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ON THE DERIVED TOWERS OF CERTAIN INCLUSIONS OF TYPE III_λ FACTORS OF INDEX 4

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Given an inclusion of type III_λ factors, $\lambda \neq 0, 1$, of index 4 and with a common discrete decomposition, we compute the principal graph of its derived tower based on that on the associated type II_1 inclusion. Applications to the classification problem of hyperfinite type III_λ subfactors are discussed.

1. Introduction. Since the introduction and development of the theory of index by V. Jones in [**J**] to study a pair of type II_1 factors, one of the main problems has been the classification, up to conjugacy, of type II_1 subfactors of the same index of the hyperfinite type II_1 factor R_0. Lately a great deal of progress has been made on this problem in [**O1**], [**P1**], [**P2**]. As these works show, the tower of higher relative commutants (also known as the derived tower) associated with an inclusion of type II_1 factors is an important conjugacy invariant finer than the index, and if the inclusion has finite depth, or more generally the generating property as introduced in [**P2**], then this invariant contains sufficient information to determine the subfactor completely.

In another development, the notion of an index has been extended by various authors, [**Ko**], [**Ln**], [**PiPo1**] to arbitrary inclusions of factors that are associated with a normal faithful conditional expectation.

In [**L1**], it is shown that the theory of index for type III_λ factors, $\lambda \neq 0, 1$, is closely related to that for type II_1 factors. In particular, when both factors are of type III_λ, $\lambda \neq 0, 1$, then such an inclusion can be studied by means of a common discrete decomposition. Motivated by the classification work on injective type III_λ factors of A. Connes by using the automorphism approach (cf. [**C1**]), a method of classifying hyperfinite type III_λ subfactors of the Powers factor R_λ is presented in [**L2**]; it consists of the study of the outer conjugacy problem of trace scaling automorphisms which act simultaneously on a pair of hyperfinite type II_∞ factors with finite index.

It is shown in Proposition 3.1 of [**L2**] that the type III_λ derived tower is always contained in the one obtained from the type II_1 pair arising from a common discrete decomposition. The difference between these two derived towers may be viewed as an obstruction for a hyperfinite type III_λ inclusion of finite depth to split as a tensor product of a type II_1 pair with the Powers factor R_λ (see Proposition 6.1 in [**L2**] and [**Ka**]). Using this criterion, it is determined in [**L2**] that, in the index less than 4 case, if the principal graph of the type III_λ derived tower is D_{2n}, E_6 or E_8, or if the type II_1 principal graph is A_n, E_6 or E_8, then the type III_λ inclusion splits into a tensor product as described above and it is therefore classified by the type II_1 pair of tensor components.

In this paper, we will study type III_λ (hyperfinite) inclusions of index 4 and with a common discrete decomposition. More specifically, we will determine the principal graph of such an inclusion based on that one of the associated type II_1 pair. These results will then be applied to determine those (hyperfinite) type III_λ $\lambda \neq 0$, 1 inclusions that split as tensor products and to construct uncountably many non-conjugate type III_λ, $\lambda \neq 0$, pairs of factors with index 4 and principal graph $A_{\infty,\infty}$.

Our computations will be based on the results established in [**GHJ**] to the effect that the principal graph of a pair of type II_1 factors of index 4 is an extended Coxeter-Dynkin diagram of type $\tilde{A}, \tilde{D}$ and $\tilde{E}$ (for finite depth inclusions) or one of the infinite graphs $A_\infty, A_{\infty,\infty}$, D_∞ (for infinite depth inclusions). The classification result of hyperfinite type II_1 pairs of index 4 in [**P2**] will also be needed. In addition, we will also use the string algebras model introduced in [**O2**] to facilitate our computations.

The basic references for this paper are [**J**], [**Ko**], [**O1**], [**O2**], [**P1**], [**P2**], [**L2**]. We will use the results, definitions and terminologies in these works freely.

2. Preliminaries. The standing assumption throughout this paper is a pair of hyperfinite type III_λ factors, $\lambda \neq 0, 1, N \subset M$ such that there is a normal faithful conditional expectation $E : M \to N$ of index 4 and such that $N \subset M$ admits a common discrete decomposition with respect to E. This means that if φ is a generalized trace on N, then $\psi = \varphi \circ E$ is a generalized trace on M and it follows that $N \subset M$ is isomorphic to $N^\varphi \times_\theta \mathbf{Z} \subset M^\psi \times_\theta \mathbf{Z}$, where θ is a trace scaling automorphism on $N^\varphi \subset M^\psi$ with mod $\theta = \lambda$ and Ind$E$ = The Jones index of $N^\varphi \subset M^\psi$ (cf. [**L1**]). Note that this decomposition is essentially unique and does not depend on the choice of φ. So, we may identify $N \subset M$ with $Q \times_\theta \mathbf{Z} \subset P \times_\theta \mathbf{Z}$ where $Q \subset P$ is a pair of type II_∞ factors of index 4 and θ is an automorphism on $Q \subset P$ with mod $\theta = \lambda$.

In order to set up the framework of our calculations of the type III_λ derived tower, we need to recall its relation with that of the associated type II_1 inclusion. The arguments below differ slightly from those in [**L2**] but appear more suitable for our purpose here.

Let $\cdots \subset Q_{k+1} \subset Q_k \subset \cdots \subset Q \subset P$ be a tunnel for $Q \subset P$ with Jones projections $\{e_{-k}\}_{k \geq 0}$, and conditional expectations $E_{-k} : Q_{k-1} \to Q_k$, here we set $Q_0 = Q$, $Q_1 = P$ and $E_0 = E$.

Since $\theta^{-1} \circ E \circ \theta = E$, $E(\theta(e_0)) = \theta(E(e_0)) = (IndE)^{-1}$; by 1.7 of [**PiPo1**], there is a unitary u in Q such that $\mathrm{Ad}u \circ \theta(e_0) = e_0$ and as a result, $\theta_1 = \mathrm{Ad}u \circ \theta$ preserves the inclusion $Q_1 \subset Q \subset P$ because $Q_1 = \{e_0\}' \cap Q$. Note also that as E_{-1} is equivariant with respect to θ_1 and as we can identify M with $P \times_{\theta_1} \mathbf{Z}$ and N with $Q \times_{\theta_1} \mathbf{Z}$, E_{-1} extends to a conditional expectation from N onto $N_1 = Q_1 \times_{\theta_1} \mathbf{Z}$; it follows that N_1 is a downward construction of $N \subset M$.

Reiterating the argument above as in [**L2**], we obtain for each $k \geq 0$ a perturbation θ_k of θ by a unitary in Q which preserves $Q_k \subset \cdots \subset Q \subset P$ and in this process, a tunnel for $N \subset M$ has been constructed whereby the k-th subfactor in this tunnel is given by $N_k = Q_k \times_{\theta_k} \mathbf{Z}$. By Proposition 3.1 in [**L2**] we have $N_k' \cap M = (Q_k' \cap P)^{\theta_k}$.

As Ind$E$ = 4, by the arguments in [**GHJ**] the principal graph of $N \subset M$ and that of $Q \subset P$ are both of the type: $\tilde{A}$, $\tilde{D}$, $\tilde{E}$, or A_∞, $A_{\infty,\infty}$, D_∞. Our aim is to determine all the possible combinations of these graphs that may occur.

Note that the family of automorphisms $\{\theta_k\}_{k\geq 0}$ when restricted to the sequence of finite dimensional algebras $\{Q'_k \cap P\}_{k\geq 0}$ satisfy the following properties:

(1) θ_k is trace preserving for each $k \geq 0$;

(2) θ_k preserves the inclusion $Q'_j \cap P \subset Q'_j \cap P$ for $0 \leq j \leq k$;

(3) θ_k extends θ_{k-1};

(4) $\theta_k(e_{-j}) = e_{-j}$ for $0 \leq j \leq k-1$.

