C_1) \leq 0$ and therefore $m(C_1) = 0$, and so $m_i(S - K) = 0$. Let C_2 be any set $\in \mathcal{M}$ with $C_2 \subset (L - S)$. Then $C_2 \cap S = \emptyset$ so that $S \subset (L - C_2)$, and $m(L) = m_e(S) \leq m(L - C_2) = m(L) - m(C_2)$, which gives $m(C_2) \leq 0$ since $m(L) < \infty$ and therefore $m(C_2) = 0$, and so $m_i(L - S) = 0$. Thus, the first line of (7.2) is established. Also, $(L - K) = (L - S) \cup (S - K)$ and $(L - S) \cap (S - K) = \emptyset$, so that by Lemma 2,

$$m_i(L - S) + m_e(S - K) = m_e(L - S) + m_i(S - K) = m(L - K).$$

This gives, using the first line of (7.2), that $m_e(S - K) = m_e(L - S) = m(L - K)$. Thus, (7.1) and (7.2) are established for $m_e(S) < \infty$. Note, when $m_e(S) < \infty$, that (7.2) followed from (7.1) in the above proof.

Consider now the case that $m_e(S) = \infty$, so that $m(\widetilde{M}) = \infty$. Write $\widetilde{M} = \bigcup_{j=1}^{\infty} M^j$ as in (2.9), where the M^j are mutually disjoint and $M^j \in \widetilde{\mathcal{M}}$ and $m(M^j) < \infty$ for every j. Then $S = \bigcup_{j=1}^{\infty} (M^j \cap S)$, and for the set $M^j \cap S$ there is by the preceding paragraph sets K^j and L^j, both $\in \widetilde{\mathcal{M}}$, with $K^j \subset (M^j \cap S) \subset L^j \subset M^j$, and formulas (7.1) and (7.2) hold with $K^j, M^j \cap S, L^j$ replacing K, S, L respectively. Now, place $K = \bigcup_{j=1}^{\infty} K^j, L = \bigcup_{j=1}^{\infty} L^j$. Then $K \subset S \subset L \subset \widetilde{M}$, and since the M^j are mutually disjoint, so are the K^j, and also so are the L^j. One has

$$m(K) = \sum_{j=1}^{\infty} m(K^j) = \sum_{j=1}^{\infty} m_i(M^j \cap S) = m_i(S)$$

by Lemma 1, formula (2.7), since $\left(X - \bigcup_{j=1}^{\infty} M^j \right) \cap S = \emptyset$, and also

$$m(L) = \sum_{j=1}^{\infty} m(L^j) = \sum_{j=1}^{\infty} m_e(M^j \cap S) = m_e(S)$$

by Lemma 1, (2.7) again. So, (7.1) is established.

Also,

$$(S - K) = \bigcup_{j=1}^{\infty}(M^j \cap (S - K)) = \bigcup_{j=1}^{\infty}((M^j \cap S) - K^j)$$

since $M^j \cap K = K^j$, and then by Lemma 1, (7.1) and (7.2) for finite m_e,

$$m_i(S - K) = \sum_{j=1}^{\infty} m_i((M^j \cap S) - K^j) = \sum_{j=1}^{\infty} 0 = 0,$$

$$m_e(S - K) = \sum_{j=1}^{\infty} m_e((M^j \cap S) - K^j) = \sum_{j=1}^{\infty} m(L^j - K^j) = m(L - K)$$

since $L - K = \bigcup_{j=1}^{\infty}(L^j - K^j)$. Likewise,

$$L - S = \bigcup_{j=1}^{\infty}(M^j \cap (L - S)) = \bigcup_{j=1}^{\infty}(L^j - (M^j \cap S))$$

since $M^j \cap L = L^j$, and then by Lemma 1, (7.1) and (7.2) for finite m_e,

$$m_i(L - S) = \sum_{j=1}^{\infty} m_i(L^j - (M^j \cap S)) = \sum_{j=1}^{\infty} 0 = 0,$$

$$m_e(L - S) = \sum_{j=1}^{\infty} m_e(L^j - (M^j \cap S)) = \sum_{j=1}^{\infty} m(L^j - K^j) = m(L - K).$$

Formulas (7.2) are established.

Concerning the last sentence of Lemma 6, one has that

$$m_i((M \cap S) - (M \cap K)) = 0 \quad \text{and} \quad m_i((M \cap L) - (M \cap S)) = 0,$$

which is the first line of (7.2) with $M \cap K, M \cap S, M \cap L$ in place of K, S, L respectively, since $(M \cap S) - (M \cap K) = M \cap (S - K) \subset (S - K)$ and $m_i(S - K) = 0$, and likewise $(M \cap L) - (M \cap S) = M \cap (L - S) \subset (L - S)$ and $m_i(L - S) = 0$. By Lemma 1,

$$m_i(M \cap S) = m_i(M \cap K) + m_i((M \cap S) - (M \cap K))$$
$$= m_i(M \cap K) + .0 = m(M \cap K),$$

and since $(M \cap L) = (M \cap S) \cup (M \cap (L - S))$, Lemma 2 gives

$$m(M \cap L) = m_e(M \cap S) + m_i(M \cap (L-S)) = m_e(M \cap S) + 0 = m_e(M \cap S);$$

thus, (7.1) holds with $M \cap K, M \cap S, M \cap L$ in place of K, S, L. Also, $(M \cap L) - (M \cap K) = ((M \cap L) - (M \cap S)) \cup ((M \cap S) - (M \cap K))$, so that by Lemma 2,

$$\begin{aligned}
&m((M \cap L) - (M \cap K)) \\
&\quad = m_e((M \cap L) - (M \cap S)) + m_i((M \cap S) - (M \cap K)) \\
&= m_e((M \cap L) - (M \cap S)) + 0 = m_e((M \cap L) - (M \cap S)), \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
&m((M \cap L) - (M \cap K)) \\
&\quad = m_i((M \cap L) - (M \cap S)) + m_e((M \cap S) - (M \cap K)) \\
&= 0 + m_e((M \cap S) - (M \cap K)) = m_e((M \cap S) - (M \cap K)).
\end{aligned}$$

So, the second line of (7.2) holds with $M \cap K, M \cap S, M \cap L$ in place of K, S, L; and also the first line of (7.2) holds. The last sentence of Lemma 6 is therefore established, and Lemma 6 is proved. $\square$

Now consider disjoint sets S_1, S_2 contained in a set S where $S \subset \widetilde{M} \in \widetilde{\mathcal{M}}$:

$$(7.3) \qquad S_1 \cap S_2 = \emptyset, (S_1 \cup S_2) \subset S \quad \text{where} \quad S \subset \widetilde{M} \in \widetilde{\mathcal{M}}.$$

By Theorem 4 and (3.19) and (3.20), there are the six non-negative quantities $a_1, a_2, d_i, \; h, g_1, g_2$ satisfying (3.21) and (3.22). The converse, as in Theorem 8, will now be discussed. Note that there is the formula

$$(7.4) \qquad\qquad a_1 + a_2 + d_i \leq m_i(S),$$

as well as $a_1 + a_2 + d_i + h + g_1 + g_2 \leq m_e(S)$.

As in §5, consider the cardinal number of points in a set S for which $m_e(S) > 0$, and let $\chi^{(\overline{p})}$ denote the least cardinal number of points in such sets S, among all sets S with $m_e(S) > 0$. This is analogous to §5, in defining $\chi^{(p)}$, and note that $\chi^{(\overline{p})} \leq \chi^{(p)}$. The measure space (X, M, m) will be said to satisfy the *basis condition for partially measurable sets* if

$$(7.5) \qquad\qquad \chi^{(\overline{p})} \geq \chi^{(f)},$$

where $\chi^{(f)}$ is defined in §5. And also suppose as in (5.2) the following.

$$(7.6) \begin{cases} \text{Every set } S \text{ with } m_e(S) > 0 \text{ contains an} \\ \text{infinity of points, besides the first sentence of (5.2).} \end{cases}$$

Let $\underline{w}^{(\overline{p})}$ denote the smallest ordinal number of cardinality $\chi^{(\overline{p})}$. There is the Theorem 9 below, analogous to Theorems 6 and 7.