Here the trace on $Q'_k \cap P$ in (1) is given by the restriction of the conditional expectation $E_{-k} \circ E_{-(k-1)} \circ \cdots \circ E_0 : P \to Q_k$. In other words, $\{\theta_k\}_{k\geq 0}$ is an element of the group $\mathcal{G}$ as defined in Section 5 of [**L2**], which may be called the group of automorphisms on the principal graph of $Q \subset P$ or the group of *standard automorphisms* with respect to the tunnel $\{Q_j\}_{j\geq 0}$. We also recall that there is a homomorphism (the *standard homomorphism*) from $\mathrm{Aut}(P, Q)$ into $\mathcal{G}$ which maps outer conjugacy classes in $\mathrm{Aut}(P, Q)$ to conjugacy classes in $\mathcal{G}$ (cf. [**L2**]).

Thus to compute the type III_λ derived tower, we need to know:

(1) the derived tower of the type II_1 inclusion;

(2) the group of automorphisms on the type II_1 principal graph;

(3) the fixed point algebras of elements of the group in (2).

According to the results of [**GHJ**], the principal graph of the derived tower of a type II_1 pair of index 4 has been determined and all that remains to be done is the calculation of (2) and (3).

To this end, we need to use the model of string algebras introduced in [**O2**] to represent the inductive system $\{Q'_k \cap P\}_{k\geq 0}$ and then obtain a formula of a family of filtered automorphisms $\{\theta_k\}_{k\geq 0}$ in terms of the strings.

3. String representation formula for automorphisms of an inductive system of finite dimensional algebras. To acquaint the reader with the notations as well as the definitions involved with string algebras, we include a brief summary of those facts that will be relevant to our task at hand. The definitions and notations are all taken from [**O2**].

Let $\mathcal{A} = \{A_n, i_k^n\}_{n\geq 0}$ be an inductive system of finite dimensional C^*-algebras such that:

(1) $A_0 = \mathbf{C}$;

(2) $i_k^n : A_k \to A_n$ is a unital $*$-homomorphism for each $k \leq n$ such

that $i_k^m \circ i_m^n = i_k^n$ if $k \leq m \leq n$.

We are going to construct a graph G as follows. Let the n-level vertices $G_n^{(0)}$ of G be the set of equivalence classes of irreducible left modules of A_n, for each $x \in G_n^{(0)}$, choose a representative $H(x) = {}_{A_n}H(x)$ which is a finite dimensional Hilbert space. The unique 0 level vertex is denoted $*$ and has representative $H(*) = \mathbf{C}$. The set of vertices of G is defined to be $G^{(0)} = \bigcup_{n\geq 0} G_n^{(0)}$.

For $x \in G_n^{(0)}$ and $y \in G_{n+1}^{(0)}$, ${}_{A_{n+1}}H(y)$ restricts to a left A_n-module by means of the $*$-homomorphism i_n^{n+1}. We let $G_{x,y}^{(1)}$ be the set of edges e with source x and range y and have cardinality the multiplicity of $A_nH(x)$ in ${}_{A_n}H(y)$, and we choose a family of isometries $H(e) : H(x) \to H(y)$ which are A_n-module morphisms and have mutually orthogonal ranges. The n-level edges of G are $G_n^{(1)} = \bigcup G_{x,y}^{(1)}$ with x, y as above, and the set of all edges of G is defined to be $G^{(1)} = \bigcup_{n\geq 0} G_n^{(1)}$. We have:

$$H(e)^*H(f) = \delta(e,f)1_{H(s(e))}$$

for any $e, f \in G^{(1)}$ with $r(e) = r(f)$, and $\sum H(e)H(e)^* = 1_{H(y)}$ for any $y \in G_{n+1}^{(0)}$, where the sum is over all $e \in G_n^{(1)}$ with $r(e) = y$.

Conversely, let G be an oriented, locally finite, connected graph with a distinguished vertex $* = *_G$, one can construct an inductive system of finite dimensional C^*-algebras known as the string algebra of G.

As above $G^{(0)}$ is the set of vertices and $G^{(1)}$ the set of edges. For $e \in G^{(1)}$, $s(e)$ and $r(e)$ denote, respectively, the source and range of e. A path is an n-tuple $\xi = (e_1, \ldots, e_n)$ of edges with $s(e_{i+1}) = r(e_i)$ and we set $s(\xi) = s(e_1), r(\xi) = r(e_n)$ and the length of ξ is $|\xi| = n$.

Two paths $\xi = (e_1, \ldots, e_n)$ and $\eta = (f_1, \ldots, f_n)$ can be composed if $r(e_n) = s(f_1)$, in this case $\xi \circ \eta = (e_1, e_2, \ldots, e_n, f_1, \ldots, f_n)$.

A string on G is an ordered pair $\rho = (\xi, \eta)$ of paths with $s(\xi) = s(\eta)$ and $r(\xi) = r(\eta)$, $|\xi| = |\eta|$. We set $\rho_+ = \xi$ and $\rho_- = \eta$. As with paths, two strings ρ and σ can be composed if $r(\rho) = s(\sigma)$ and in this case, $\rho \circ \sigma = (\rho_+ \circ \sigma_+, \rho_- \circ \sigma_-)$.

If ρ and σ are two strings of the same length, the product of ρ with σ is defined to be: $\rho\sigma = \delta(\rho_-, \sigma_+)(\rho_+, \sigma_-)$ and the $*$-operation: $\rho^* = (\rho_-, \rho_+)$. Let A_n be the linear span of all strings with source $*$ and of length n, then under the product and $*$-operation defined above, A_n is a finite dimensional (due to the local finiteness of G)

complex unital C^*-algebra.

We define next $i_k^n : A_k \to A_n, k \leq n$ by $i_k^n(\rho) = \sum_\xi \rho \circ (\xi, \xi)$, the sum is over all paths ξ of length $n-k$ with $s(\xi) = r(\rho)$. It can be easily checked that i_k^n is a unital $*$-homomorphism and for $k \leq m \leq n$, $i_m^n \circ i_k^m = i_k^n$.

Thus starting from an oriented, pointed, locally finite and connected graph G, one can construct the string algebra of G which is an inductive system of finite dimensional C^* -algebras.

The following result of **[O2]** shows that any inductive system of finite dimensional C^* -algebras can be put in string form.

THEOREM 3.1. *Let $\mathcal{A} = (A_n, i_k^n)_{n\geq 0}$ be an inductive system of finite dimensional algebras with $A_0 = \mathbf{C}$ and unital $*$-homomorphisms i_k^n. Then $\mathcal{A}$ is isomorphic to the inductive system of string algebras of the Bratteli diagram G of $\mathcal{A}$.*

The isomorphism in Theorem 3.1 is defined as follows:

Write, as before,

$$A_n = \sum_{x \in G_n^{(0)}} H(x) \otimes H(x)^*.$$

For each string ρ of length n on G, define $\theta_n(\rho) = H(\rho_+)\xi_0 \otimes (H(\rho_-)\xi_0)^*$, where $H(\rho_+)$ is the isometry $H(e_n) \circ \cdots \circ H(e_1)$ if $\rho_+ = (e_1, \ldots, e_n)$, $H(\rho_-)$ is defined similarly, $\xi_0 \in H(*) = \mathbf{C}$ is a fixed unit vector and $\xi \otimes \eta^*$ stands for the rank one operator sending ζ to $\langle \zeta | \eta \rangle \xi$, then θ is the isomorphism between $\mathcal{A}$ and the string algebra of its Bratteli diagram.

Now let $\alpha = \{\alpha_n\}_{n\geq 0}$ be a system of automorphisms of $\{A_n, i_k^n\}_{n\geq 0}$ such that $\alpha_{n+1} \circ i_n^{n+1} = i_n^{n+1} \circ \alpha_n$ for each n. We will establish a formula that expresses the action of α in terms of the strings. First we prove that α induces an automorphism on the graph G (the Bratteli diagram) of $\{A_n, i_k^n\}_{n\geq 0}$, under the extra assumption that G has at most one edge between any 2 vertices on consecutive levels.