THEOREM 9. *Suppose that* $(X, \mathcal{M}, m)$ *is a measure space satisfying the basis condition* (7.5) *for partially measurable sets, and also* (7.6). *Consider a set* S *in* X *with* $m_e(S) > 0$ *and* $S \subset \widetilde{M}$ *where* $\widetilde{M} \in \widetilde{\mathcal{M}}$, *and let* L *be a set* $\in \widetilde{\mathcal{M}}$ *with*

$$S \subset L \quad and \quad m(L) = m_e(S), \quad m_i(L - S) = 0,$$

as in Lemma 6. Then for any cardinal number $\leq \chi^{(\overline{p})}$ *and* ≥ 2, *there are that number of mutually disjoint sets* Z_β *contained in* S, *where* β *runs over all non-negative ordinal numbers of cardinality* $<$ *that cardinal number, such that*

$$(7.7) \qquad S = \bigcup_\beta Z_{\underline{\beta}} \quad and \quad m_i(Z_{\underline{\beta}}) = 0,$$

$$m_i(L - Z_{\underline{\beta}}) = 0 \quad for \ all \quad \underline{\beta}.$$

Proof. Let $\mathcal{C}^{(f)}$ be a collection of sets $C_{\underline{\eta}} \in \mathcal{M}$ with $m(C_{\underline{\eta}}) > 0$, which form a measure basis for $(X, \mathcal{M}, m)$, there being $\chi^{(f)}$ number of sets $C_{\underline{\eta}}$ in the collection $\mathcal{C}^{(f)}$. Consider all those sets $C_{\underline{\eta}}$ in $\mathcal{C}^{(f)}$ for which $m_e(C_{\underline{\eta}} \cap S) > 0$; the totality of all such $C_{\underline{\eta}}$ forms a collection $\mathcal{C}^{(\overline{f})}$, where $\mathcal{C}^{(\overline{f})}$ is a part of $\mathcal{C}^{(f)}$. Let $\chi^{(\overline{f})}$ denote the cardinal number of sets $C_{\underline{\eta}}$ in $\mathcal{C}^{(\overline{f})}$, so that $\chi^{(\overline{f})} \leq \chi^{(f)}$, and let $\underline{w}^{(\overline{f})}$ be the smallest ordinal number of cardinality $\chi^{(\overline{f})}$. The totality of all the $C_{\underline{\eta}}$ in $\mathcal{C}^{(\overline{f})}$ can be written as $C_{\underline{\eta}_\alpha}$, where $\underline{\alpha}$ runs over all non-negative ordinal numbers $< \underline{w}^{(\overline{f})}$, and designate the $C_{\underline{\eta}_\alpha}$ more simply as $\overline{C_{\underline{\alpha}}}$. Thus, the collection $\mathcal{C}^{(\overline{f})}$ consists of the sets $\overline{C_{\underline{\alpha}}}$ with

$$m_e(\overline{C_{\underline{\alpha}}} \cap S) > 0, \quad \text{for all non-negative ordinals} \quad \underline{\alpha} < \underline{w}^{(\overline{f})};$$

and the measure basis collection $\mathcal{C}^{(f)}$ consists of $\mathcal{C}^{(\overline{f})}$ and also all C_η for which $m_e(C_\eta \cap S) = 0$.

Now, proceed as in the proof of Theorem 6, with $\chi^{(\overline{p})}$ and $\underline{w}^{(\overline{p})}$ (where $\underline{w}^{(\overline{p})}$ is the smallest ordinal number of cardinality $\chi^{(\overline{p})}$), and $\chi^{(\overline{f})}$ and $\underline{w}^{(\overline{f})}$, replacing $\chi^{(p)}$ and $\underline{w}^{(p)}$, and $\chi^{(f)}$ and $\underline{w}^{(f)}$, respectively. As in Lemma 5, but with $\overline{C_\alpha} \cap S$ replacing C_α, one proves that there are points y_α and $x_{(\alpha, \beta)}$ in S, for all ordinals $\underline{\alpha} < \underline{w}^{(\overline{f})}$ and all ordinals $\underline{\beta} < \underline{w}^{(\overline{p})}$, which are all *different* points, and

$$(7.8) \qquad y_\alpha \in (\overline{C_\alpha} \cap S), x_{(\alpha, \beta)} \in (\overline{C_\alpha} \cap S),$$

$$\text{for all } \underline{\alpha} < \underline{w}^{(\overline{f})} \text{ and all } \underline{\beta} < \underline{w}^{(\overline{p})}.$$