As before, we write $A_n = \sum_{x \in G_n^{(0)}} H(x) \otimes H(x)^*$, where each $H(x)$ is an irreducible left A_n-module. For each $x \in G_n^{(0)}$, define a new action of A_n on $H(x)$ by $a \cdot \xi = \alpha_n(a)\xi$ for $a \in A_n$ and $\xi \in H(x)$. Since $H(x)$ is also irreducible under this new action, there is a unique $x' \in G_n^{(0)}$ and a unitary $U_{x,x'} : H(x') \to H(x)$ such that

$$U_{x,x'} a U_{x,x'}^* = \alpha_n(a) \quad \text{for all} \quad a \in A_n.$$

Define $\alpha(x) = x'$ and set $U_n = \sum_{x\in G_n^{(0)}} U_{x,\alpha(x)}$, then for all $a \in A_n$, $\alpha_n(a) = U_n a U_n^*$.

Thus to each $x \in G_n^{(0)}$, there is a unique $\alpha(x) \in G_n^{(0)}$ such that $\alpha(H(x)\otimes H(x)^*) = H(\alpha(x))\otimes H(\alpha(x))^*$ and α induces a bijection on $G_n^{(0)}$.

To define the action of $\alpha = \{\alpha_n\}_{n\geq 0}$ on the edges of G, we fix an $x \in G_{n-1}^{(0)}$ and let ξ, η be arbitrary (nonzero) vectors of $H(x)$. We have

$$\begin{aligned}
i_{n-1}^n(\xi\otimes\eta^*) &= \sum_{y\in G_n^{(0)},\, e\in G_{x,y}^{(0)}} H(e)\xi\otimes(H(e)\eta)^* \\
\alpha_n(i_{n-1}^n(\xi\otimes\eta^*)) &= AdU_n\Bigg(\sum_{y\in G_n^{0},\, e\in G_{x,y}^{(1)}} H(e)\xi\otimes(H(e)\eta)^*\Bigg) \\
&= \sum_{y\in G_n^{(0)},\, e\in G_{x,y}^{(1)}} U_nH(e)\xi\otimes(U_nH(e)\eta)^*
\end{aligned}$$

and

$$i_{n-1}^n(\alpha_{n-1}(\xi\otimes\eta^*)) = \sum_{y\in G_n^{(0)},\, f\in G_{\alpha(x),\alpha(y)}^{(1)}} H(f)U_{n-1}\xi\otimes(H(f)U_{n-1}\eta)^*.$$

Since $\alpha_n\circ i_{n-1}^n = i_{n-1}^n\circ\alpha_{n-1}$, we get:

$$\sum_{y\in G_n^{(0)}\, e\in G_{x,y}^{(1)}} U_nH(e)\xi\otimes(U_nH(e)\eta)^* = \sum_{y\in G_n^{(0)},\, f\in G_{\alpha(x),\alpha(y)}^{(1)}} H(f)U_{n-1}\xi\otimes(H(f)U_{n-1}\eta)^*.$$

With a fixed y in $G_n^{(0)}$ and $e \in G_{x,y}^{(1)}$, the range of $U_nH(e)$ is contained in $H(\alpha(y))$, and so we have:

$$\langle U_nH(e)\eta|U_nH(e)\eta\rangle U_nH(e)\xi = \langle U_nH(e)\eta|H(f)U_{n-1}\eta\rangle H(f)U_{n-1}\xi,$$

for some $f \in G_{\alpha(x),\alpha(y)}^{(1)}$, which is unique by our hypothesis. By Schur's lemma, $U_{n-1}^*H(f)^*U_nH(e)$ is a complex scalar which we denote by $W(e,f)$. Thus the equality above becomes:

$$U_nH(e)\xi = W(e,f)H(f)U_{n-1}\xi.$$

Hence $W(e,f)$ has modulus 1 and $U_nH(e) = W(e,f)H(f)U_{n-1}$. Defining $\alpha(e)$ to be f, we have then constructed an automorphism of the graph G. We will write $W(e)$ instead of $W(e,\alpha(e))$ and define $W(\xi) = W(e_1)W(e_2)\cdots W(e_n)$ for a path $\xi = (e_1,\dots,e_n)$.

We should point out that the preceding argument is actually a special case of the notion of a connection introduced in **[O1]** where more general filtered $*$-homomorphisms are considered. However, being able to represent the action of α on the string algebra in terms of its induced automorphism on the graph turns out to be rather useful for our computations.

As mentioned before, there is a $*$-isomorphism $\theta = \{\theta_n\}$ from the string algebra of G onto $\{A_n, i_k^n\}$ given by:

$$\theta_n(\rho) = H(\rho_+)\xi_0 \otimes (H(\rho_-)\xi_0)^*,$$

where $\rho = (\rho_+,\rho_-)$ is a string of length n and ξ_0 is a fixed unit vector in $H(*) = \mathbf{C}$. With $\rho_+ = (e_1,\dots,e_n)$ and $\rho_- = (f_1,\dots,f_n)$, we have:

$$\begin{aligned}\alpha_n(\theta_n(\rho)) &= U_nH(\rho_+)\xi_0 \otimes (U_nH(\rho_-)\xi_0)^* \\ &= U_nH(e_n)\circ\cdots\circ H(e_1)\xi_0 \otimes (U_nH(f_n)\cdots H(f_1)\xi_0)^*.\end{aligned}$$

Since we may choose U_0 so that $U_0\xi_0 = \xi_0$, we have:

$$\begin{aligned}U_nH(e_n)\cdots H(e_1)\xi_0 &= U_nH(e_n)U_{n-1}^*U_{n-1}\cdots U_1^*U_1H(e_1)U_0^*U_0\xi_0 \\ &= W(e_n)H(\alpha(e_n))\cdots W(e_1)H(\alpha(e_1))\xi_0 \\ &= W(e_n)\cdots W(e_1)H(\alpha(\rho_+))\xi_0 \\ &= W(\rho_+)H(\rho_+)\xi_0.\end{aligned}$$

A similar calculation also works for $U_nH(f_n)\cdots H(f_1)\xi_0$. It follows that

$$\begin{aligned}\alpha_n(\theta_n(\rho)) &= W(\rho_+)H(\alpha(\rho_+)\xi_0 \otimes (W(\rho_-)H(\alpha(\rho_-)\xi_0)^* \\ &= W(\rho_+)\overline{W(\rho_-)}\theta_n(\alpha_n(\rho)).\end{aligned}$$

With a slight abuse of notation, we will omit the θ_n's in our formula and will just write:

$$\alpha(\rho_+,\rho_-) = W(\rho_+)\overline{W(\rho_-)}(\alpha(\rho_+),\alpha(\rho_-)).$$

Note however that α on the right denotes the induced automorphism on the graph of $\{A_n, i_k^n\}$, whereas α on the left is the actual automorphism on the string algebra. It turns out that the induced automorphism on the graph can be easily computed in most cases, once we know the type of the graph.

4. The type III_λ derived tower. In preparation for the computations of the type III_λ derived tower, we need to recall the following results in [**GHJ**] and the classification results in [**P2**], [**O3**], [**IK**], [**Ka2**].

THEOREM 4.1. (cf. 4.6.7 in [**GHJ**], [**P2**]) *Let $B \subset A$ be a pair of type II_1 factors with index 4, then the principal graph of $B \subset A$ is either an extended Coxeter-Dynkin diagram of type $\tilde{A}, \tilde{D}, \tilde{E}$ or one of the infinite graphs A_∞, $A_{\infty,\infty}$, D_∞.*

Each of these graphs can be realized as the principal graph of an inclusion of the form $R_0^G \subset [R_0 \otimes M_2(\mathbf{C})]^G$, where G is a subgroup of $SU(2)$, the latter acts by way of an infinite tensor product action on $R_0 \subset R_0 \otimes M_2(\mathbf{C})$.

More specifically, we have the following correspondence:

The cyclic group $\mathbf{Z}_n$ corresponds to $\tilde{A}_n$, the dihedral group of $2(n-2)$ elements to $\tilde{D}_n, n \geq 4$; the tetrahedral group ($\cong A_4$) to $\tilde{E}_6$, the octahedral group ($\cong S_4$) to $\tilde{E}_7$ and the icosahedral group ($\cong A_5$) to $\tilde{E}_8$. For the infinite graphs, $SU(2)$ corresponds to $A_\infty, \mathbf{T}$ to $A_{\infty,\infty}$ and the infinite dihedral group to D_∞.

For details of the classification of the hyperfinite type II_1 subfactors of index 4, the reader is referred to [**P2**], [**O3**], [**IK**], [**Ka2**].

Let $B \subset A$ be type II_1 hyperfinite factors with finite index and finite depth, $\{B_j\}_{j\geq 0}$ a spanning tunnel of $B \subset A$ with Jones projections $\{e_{-j}\}_{j\geq 0}$ such that $B_0 = B$. Recall that by [**P1**], [**O2**], such a tunnel exists with the property $B_j' \cap A \uparrow A$. Let $\mathcal{G}$ be the group of standard automorphisms corresponding to the tunnel $\{B_j\}_{j\geq 0}$ (cf. Section 2). For each $\alpha \in \mathcal{G}$, α can be extended to an automorphism on $B \subset A$. Assuming that B^α and A^α are factors, then $[A^\alpha : B^\alpha] = [A : B]$ and $\{B_j^\alpha\}$ is a tunnel for $B^\alpha \subset A^\alpha$. Recall that by [**GHJ**], $B \subset A$ has finite depth if there is $j > 0$ such that the Jones projection e_{-j} has central support 1 in $B_j' \cap A$.

LEMMA 4.2. *Let $B \subset A$ and $\alpha \in \mathcal{G}$ be as above. Then $B^\alpha \subset A^\alpha$ has finite depth if and only if α has finite order.*

Proof. Suppose that $B^\alpha \subset A^\alpha$ has finite depth, then as $B \subset A$ and $B^\alpha \subset A^\alpha$ share the same Jones projections and $(B_j^\alpha)' \cap A^\alpha \subset B_j' \cap A$, $B \subset A$ also has finite depth. It is straightforward to check that $(B_j' \cap A)^\alpha \uparrow A^\alpha$. By Lemma 4.2.4 in [**GHJ**], the finite depth condition of $B^\alpha \subset A^\alpha$ and $B \subset A$ implies that the periodicity assumption in Theorem 1.5 of [**Wen**] is satisfied. Hence $[A : A^\alpha]$ is finite and is equal to the period k of α.

Conversely, suppose that $B \subset A$ has finite depth and $\alpha^k = Id$ for some positive k. As above, since $B_j' \cap A \uparrow A$, $(B_j' \cap A)^\alpha \uparrow A^\alpha$.

For each j, let G_j be the embedding matrix for $(B_j' \cap A)^\alpha \subset B_j' \cap A$. By Theorem 1 of [**PiPo2**], $\|G_j\|^2 \leq [A : A^\alpha]$ for all j. Since $\{B_j' \cap A\}$ is periodic, such an inequality would not hold unless $\{(B_j' \cap A)^\alpha\}$ is also periodic. Since the Jones projections e_{-j} for $B^\alpha \subset A^\alpha$ are also the Jones projections for $B_j' \cap A^\alpha$ and $(B_j' \cap A)^\alpha = B_j' \cap A^\alpha \subset (B_j^\alpha)' \cap A^\alpha$; we conclude that $B^\alpha \subset A^\alpha$ has finite depth. $\square$

We would like to point out that with a little more work, we can actually show that $B_j' \cap A^\alpha = (B_j^\alpha)' \cap A^\alpha$ for all j.

The following lemma is a formula for the l^2-norm for the trace vectors for the algebras defined by $\tilde{A}_n$ and $\tilde{D}_n$. The proof consists of a straightforward inductive argument and will be omitted. Note that $\tilde{A}_n$ here has $2n$ vertices.

LEMMA 4.3. *Let ξ_n^j and η_n^j be the trace vectors on the j^{-th} floor of the finite dimensional algebras determined by the graphs $\tilde{A}_n$ and $\tilde{D}_n$ respectively. Then for any $j \geq n$, $\|\xi_n^j\|^2 = \dfrac{n}{2^{2j}}$ and $\|\eta_n^j\|^2 = \dfrac{2n-4}{2^{2j}}$.*

We can now state the main theorem of the paper.

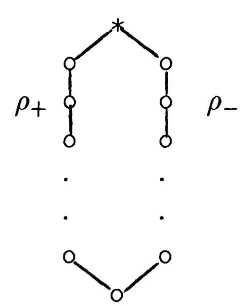
THEOREM 4.4. *Let $N \subset M$ be a pair of hyperfinite type III_λ factors with index 4 and a common discrete decomposition. The type III_λ principal graph can be determined according to the following table:*

	Type II_1 graph	$\mathcal{G}$	*Type III_λ graph*
(1)	$\tilde{A}_n, n \geq 2$	$\mathbf{T} \times_\sigma \mathbf{Z}_2$	$\tilde{A}_n$, $\tilde{A}_{nl}$, $l \geq 1$, $A_{\infty,\infty}$, $\tilde{D}_{n+2}$
(2)	$\tilde{A}_1$	$SO(3, \mathbf{R})$	$\tilde{A}_n$, $n \geq 1$, $A_{\infty,\infty}$
(3)	$\tilde{D}_n, n \geq 5$	$\mathbf{Z}_2$	$\tilde{D}_n$, $\tilde{D}_{2n-2}$
(4)	$\tilde{D}_4$	$\mathbf{Z}_2$ *or* S_3	$\tilde{D}_4$, $\tilde{D}_6$, $\tilde{E}_6$
(5)	$\tilde{E}_6$	$\mathbf{Z}_2$	$\tilde{E}_6$, $\tilde{E}_7$
(6)	$\tilde{E}_7$	*trivial*	$\tilde{E}_7$
(7)	$\tilde{E}_8$	*trivial*	$\tilde{E}_8$
(8)	D_∞	*trivial*	D_∞
(9)	A_∞	*trivial*	A_∞
(10)	$A_{\infty,\infty}$	$\mathbf{Z}_2$	$A_{\infty,\infty}$, D_∞

Proof. Throughout the proof, we will let $B \subset A$ be the associated type II_1 pair of $N \subset M$, $\Gamma_{(A,B)}$, $\Gamma_{(M,N)}$ will denote the principal graphs and we set $Y_k = B_k' \cap A$ where B_k is a tunnel of $B \subset A$ together with the Jones projections $\{e_{-j}\}_{j \geq 0}$.

(1) $\Gamma_{(A,B)} = \tilde{A}_n$, $n \geq 2$.

First we determine the corresponding group $\mathcal{G}$ for $\tilde{A}_n$. Consider the graph $\tilde{A}_n$ with the string $\rho = (\rho_+, \rho_-)$ as labeled:



Write $\rho_+ = (\xi_1, \dots, \xi_n)$, $\rho_- = (\eta_1, \dots, \eta_n)$.

Let $\alpha = \{\alpha_k\}_{k \geq 0} \in \mathcal{G}$. Considering the induced automorphism by α on the graph $\mathcal{G}$, there are two possibilities:
(i) $\alpha(\xi_1) = \xi_1$ and $\alpha(\eta_1) = \eta_1$;
(ii) $\alpha(\xi_1) = \eta_1$ and $\alpha(\eta_1) = \xi_1$.

If α is of the first type, then $\alpha(\xi_i) = \xi_i$ and $\alpha(\eta_i) = \eta_i$ for $1 \leq i \leq n$ and hence α_{n-1} is the identity of Y_{n-1}, whereas on Y_n, the action of α_n can be described as follows: it is the identity on $Y_{n-1}e_{-n}Y_{n-1}$

and is given by Adu (locally) on the direct summands orthogonal to $Y_{n-1}e_{-n}Y_{n-1}$, where

$$u = \begin{bmatrix} 1 & 0 \\ 0 & \overline{w} \end{bmatrix}$$

and $w = W(\rho_+)\overline{W(\rho_-)}$, where $\alpha(\rho_+, \rho_-) = W(\rho_+)\overline{W(\rho_-)}(\rho_+, \rho_-)$ (Cf. Section 3). Since $Y_{k+1} = Y_k e_{-(k+1)} Y_k$, for $k \geq n$, α is completely determined by its action on Y_n, it follows that elements of $\mathcal{G}$ that are of the first type are represented by elements of the unit circle $\mathbf{T}$.