The proof of this is the same as the proof of Lemma 5, with $\overline{C_\alpha} \cap S$ replacing C_α in the proof, noting that $m_e(\overline{C_\alpha} \cap S) > 0$ so that $\overline{C_\alpha} \cap S$ contains at least $\chi^{(\overline{p})}$ points, and $\chi^{(\overline{p})} \geq \chi^{(f)} \geq \chi^{(\overline{f})}$ by (7.5).

Proceeding as in the proof of Theorem 6 after the proof of Lemma 5, define the set

$$(7.9) \qquad Z_{\underline{\beta}} \quad \text{as in (5.9) for} \quad 0 < \underline{\beta} < \underline{w}^{(\overline{p})},$$

$$\text{and} \quad Z_0 = S - \bigcup_{1 \leq \underline{\beta} < \underline{w}^{(\overline{p})}} Z_{\underline{\beta}},$$

so that $S = \bigcup_\beta Z_{\underline{\beta}}$ as in (7.7). Let M be a set $\in \mathcal{M}$ with $M \subset Z_{\underline{\beta}}$. If $m(M) > 0$, there is a set $C_\eta \in \mathcal{C}^{(f)}$ with $m(C_\eta) > 0$ and $M \supset C_\eta$, so that $C_\eta \subset Z_{\underline{\beta}} \subset S$. Since $C_\eta \cap S = C_\eta$, C_η is a $\overline{C_\alpha} \in \mathcal{C}^{(\overline{f})}$, and the same contradiction as in the proof of Theorem 6 shows that $m(M) = 0$. Thus, $m_i(Z_{\underline{\beta}}) = 0$ as in (7.7).

Concerning the set $L - Z_{\underline{\beta}}$, let M be a set $\in \mathcal{M}$ with $M \subset (L - Z_{\underline{\beta}})$. If $m(M) > 0$, there is a set $C_\eta \in \mathcal{C}^{(f)}$ with $m(C_\eta) > 0$ and $M \supset C_\eta$, so that $C_\eta \subset (L - Z_{\underline{\beta}})$. If $m_e(C_\eta \cap S) > 0$, then C_η is a $\overline{C_\alpha} \in \mathcal{C}^{(\overline{f})}$, and $\overline{C_\alpha} \subset (L - Z_{\underline{\beta}})$ with $m(\overline{C_\alpha}) > 0$. But for $\underline{\beta} > 0, x_{(\alpha, \beta)} \in \overline{C_\alpha}$ and $x_{(\alpha, \beta)} \notin (L - Z_{\underline{\beta}})$ by (7.8) and (7.9) and (5.9), so that $\overline{C_\alpha} \not\subset (L - Z_{\underline{\beta}})$; and for $\underline{\beta} = 0, y_\alpha \in \overline{C_\alpha}$ by (7.8) but $y_\alpha \in Z_0$ by (7.9) and (5.9), so that $\overline{C_\alpha} \not\subset (L - Z_0)$. These contradictions show that $m_e(C_\eta \cap S) = 0$. Let Q be a set $\in \mathcal{M}$ with $C_\eta \supset Q \supset (C_\eta \cap S)$ and $m(Q) = m_e(C_\eta \cap S) = 0$. Then $m(C_\eta - Q) = m(C_\eta) > 0$, and

$(C_\eta - Q) \cap S = \emptyset$; and $(C_\eta - Q) \subset C_\eta \subset (L - Z_\beta) \subset L$, so that $(C_\eta - Q) \subset (L - S)$. Since $\overline{m}_i(L - S) = 0$, one has $\overline{m}(C_\eta - Q) = 0$. This contradicts that $m(C_\eta - Q) > 0$. Therefore the assumption that $m(M) > 0$ is false, and so $m(M) = 0$. This is for every $M \in \mathcal{M}$ with $M \subset (L - Z_\beta)$, so that $m_i(L - Z_\beta) = 0$. Thus, (7.7) is established, and Theorem 9 is proved for a cardinal number $= \chi^{(\overline{p})}$.