If α is an automorphism of the second type, then $\alpha(\xi_i) = \eta_i$ for $1 \leq i \leq n$ and $\alpha(\rho_+, \rho_-) = W(\rho_+)\overline{W(\rho_-)}(\rho_-, \rho_+)$. It is easy to check that α has order 2 and is the product of an automorphism of the first type with the automorphism σ defined by $\sigma(\rho_+, \rho_-) = (\rho_-, \rho_+)$.

If α is an automorphism of the first type then $\sigma\alpha\sigma^{-1} = \alpha^{-1}$. Hence $\mathcal{G}$ is contained in the semidirect product of $\mathbf{T}$ by $\mathbf{Z}_2$ via this action.

Conversely, to prove that $\mathcal{G} = \mathbf{T} \times_\sigma \mathbf{Z}_2$, we recall that by a result in [**PiPo1**], the inclusion $B \subset A$ is locally trivial in the sense that there exist a type II_1 factor P isomorphic to R_0, and an automorphism θ of P with outer period n such that $B \subset A$ has the form: $\{x \oplus \theta(x) : x \in P\} \subset P \otimes M_{2\times 2}(C)$. For each $w \in \mathbf{T}$, consider $\alpha \in Aut(A, B)$ defined by:

$$\alpha_w = Ad \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}.$$

We claim that α_w acts trivially on the principal graph $\tilde{A}_n$ if and only if $w^n = 1$. For by [**L1**] and [**Ka**], α_w acts trivially on the principal graph if and only if there is a sequence of unitaries $\{u_k\}$ in P such that

$$\alpha_w = \lim_{k \to \infty} Ad \begin{bmatrix} u_k & 0 \\ 0 & \theta(u_k) \end{bmatrix}.$$

This means that $u = \{u_k\}$ is in P_ω, the algebra of central sequences of P and $\theta_\omega(u)u^* = w$ in P_ω. But as θ_ω has period n in P_ω, the preceding assertion holds if and only if $w^n = 1$ by [**C2**].

Thus if $w^n \neq 1$, then α_w acts non-trivially on the graph and the corresponding element in $\mathcal{G}$ is of the first type. As a result, $\mathbf{T} \subset \mathcal{G}$. By Connes' classification result of automorphisms on the hyperfinite type II_1 factor, θ is outer conjugate to θ^{-1}, say $\theta \circ \varphi =$

$Adu \circ \varphi \circ \theta^{-1}$, where $\varphi \in \mathrm{Aut}P$ and u is a unitary in P, it follows that the automorphism $\tilde{\sigma}$ on $B \subset A$ defined by:

$$\tilde{\sigma} = Ad \begin{bmatrix} 0 & 1 \\ u & 0 \end{bmatrix} \circ (\varphi\theta^{-1} \otimes Id)$$

makes sense and induces a non-trivial element of $\mathcal{G}$, namely the element σ defined above which acts on $\mathbf{T}$ as a period 2 automorphism. Hence we have determined that $\mathcal{G} \cong \mathbf{T} \times_\sigma \mathbf{Z}_2$.

We can now determine the fixed point algebra of an element $\alpha = \{\alpha_k\}$ of $\mathcal{G}$. If α is of the first type, then it is clear that α_{n-1} is the identity on Y_{n-1} and that $Y_n^\alpha = Y_{n-1}e_{-n}Y_{n-1} \oplus \mathbf{C} \oplus \mathbf{C}$. Thus the Bratteli diagram of $\{Y_k^\alpha\}_{k=0}^n$ is Pascal's triangle and according to the list in [**GHJ**], this implies that the principal graph for $\{Y_k^\alpha\}_{k\geq 0}$ is either $\tilde{A}_m$ or $A_{\infty,\infty}$. In fact, by Lemma 4.2, the graph is $\tilde{A}_m$ if and only if α has finite order, say l. By the Wenzl index formula, we have: $l = [A : A^\alpha] = \dfrac{\|\xi_m^j\|^2}{\|\xi_n^j\|^2}$, for j large enough, where ξ_m^j and ξ_n^j are, respectively, the trace vectors of the j^{-th} floor of the algebras determined by $\tilde{A}_m$ and $\tilde{A}_n$. It follows from Lemma 4.3 that $l = \dfrac{m}{n}$.

If α has infinite order, then the graph for $\{Y_k^\alpha\}$ is $A_{\infty,\infty}$.

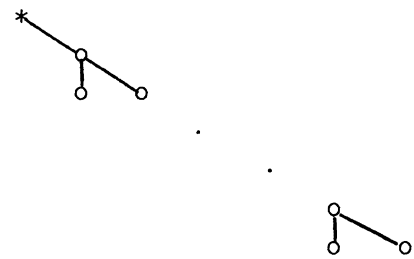
If α is of the second type, then it is easily checked that $Y_0^\alpha, Y_1^\alpha, Y_2^\alpha$ are, respectively, $\mathbf{C}$, $\mathbf{C}$, $\mathbf{C}^4$, for $n = 2$ and $\mathbf{C}$, $\mathbf{C}$, $\mathbf{C}^3$ for $n \geq 3$. Hence the principal graph of $\{Y_k^\alpha\}$ is $\tilde{D}_4$ if $n = 2$ or $\tilde{D}_m$ or D_∞ if $n \geq 3$.

By Lemma 4.2, the graph is $\tilde{D}_m$ for some m. We refer to the figure of $\tilde{A}_n$ as before. A simple calculation shows that the direct summand of Y_n formed by the paths ρ_+ and ρ_- gives rise to a 2-dimensional subspace in Y_n^α, which is in the orthogonal complement of $Y_{n-1}^\alpha e_{-n} Y_{n-1}^\alpha$. This implies that the principal graph of $\{Y_k^\alpha\}_{k\geq 0}$ has at least two vertices at the n^{-th} level. By looking at the list provided in [**GHJ**], we can conclude that the principal graph of $\{Y_k^\alpha\}$ is $\tilde{D}_{n+2}$. We can also get the same result by using the Wenzl index formula.

(2) $\Gamma_{A,B} = \tilde{A}_1$.

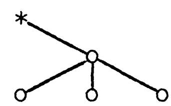
In this case, observe that since $B' \cap A = M_{2\times 2}(\mathbf{C})$ and $B \subset A$ has depth 1, any element of $\mathcal{G}$ is determined by its action on $B' \cap A$ and hence by a 2×2 unitary matrix, up to scalars of moduli 1. It follows that $\mathcal{G}$ can be identified with $U_{2\times 2}(\mathbf{C})/\mathbf{T}$, which is isomorphic to $SU(2)/\{I,-I\}$, and hence to $SO(3,\mathbf{R})$. Now the same kind of arguments as in (1) can be applied to get the type III graph, which is either $\tilde{A}_n$ or $A_{\infty,\infty}$.

(3) $\Gamma_{(A,B)} = \tilde{D}_n,\ n \geq 5$.



It is clear that in this case, $\mathcal{G} \cong \mathbf{Z}_2$. Let $\alpha = \{\alpha_k\}$ be the nontrivial element of $\mathcal{G}$, then α_{n-2} is the identity and thus the principal graph of $\{Y_k^\alpha\}$ is either $\tilde{D}_m$ or D_∞. Again Lemma 4.2 eliminates D_∞. Using the Wenzl index formula, we can determine m to be $2n-2$.

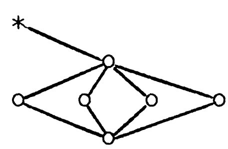
(4) $\Gamma_{(A,B)} = \tilde{D}_4$.