Theorem 9 for a cardinal number $< \chi^{(\overline{p})}$ and ≥ 2 is proved as in the proof of Theorem 7, using $L - Z_1, L - Z_0$, and $(L - \hat{Z}_0)$ in place of $X - Z_1, X - Z_0$, and $(X - \hat{Z}_0)$ respectively. The proof of Theorem 9 is completed. □

Incidentally, Theorem 9 shows that even if $m_i(S) = 0, S$ can be decomposed as in (7.7). For example, any particular Z_β in (7.7) can itself be decomposed as in (7.7).

Now, there is Theorem 4, and analogous to Theorem 8 there is the following theorem.

THEOREM 10. *Suppose that $(X, \mathcal{M}, m)$ is a partitionable measure space (this is defined in the first paragraph of §4) satisfying the basis condition (7.5) for partially measurable sets. Let S be a set for which*

$$m_e(S) > 0 \quad and \quad S \subset \widetilde{M} \quad for \ some \quad \widetilde{M} \in \widetilde{\mathcal{M}}.$$

Then, given any six non-negative real numbers or ∞, namely

$$a_1, a_2, d_i, h, g_1, g_2 \quad all \quad \geq 0, \quad satisfying$$

(7.10) $\quad \begin{cases} a_1 + a_2 + d_i + h + g_1 + g_2 \leq m_e(S), \quad and \\ a_1 + a_2 + d_i \leq m_i(S), \end{cases}$

there is a pair of disjoint sets S_1, S_2 contained in S such that

(7.11) $\begin{cases} m_i(S_1) = a_1, \quad m_i(S_2) = a_2, \quad d_i(S_1, S_2) = d_i, \\ g_1(S_1, S_2) = g_1, \quad g_2(S_1, S_2) = g_2, \quad h(S_1, S_2) = h. \end{cases}$

And formulas (6.1) hold.

Proof. Concerning the set S one has Lemma 6, including the sets K and L in $\widetilde{\mathcal{M}}$ satisfying (7.1) and (7.2). Consider first the case

that $m_e(S) < \infty$, so that $a_1, a_2, d_i, h, g_1, g_2$ are all finite by (7.10). Select three mutually disjoint sets

$$
(7.12) \quad
\begin{cases}
A_1, A_2, \text{ and } D_i, \text{ mutually disjoint and } \subset K \\
\text{and } \in \widetilde{\mathcal{M}}, \text{ with} \\
\quad m(A_1) = a_1, \quad m(A_2) = a_2, \quad m(D_i) = d_i.
\end{cases}
$$

This is possible by the second formula of (7.10) and Theorem 5 and the paragraph preceding Lemma 4 in §4. Place

$$
(7.13) \quad
\begin{cases}
\overline{K} = K - (A_1 \cup A_2 \cup D_i), \\
\overline{L} = L - (A_1 \cup A_2 \cup D_i), \quad \text{and} \\
\overline{S} = S - (A_1 \cup A_2 \cup D_i),
\end{cases}
$$

so that $\overline{K} \subset \overline{S} \subset \overline{L} \subset \widetilde{M}$. By Lemma 1,

$$m_i(\overline{S}) = m_i(S) - (a_1 + a_2 + d_i) = m(K) - (a_1 + a_2 + d_i) = m(\overline{K}), \quad \text{and}$$

$$m_e(\overline{S}) = m_e(S) - (a_1 + a_2 + d_i) = m(L) - (a_1 + a_2 + d_i) = m(\overline{L}),$$

$$\text{and } \overline{S} - \overline{K} = S - K, \quad \overline{L} - \overline{S} = L - S, \text{ and } \overline{L} - \overline{K} = L - K.$$