In this case, by Corollary 1 of [**P2**], $B \subset A$ is isomorphic to either $R_0 \subset R_0 \times \mathbf{Z}_4$ or $R_0 \subset R_0 \times \mathbf{Z}_2 \oplus \mathbf{Z}_2$ so that the corresponding group $\mathcal{G}$ is either $\mathrm{Aut}(\mathbf{Z}_4) \cong \mathbf{Z}_2$ or $\mathrm{Aut}(\mathbf{Z}_2 \oplus \mathbf{Z}_2) \cong S_3$ (cf. Theorem 6.3 in [**L2**]).

In either case, if $\alpha = \{\alpha_k\} \in \mathcal{G}$ is an element of order 2, then the same argument as in (2) shows that the principal graph of $\{Y_k^\alpha\}$ is $\tilde{D}_6$.

Suppose now that $\mathcal{G} \cong S_3$ and $\alpha = \{\alpha_k\}$ is an element of order 3 of $\mathcal{G}$. Consider the Bratteli diagram of $Y_0 \subset Y_1 \subset Y_2 \subset Y_3$:

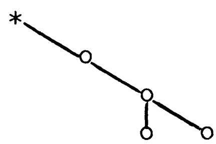


On Y_3, viewed as the string algebra spanned by the strings formed with the paths $\xi_i, 1 \leq i \leq 4$, we may assume that $\alpha(\xi_1) = \xi_2$, $\alpha(\xi_2) = \xi_3$, $\alpha(\xi_3) = \xi_1$ and $\alpha(\xi_4) = \xi_4$.

It is then easily checked that $Y_0^\alpha \cong \mathbf{C}, Y_1^\alpha \cong \mathbf{C}$, and $Y_2^\alpha \cong \mathbf{C}^2$. Now an arbitrary string in Y_3 has the form $\rho = \sum_{i,j=1}^4 c_{ij}(\xi_i, \xi_j)$, where $c_{ij} \in \mathbf{C}$. Using the representation formula for α on the strings established in Section 2 we have:

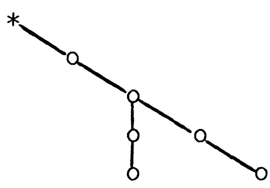
$$\alpha(\rho) = \sum_{i,j=1}^{4} c_{i,j} W(\xi_i)\overline{W(\xi_j)}(\alpha(\xi_i), \alpha(\xi_j)).$$

Thus $\alpha(\rho) = \rho$ if and only if $c_{ij}W(\xi_i)\overline{W(\xi_j)} = c_{\alpha(i)\alpha(j)}$ for $1 \leq i, j \leq 4$. Writing out these equations explicitly, the dimension of Y_3^α is found to be 6. On the other hand, since Y_3^α already contains $Y_2^\alpha e_{-3} Y_2^\alpha \cong M_2(\mathbf{C})$, we can conclude that $Y_3^\alpha \cong M_2(\mathbf{C}) \oplus \mathbf{C} \oplus \mathbf{C}$. This shows that the Bratteli diagram for $Y_0^\alpha \subset Y_1^2 \subset Y_2^\alpha \subset Y_3^\alpha$ is:



Upon inspection of the list in [**GHJ**], we infer that the principal graph for $\{Y_k^\alpha\}$ must be $\tilde{E}_6$ or $\tilde{E}_8$. But in order for the Wenzl index formula to hold, the graph must be $\tilde{E}_6$.

(5) $\Gamma_{(A,B)} = \tilde{E}_6$.



As is easily seen, $\mathcal{G} \cong \mathbf{Z}_2$. Let α be the nontrivial element of $\mathcal{G}$, then $Y_0^\alpha \cong \mathbf{C}, Y_1^\alpha \cong \mathbf{C}, Y_2^\alpha \cong \mathbf{C}^2$ and $Y_3^\alpha \cong M_2(\mathbf{C}) \oplus \mathbf{C}$. Using the string representation of α, a simple computation similar to that in (3) shows that the dimension of Y_4^α is 15. Since Y_4^α already contains $Y_3^\alpha e_{-4} Y_3^\alpha \cong M_2(\mathbf{C}) \oplus M_3(\mathbf{C})$, we see that Y_4^α is isomorphic to $M_2(\mathbf{C}) \oplus M_3(\mathbf{C}) \oplus \mathbf{C} \oplus \mathbf{C}$. As in (3), we can now conclude that the principal graph for $\{Y_k^\alpha\}$ is $\tilde{E}_7$.

The proofs of the remaining cases are quite simple and are left as an exercise. □

REMARK 4.5. From Theorem 4.4 (1), we see that hyperfinite type II_1 subfactors with $\tilde{D}_n$ as principal graph can be realized as fixed point algebras of a standard automorphism of period 2 on an inclusion having $\tilde{A}_{n-2}$ as principal graph. Such a result has also been obtained in [**IK**] independently, but the automorphism used in [**IK**] is different from ours.

5. Applications.

Theorem 4.4 has many consequences regarding the classification of type III_λ subfactors of index 4 of the Powers factor R_λ. First, we are going to determine those hyperfinite inclusions satisfying the hypotheses in 4.4 that split as tensor products.

A criterion for the splitting is obtained in [**L2**] when the inclusion is irreducible and has finite depth.

THEOREM 5.1. (cf. Theorem 6.1 [**L2**]) *Let $N \subset M$ be an irreducible type III_λ inclusion having finite index, finite depth and a common discrete decomposition. If the derived tower of $N \subset M$ is equal to that of the corresponding type II_1 pair, then $N \subset M$ splits as a tensor product.*

Applying this criterion to the case of index 4, we have the following.

COROLLARY 5.2.. *Let $N \subset M$ be a hyperfinite type III_λ inclusion with index 4 and a common discrete decomposition. Then there exist hyperfinite type II_1 factors $B \subset A$ such that $N \subset M$ is isomorphic to $B \otimes R_\lambda \subset A \otimes R_\lambda$ if one of the following conditions is satisfied:*

(a) The principal graph of the derived tower of $N \subset M$ is $\tilde{E}_8$.

(b) The principal graph of the derived tower of the associated type II_1 inclusion is $\tilde{E}_7$, $\tilde{E}_8$.

Proof. (a) If the type III graph is $\tilde{E}_8$, then by Theorem 3.2, the principal graph of the type II_1 inclusion coming from the common discrete decomposition agrees with that of $N \subset M$ and so by Theorem 5.1, $N \subset M$ splits as a tensor product type inclusion.

(b) If any of the conditions in (b) is satisfied, then since the corresponding group $\mathcal{G}$ for each of these graphs is trivial, the principal graph of $N \subset M$ agrees with that of the associated type II_1 inclusion. By Theorem 5.1, $N \subset M$ splits as a tensor product. $\square$

According to the classification result in [**P2**], [**O3**], each of the graphs mentioned in Corollary 5.2 determines a unique subfactor of R_0, and as a result a hyperfinite type III_λ inclusion satisfying one of the conditions in Corollary 5.2 is also unique.

As a glance at the table in Theorem 4.4 might suggest, the case of the graph $\tilde{A}_n$ is interesting for its group $\mathcal{G}$ differs from the rest in that it is infinite and we are going to show that in this case we can construct a family of uncountably many pair of non-conjugate type III_λ, $\lambda \neq 0$, with index 4 and having $A_{\infty,\infty}$ as principal graph.

Keeping the same notations as in Theorem 4.4, we first establish a simple lemma regarding the conjugacy classes of the elements in the group $\mathcal{G}$ for the graph $\tilde{A}_n$, $n \geq 1$.

LEMMA 5.3. *Let $\mathcal{G}$ be the group of standard automorphisms corresponding to the graph $\tilde{A}_n$.*

a) For $n \geq 2$, $\mathcal{G} \cong \boldsymbol{T} \times_\sigma \boldsymbol{Z}_2$ and we have:

i) Two elements α, β in $\boldsymbol{T}$ are conjugate in $\mathcal{G}$ if and only if $\alpha = \beta^{\pm 1}$;

ii) For any element α in $\boldsymbol{T}$, $\alpha\sigma$ is conjugate to σ.

b) For $n = 1$, $\mathcal{G} \cong SU(2,)/\{I - I\}$, and α, β are conjugate in $\mathcal{G}$ if and only if $\alpha = \beta^{\pm 1}$.

Proof. a) First, let us consider the case of $\tilde{A}_n$ with $n \geq 2$. Let α and β be two distinct elements of $\mathbf{T}$ which are conjugate in $\mathbf{T} \times_\sigma \mathbf{Z}_2$. Let θ be such that $\theta\alpha\theta^{-1} = \beta$. We may assume that θ is of the form $\sigma\tilde{\theta}$ where $\tilde{\theta}$ is some element of $\mathbf{T}$. We then have:

$$\begin{aligned}\beta &= \theta\alpha\theta^{-1} \\ &= \sigma\tilde{\theta}\alpha\tilde{\theta}^{-1}\sigma^{-1} \\ &= \sigma\alpha\sigma^{-1}, \textit{ as } \mathbf{T} \textit{ is abelian}, \\ &= \alpha^{-1},\end{aligned}$$

which proves i).

For any $\alpha \in \mathbf{T}$, let $\beta \in \mathbf{T}$ be such that $\beta^2 = \alpha$. Then $(\beta\sigma)\sigma(\beta\sigma)^{-1} = \beta^2\sigma = \alpha\sigma$ and ii) follows.

b) In the case of $\tilde{A}_1$, recall that $\mathcal{G}$ is isomorphic to $SU(2)/\{I, -I\}$. For any $u \in SU(2)$, it is easy to see that u is conjugate in $\mathcal{G}$ to $\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$, for some $t \in \mathbf{R}$. Thus α and β are conjugate in $\mathcal{G}$ if and only if $\alpha = \beta^{\pm 1}$.

It follows that there are uncountably many non-conjugate elements in $\mathcal{G}$ corresponding to $\tilde{A}_n$. □

COROLLARY 5.4. *For each* $0 < \lambda < 1$, *and each* $n \geq 1$, *there is an uncountable family of non-conjugate type* III_λ *subfactors of* R_λ *of index* 4 *such that the type* III_λ *principal graph is* $A_{\infty,\infty}$ *and the type* II_1 *principal graph is* $\tilde{A}_n$.

Proof. Starting with a hyperfinite type II_1 pair $B \subset A$ having $\tilde{A}_n$ as principal graph (such a pair exists by [**GHJ**]). By Lemma 5.3, there exist uncountably many non-conjugate automorphisms on the derived tower of $B \subset A$ that correspond to the elements of $\mathbf{T}$ with infinite order. As these automorphisms are trace-preserving, they extend to automorphisms on $B \subset A$ which are non-conjugate and aperiodic, as shown in Section 5 of [**L2**]. For each element α of this family, we form the automorphism $\alpha \otimes \theta$ on $B \otimes R_{0,1} \subset A \otimes R_{0,1}$, where θ is the unique automorphism on $R_{0,1}$ with mod $\theta = \lambda$. Now the type III_λ inclusions formed by taking the crossed products:

$$B \otimes R_{0,1} \times_{\alpha\otimes\theta} \mathbf{Z} \subset A \otimes R_{0,1} \times_{\alpha\otimes\theta} \mathbf{Z}$$

are mutually non-isomorphic and have $A_{\infty,\infty}$ as principal graph. □

Using the continuous decomposition for type III_1 factors, we have:

COROLLARY 5.5. *For each* $n \geq 1$, *there exists an uncountable family of non-isomorphic type* III_1 *subfactors of the Araki-Woods factor with index* 4 *such that the type III graph is* $A_{\infty,\infty}$ *and the type II graph is* $\tilde{A}_n$

Proof. Let $B \subset A$ be the pair of hyperfinite type II_1 factors having $\tilde{A}_n$, $n \geq 1$, as principal graph. Then as we know, the corresponding group of automorphisms contains $\mathbf{T}$ as a subgroup. Denote the elements in the subgroup $\mathbf{T}$ by $\{\alpha_\gamma\}_{\gamma\in\mathbf{T}}$ and let $\{\theta_t\}_{t\in\mathbf{R}}$ be the (unique) one-parameter trace-scaling automorphism group of the hyperfinite type II_∞ factor $R_{0,1}$ (Cf. [**C4**], [**Ha**]). For $\varepsilon > 0$ and $t \in R$, consider the one-parameter group $\{\alpha_{\exp i\varepsilon t} \otimes \theta_t\}$ acting on the pair $B \otimes R_{0,1} \subset A \otimes R_{0,1}$. Since $\{\alpha_{\exp i\varepsilon t}\}$ are non-conjugate for different $\varepsilon's$, the type III_1 pair of factors obtained by taking the crossed products are thus pairwise non-isomorphic. From Theorem 4.4, the principal graph of these pairs of factors is $A_{\infty,\infty}$. □

Actually, as the following proposition shows, the examples constructed in Corollary 5.5 exhaust all the one-parameter standard automorphisms on the hyperfinite type II_1 pair $B \subset A$ with $\tilde{A}_n$ as principal graph, for any $n \geq 1$.

PROPOSITION 5.6. *Let $\{\theta_t\}_{t\in R}$ be a one-parameter group of automorphisms on $B \subset A$. Let $\varphi(t) \in \mathcal{G}$ denote the standard automorphism corresponding to θ_t. Then we have:*

i) $\varphi(t) = Id$, $\forall t \in \boldsymbol{R}$, or

ii) $\varphi(t) \sim e^{\frac{2\pi i t}{t_0}}$ for $n \geq 2$, or $\varphi(t) \sim \begin{bmatrix} e^{\frac{2\pi i t}{t_0}} & 0 \\ 0 & e^{-\frac{2\pi i t}{t_0}} \end{bmatrix}$ for $n = 1$, where $t_0 \neq 0$ and $t_0\boldsymbol{Z} = \{t \in \boldsymbol{R} : \varphi(t) = Id\}$.

In the latter case, for any two one-parameter groups $\theta^{(1)}$ and $\theta^{(2)}$ on $B \subset A$, $\varphi(\theta^{(1)})$ and $\varphi(\theta^{(2)})$ are conjugate in $\mathcal{G}$ if and only if $t_0^{(1)} = \pm t_0^{(2)}$.

Proof. Suppose that $\varphi(t) \neq Id$ for some $t \in \mathbf{R}$, then the kernel $\{t \in \mathbf{R} : \varphi(t) = Id\}$ is a closed subgroup of $\mathbf{R}$ due to the continuity of the standard homomorphism (Cf. [**L2**]), and so it is of the form $t_0\mathbf{Z}$, for some non-zero t_0. It is then elementary to show that $\varphi(t) \sim e^{\frac{2\pi i t}{t_0}}$ for $n \geq 2$ or $\varphi(t) \sim \begin{bmatrix} e^{\frac{2\pi i t}{t_0}} & 0 \\ 0 & e^{-\frac{2\pi i t}{t_0}} \end{bmatrix}$ for $n = 1$. The rest of the statement in ii) follows from Lemma 5.3. □

For the existence of one-parameter group on $B \subset A$ with prescribed standard invariants, we note that if for each $s \in \mathbf{R}$, we set $\theta_t^{(s)} = Ad \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i s t} \end{bmatrix}$, then from the proof of Theorem 4.4, $\varphi(\theta_t^{(s)}) =$

Id if and only if nst is an integer, thus by Proposition 5.6, $\varphi(\theta^{(s)})$ is classified, up to conjugacy, by $\dfrac{1}{ns}$. It follows that if $s \neq \pm s'$, then $\theta^{(s)}$ and $\theta^{(s')}$ are non-conjugate.

REMARK 5.7. As will be explained in the next section, the results of Corollaries 5.4 and 5.6 follow readily from known results on the classification of automorphisms on hyperfinite type III_λ factors in [**KST**]. However, it is reassuring to know that the standard invariant can also be used to study the conjugacy problem for automorphisms via the subfactor approach.

6. Remarks. Finally we would like to explain the results of Theorem 4.4 and its corollaries from the perspective of the classical invariants for automorphisms on a single factor that were introduced in [**C3**], [T]. First we would like to recall the construction of locally trivial subfactors, which appear in [**J**], [**PiPo1**], [**P2**]. We thank Prof. S. Popa for suggesting to us the possible relation between this construction and our results.