These state that the sets $\overline{K}$ and $\overline{L}$ are the K and L of Lemma 6 for the set $\overline{S}$, satisfying (7.1). And (7.2) is satisfied for $\overline{S}$ and $\overline{K}, \overline{L}$ because (7.2) followed from (7.1) in the proof of Lemma 6. Since

$$(7.14) \quad m(\overline{L}) = m_e(S) - (a_1 + a_2 + d_i) \geq h + g_1 + g_2,$$

by the first of (7.10), select three mutually disjoint sets

$$
(7.15) \quad
\begin{cases}
H, G_1, \text{and } G_2, \text{ mutually disjoint} \\
\text{and } \subset \overline{L} \text{ and } \in \widetilde{\mathcal{M}}, \text{ with} \\
\quad m(H) = h, \quad m(G_1) = g_1, \quad m(G_2) = g_2.
\end{cases}
$$

This is possible by (7.14), and Theorem 5 and the paragraph preceding Lemma 4 in §4.

Now, apply Theorem 9 for the set $\overline{S}$. If $m_e(\overline{S}) > 0$, then since $\overline{S} \subset \widetilde{M} \subset \widetilde{\mathcal{M}}$, every such $\overline{S}$ contains an uncountable infinity of points. For, a point of $\overline{S}$ is contained in a set $M_0 \subset \widetilde{M}$ with $M_0 \in \mathcal{M}$ and $m(M_0) = 0$, by Lemma 4. If there are only a countable

number of points in $\overline{S}$, then $\overline{S}$ would be contained in a countable union of such sets, which is a set in $\mathcal{M}$ of m-measure 0, so that $m_e(\overline{S}) = 0$, contrary to $m_e(\overline{S}) > 0$. Thus, the condition (7.6) is satisfied, and apply Theorem 9 for 3 sets. There are three mutually disjoint sets Z_1, Z_2, and Z_3 in $\overline{S}$, with $\overline{S} = Z_1 \cup Z_2 \cup Z_3$, satisfying (7.7) using $\overline{S}$ and $\overline{L}$ for these three sets. If $m_e(\overline{S}) = 0$, choose any three mutually disjoint sets in $\overline{S}$, possibly including $\emptyset$, whose union is $\overline{S}$, for Z_1, Z_2, Z_3. Also, by Theorem 9 for 2 sets, there are two disjoint sets Z and $D_i - Z$ in D_i with $m_i(Z) = 0$ and $m(D_i - Z) = 0$. Place

$$(7.16) \quad \begin{cases} S_1 = A_1 \cup Z \cup (H \cap Z_1) \cup (G_1 \cap Z_1), \\ S_2 = A_2 \cup (D_i - Z) \cup (H \cap Z_2) \cup (G_2 \cap Z_2), \\ \text{so that} \quad S_1 \cup S_2 = A_1 \cup A_2 \cup D_i \\ \qquad \cup (H \cap (Z_1 \cup Z_2)) \cup (G_1 \cup Z_1) \cup (G_2 \cap Z_2). \end{cases}$$

The sets $A_1, A_2, D_i, H, G_1, G_2$ are mutually disjoint sets $\in \widetilde{\mathcal{M}}$, so that by Lemma 1,

$$(7.17) \quad \begin{cases} m_i(S_1) \quad = m(A_1) + 0 + 0 + 0 = m(A_1), \\ m_i(S_2) \quad = m(A_2) + 0 + 0 + 0 = m(A_2), \\ m_i(S_1 \cup S_2) \quad = m(A_1) + m(A_2) + m(D_i) + 0 \\ \qquad\qquad +0 + 0 = m(A_1) + m(A_2) + m(D_i), \end{cases}$$

since $H \cap (Z_1 \cup Z_2) = (H \cap (\overline{S} - Z_3)) \subset (\overline{S} - Z_3) \subset (\overline{L} - Z_3)$ and $m_i(\overline{L} - Z_3) = 0$ by (7.7) for $\overline{S}, \overline{L}$. Concerning exterior measures for (7.16), one has by Lemma 2

$$m_i(H - (H \cap Z_1)) + m_e(H \cap Z_1) = m(H),$$

and $H - (H \cap Z_1) = (H - Z_1) \subset (\overline{L} - Z_1)$ so that $m_i(H - (H \cap Z_1)) = 0$ because $m_i(\overline{L} - Z_1) = 0$. Therefore, $m_e(H \cap Z_1) = m(H)$. Likewise for the other terms in applying m_e to (7.16), including Z and $D_i - Z$ contained in D_i; and $m_e(H \cap (Z_1 \cup Z_2)) = m(H)$ since $(H \cap Z_1) \subset (H \cap (Z_1 \cup Z_2)) \subset H$. There results, by Lemma 1:

$$(7.18) \quad \begin{cases} m_e(S_1) \quad = m(A_1) + m(D_i) + m(H) + m(G_1), \\ m_e(S_2) \quad = m(A_2) + m(D_i) + m(H) + m(G_2), \\ m_e(S_1 \cup S_2) \quad = m(A_1) + m(A_2) + m(D_i) \\ \qquad\qquad + m(H) + m(G_1) + m(G_2). \end{cases}$$

Concerning the functionals of the disjoint pair S_1, S_2 for the case that $m_e(S) < \infty$, the quantities $a_1, a_2, d_i, h, g_1, g_2$ are finite by (7.10). One has from (3.2), (7.17), and (7.12) that

$$(7.19) \qquad d_i(S_1, S_2) = m(D_i) = d_i.$$

Also, from (3.2), (7.17), (7.18) and (7.12), (7.15), one has the first two lines of (7.20):

(7.20)

$$\begin{cases} g_1(S_1, S_2) & = (a_1 + a_2 + d_i + h + g_1 + g_2) - a_1 \\ & \quad -(a_2 + d_i + h + g_2) = g_1 = m(G_1), \\ g_2(S_1, S_2) & = (a_1 + a_2 + d_i + h + g_1 + g_2) - a_2 \\ & \quad -(a_2 + d_i + h + g_1) = g_2 = m(G_2), \\ \text{and} \quad h(S_1, S_2) & = h = m(H). \end{cases}$$

The third line of (7.20) follows similarly from (3.2), (7.17), (7.18) and (7.12), (7.15). Thus, the equations (7.11) and (6.1) are established, and Theorem 10 is proved for the case that $m_e(S) < \infty$.

If $m_e(S) = \infty$, then $m(\widetilde{M}) = \infty$ since $S \subset \widetilde{M} \in \widetilde{\mathcal{M}}$, and express $\widetilde{M}$ as in (2.9): $\widetilde{M} = \bigcup_{j=1}^{\infty} M^j$ with $m(M^j) < \infty$ for all j and the M^j are mutually disjoint. Then $S = \widetilde{M} \cap S = \bigcup_{j=1}^{\infty} (M^j \cap S)$, and apply what has been proved above to the set $M^j \cap S$, using Lemma 6. By Lemma 6 with $M^j \cap K, M^j \cap S, M^j \cap L$ in place of K, S, L, one has by (7.1) that

$$m(M^j \cap K) = m_i(M^j \cap S), \quad m(M^j \cap L) = m_e(M^j \cap S),$$

and by (7.2),

$$m_i((M^j \cap S) - (M^j \cap K)) = 0, \quad m_i((M^j \cap L) - (M^j \cap S)) = 0,$$

$$m_e((M^j \cap S) - (M^j \cap K))$$
$$= m_e((M^j \cap L) - (M^j \cap S)) = m((M^j \cap L) - (M^j \cap K)).$$

Using formulas (7.19) and (7.20), there is obtained

$$d_i(M^j \cap S_1, M^j \cap S_2) = m(M^j \cap D_i),$$

$$g_1(M^j \cap S_1, M^j \cap S_2) = m(M^j \cap G_2),$$

$$g_2(M^j \cap S_1, M^j \cap S_2) = m(M^j \cap G_2),$$

$$h(M^j \cap S_1, M^j \cap S_2) = m(M^j \cap H).$$

By Theorem 1, noting that $\left(X - \bigcup\limits_{j=1}^{\infty} M^j\right) \cap (S_1 \cup S_2) = \emptyset$ since