Let P be a factor, and $\theta \in Aut(P)$, consider $N = \{x \oplus \theta(x);\, x \in P\} \subset P \otimes M_{2\times 2}(C) = M$. Let $E : M \to N$ be defined by :

$$E\left(\begin{bmatrix} a\, b \\ c\, d \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} a + \theta^{-1}(d) & 0 \\ 0 & \theta(a) + d \end{bmatrix}.$$

It is then simple to check that E is a normal faithful conditional expectation and that $\operatorname{Ind}E = 4$ by the local index formula.

If P is of type III_λ, $\lambda \neq 0, 1$, then so are $N \subset M$. However, it is not true in general, that $N \subset M$ admits a common discrete decomposition. In fact, we have the following:

PROPOSITION 6.1.. *$N \subset M$ admits a common discrete decomposition if and only if* $\operatorname{mod}(\theta) = Id$.

Proof. Recall that $N \subset M$ has a common discrete decomposition if and only if a generalized trace on N gives rise to a generalized trace on M by composing with E.

Suppose that there is a common discrete decomposition for $N \subset M$. Let φ be a generalized trace on P, then the balanced weight

$\varphi \oplus \varphi$ is also a generalized trace on N. Let μ be such that $F_{-\ln\mu} = mod(\theta)$ with $\lambda < \mu \leq 1$, where F denotes the flow of weights. Replacing, if necessary, θ by a unitary perturbation of itself, we may assume that $\varphi \circ \theta = \mu\theta$. Let $\psi = (\varphi \oplus \varphi) \circ E$, we have: $\psi = \dfrac{1+\mu}{2}\varphi \oplus \dfrac{1+\mu^{-1}}{2}\varphi$. Since ψ is a generalized trace on M, $\sigma_T^\psi = Id_M$, where $T = -\dfrac{2\pi}{\ln\lambda}$, and for all t, we have:

$$\sigma_t^\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \sigma_t^\psi(a) & \sigma_t^\psi(b)u_t^* \\ u_t\sigma_t^\psi(c) & \sigma_t^\psi(d) \end{bmatrix},$$

where $u_t = [D(\dfrac{1+\mu^{-1}}{2}\varphi) : D(\dfrac{1+\mu}{2}\varphi)]_t = \left(\dfrac{1+\mu^{-1}}{1+\mu}\right)^{it}$. It then follows that $\mu^{iT} = 1$ and hence $mod(\theta) = Id$.

The converse is clear. □

In general, if the type III_λ inclusion $E : M \to N$ does not have a common discrete decomposition, one can always perturb E by a positive invertible element in $N' \cap M$ so that $N \subset M$ will have a common discrete decomposition with respect to this new conditional expectation .

Now let P be a hyperfinite type III_λ factor, $\theta \in Aut(P)$ with $\mathrm{mod}\theta = \mathrm{Id}$, φ a generalized trace on P. Then we may assume that $\varphi \circ \theta = \theta$ and so $\theta(P^\varphi) = P^\varphi$. Let $p_a(\theta) = n$ and $p_o(\theta) = m$ be the asymptotic and outer periods of θ. By [**C3**], $\theta = \theta'\sigma_s^\varphi$, with $p_a(\theta') = p_o(\theta') = n$ and $\theta'(P^\varphi) = P^\varphi$. Observe that if $N(\theta) \subset M$ denotes the locally trivial inclusion constructed using θ, then the type II_∞ inclusion from the common discrete decomposition of $N \subset M$ is locally trivial and is determined by the restriction of θ on P^φ.

Suppose first that m and n are both positive. As θ^n is centrally trivial, by [**KST**], there is some $t \in \dfrac{R}{T\mathbf{Z}}$, $T = -\dfrac{2\pi}{\ln\lambda}$, such that $\epsilon(\theta^n) = \epsilon(\sigma_t^\varphi)$. Note that $m = nk$ for some positive k and σ_{kt}^φ is inner. Hence $t \in \{T, \dfrac{T}{k}, \ldots, \dfrac{k-1}{k}T\}$. Once t is fixed, then by [**C3**], [**KST**], θ is classified by : $p_o(\theta')$, $t(\theta)$ and $\gamma(\theta')$. Note that we actually have $t \in \{\dfrac{l}{k}T;\ 1 \leq l \leq k,\ and\,(l,k) = 1\}$. In other words, t is a primitive k^{-th} root of unity, and $\gamma(\theta')$ is an n^{-th} root of unity. Thus with m and n fixed, the number of non-conjugate automorphisms is $n\times$ the number of primitive k^{-th} roots of unity.

From the subfactor point of view via the locally trivial factor construction, this corresponds to the case when the type III_λ graph is $\tilde{A}_m$ and the type II_1 graph is $\tilde{A}_n$. Identify each of the primitive k^{-th} root of unity with its conjugate and let l be the number of such pairs, then by Theorem 4.4, the number of non-conjugate type III_λ subfactors that arise this way is at least $n \times l$, which is less than the number of non-conjugate automorphisms with the same asymptotic and outer periods as explained above.

This discrepancy is due to the fact that the locally trivial factor constructions from θ and θ^{-1} are always conjugate by means of the isomorphism that switches the diagonal entries of $P \otimes M_{2\times 2}(C)$; whereas in general, θ and θ^{-1} are not outer conjugate.

If $p_o(\theta) = 0$ and $p_a(\theta) = n > 0$, then this corresponds to the case when the type III_λ graph is $A_{\infty,\infty}$, and the type II_1 graph is $\tilde{A}_n$. By [**C3**], $\theta = \theta' \sigma_t^\varphi$ with $p_o(\theta') = n$ and $t \notin TZ$. Hence there are uncountably many non-conjugate automorphisms with these conditions. This is consistent with our results since in this case, the standard invariant is given by an element of the circle of infinite order.

If m and n are both zero, then θ is unique up to conjugacy by [**C1**]. This corresponds to the case where the type III_λ graph and the type II_1 graph are both $A_{\infty,\infty}$. In this case, θ is outer conjugate to $Id_P \otimes s_0$. This result is also consistent with Theorem 4.4.

Suppose now that P is of type III_1 and $\theta \in \mathrm{Aut}(P)$. As in [**KST**], we may assume that θ admits an invariant dominant weight φ so that $\theta(u_s) = u_s$, for all $s \in R$, where $\{u_s\}$ is the one-parameter unitary group in P associated with the continuous decomposition of P given by φ. Hence $\theta(P^\varphi) = P^\varphi$.

Since any inclusion of type III_1 factors admit a common continuous decomposition (Cf. [**L1**]), the type II_∞ inclusion of the continuous decomposition of $N(\theta) \subset M = P \otimes M_{2\times 2}(C)$ is also locally trivial and is determined by the restriction of θ to P^φ.

If $p_o(\theta) = 0$ and $p_a(\theta) = n$, then $p_o(\theta|P^\varphi) = n$. In this case, the type III_1 graph is $A_{\infty,\infty}$ and the type II_1 graph is $\tilde{A}_n$. By a result in [**KST**], $\theta^n = \mathrm{Ad}w \circ \sigma_t^\varphi$ and thus θ is classified by $t \in R$, and the obstruction of $\theta|P^\varphi$, once n is fixed. This explains the result of

Corollary 5.5.

It would be interesting to pinpoint the exact relationship between the classical invariants for automorphisms and the standard invariant that is constructed from considering either the common discrete or continuous decomposition for factor-subfactor pairs obtained from the locally trivial factor construction.

Note Added in Proof. After the completion of this paper, we received the preprint **[P3]** of S. Popa, in which it was shown, among other results, that trace-scaling automorphisms on a strongly amenable inclusion of hyperfinite type II_∞ factors with finite index are classified by their standard images. In particular, Theorem 4.4 provides a list of all inclusions of hyperfinite type III_λ, $0 < \lambda < 1$, with index 4 and a common discrete decomposition.

We also received the preprint **[EKa]** in which another kind of symmetries on the principal graph is considered.

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