$(S_1 \cup S_2) \subset \widetilde{M} = \bigcup\limits_{j=1}^{\infty} M^j,$

$$(7.21) \quad \begin{cases} d_i(S_1, S_2) = \sum\limits_{j=1}^{\infty} m(M^j \cap D_i) = m(D_i) = d_i, \\[2mm] g_1(S_1, S_2) = \sum\limits_{j=1}^{\infty} m(M^j \cap G_1) = m(G_1) = g_1, \\[2mm] g_2(S_1, S_2) = \sum\limits_{j=1}^{\infty} m(M^j \cap G_2) = m(G_2) = g_2, \\[2mm] h(S_1, S_2) = \sum\limits_{j=1}^{\infty} m(M^j \cap H) = m(H) = h, \end{cases}$$

since all the sets involved are contained in $\widetilde{M} = \bigcup\limits_{j=1}^{\infty} M^j$ using (7.12) and (7.15). Also, $m_i(S_1) = a_1, m_i(S_2) = a_2$ from (7.17). Theorem 10 is proved. $\qquad\qquad\square$

Theorem 10 is analogous to Theorem 8, but for two disjoint sets contained in any set $S \subset \widetilde{M} \in \widetilde{\mathcal{M}}$. (Note that in Theorems 10 and 8, one could also have $m_e(S) = 0$ or $m(\widetilde{M}) = 0$ by selecting the empty set $\emptyset$ for both S_1 and S_2.)

Incidentally, the proof of Theorem 10 above also gives a proof of Theorem 8. Theorem 8 is the case of Theorem 10 when $S = \widetilde{M} \in \widetilde{\mathcal{M}}$. Then $m_i(S) = m_e(S) = m(\widetilde{M})$, and the second line of (7.10) is a consequence of the first line of (7.10), and also $K = S = L = \widetilde{M}$ in the proof of Theorem 10. (Note that if one makes the extension of $(X, \mathcal{M}, m)$ to the measure space $(X, \mathcal{M}^*, m^*)$ as in §6, then the $\chi^{(\overline{p})}$ for $(X, \mathcal{M}^*, m^*)$ may possibly be smaller than the $\chi^{(\overline{p})}$ for $(X, \mathcal{M}, m)$.)

Consider Lebesgue or Borel measure on the real number line or in Euclidean n-space as the measure space. Condition (7.6) is satisfied, and if one assumes the continuum hypothesis of set theory, that

$2^{\chi_0} = \chi_1$, then $\chi^{(\overline{p})} = \chi^{(f)} = 2^{\chi_0}$ and (7.5) is satisfied. The measure space is also partitionable, so that the results of Theorems 9 and 10 hold for Lebesgue or Borel measure, if one assumes the continuum hypothesis of set theory.

If one does not assume the continuum hypothesis, there is an extension of Lebesgue measure, or it is Lebesgue measure itself, for which $\chi^{(\overline{p})} = \chi^{(f)} = 2^{\chi_0}$. Namely, consider the collection $\widehat{\mathcal{M}}$ of all sets of the form $(L \cup R_1) - R_2$ where L is Lebesgue measurable and R_1, R_2 are both sets of fewer than a continuum number 2^{χ_0} of points, and place $\hat{m}((L \cup R_1) - R_2) = m(L)$. One can show that $\widehat{\mathcal{M}}$ is a countable ring, and that $\hat{m}$ is uniquely defined in $\widehat{\mathcal{M}}$ and is countably additive, so that a measure space is formed. A set R with fewer than 2^{χ_0} points has $m_i(R) = 0$, and I believe it is not known what $m_e(R)$ might be if one does not assume the continuum hypothesis; but $R \in \widehat{\mathcal{M}}$ and $\hat{m}(R) = 0$ in the above measure space. For this measure space $\chi^{(\overline{p})} = 2^{\chi_0} = \chi^{(\overline{f})}$ and the results of Theorems 9 and 10 hold. Also, it is reasonable to consider the measure space $(X, \widehat{\mathcal{M}}, \hat{m})$ since for a set R with fewer than a continuum number 2^{χ_0} of points, on the real number line or in Euclidean space, there are a continuum number of mutually disjoint translations of R. It is reasonable to consider R to be of measure 0.

Bibliography

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