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ORDER OF THE IDENTITY OF THE STABLE SUMMANDS OF $\Omega^{2k}S^{2n+1}$

PAUL SILBERBUSH

We obtain upper bounds for the stable suspension orders of the summands in Snaith's stable decomposition of $\Omega^{2k}S^{2n+1}$, for $2k < 2n+1$, and localized at a prime p. These finite, torsion complexes also occur as subquotients of May's filtration of $\Omega^{2k}S^{2n+1}$, and as Thom spaces of canonical vector bundles. The results are obtained by induction, using results of Toda on the stable orders of stunted real projective spaces (for $p = 2$) and certain cofibrations of Mahowald to start the induction and proceding by stabilizing factorizations of power maps on $\Omega^{2k}S^{2n+1}$ due to James and Selick for $p = 2$ and Cohen-Moore-Neisendorfer for $p > 2$.

Introduction. For suitable topological spaces X and Y let $[X, Y]$ be the set of homotopy classes of pointed maps from X to Y. It is well known that $[X, Y]$ is a group if the domain is a suspension space and an abelian group if it is a double suspension. In particular, $[\Sigma X, \Sigma X]$ is a group and we say that X has suspension order k if the identity map of ΣX has order k in this group and that X has stable order k if the identity map of $\Sigma^m X$ has order k for m sufficiently large. It is not hard to show that a finite CW-complex whose reduced integral homology is all torsion has finite suspension order.

In this paper we compute upper bounds for the stable orders of a particular class of finite complexes. A theorem of Snaith [**Sn**] gives that $\Omega^n\Sigma^n X$ is stably equivalent to a wedge $\bigvee_{j\geqslant 1} D_j(\Omega^n\Sigma^n X)$, where $D_j(\Omega^n\Sigma^n X)$ is the j-adic construction on X, defined by $D_j(\Omega^n\Sigma^n X) = \mathcal{C}_n(j)\times_{\Sigma_j} X^{[j]}/\mathcal{C}_n(j)\times_{\Sigma_j} *$, where $\mathcal{C}_n(j)$ is the space of "j little n-cubes" in $\mathbf{R}^n$. The spaces $D_j(\Omega^n\Sigma^n X)$ also occur as the quotients in the canonical filtration of May's combinatorial model of $\Omega^n\Sigma^n X$, described in [**Ma**]. Specializing further, after localizing at any prime $p \geqslant 2$, we give upper bounds for the stable order of $D_j(\Omega^{2k}S^{2n+1})$ for $j \geqslant 2, k \geqslant 1$ and $2k < 2n+1$. In this case, the spaces $D_j(\Omega^{2k}S^{2n+1})$ also admit a description as Thom spaces of Whitney sums of certain bundles over the spaces $\mathcal{C}_{2k}(j) \times_{\Sigma_j} *$. It is shown in [**CCKN**] that these bundles have finite order, and this leads to a periodicity among the stable homotopy types of $D_j(\Omega^{2k}S^{2n+1})$ as n varies, which we will make extensive use of.

We will show that for j not a power of p, an upper bound for the stable order of $D_j(\Omega^{2k}S^{2n+1})$ is determined by upper bounds for the stable orders of $D_{p^m}(\Omega^{2k}S^{2n+1})$, and thus it suffices to compute upper bounds for the stable orders of the latter spaces. To accomplish this, we will use double induction on m and k.

For the case $m = 1$, the p-adic pieces $D_p(\Omega^{2k}S^{2n+1})$ may be identified (for $p = 2$) with suspensions of stunted real projective spaces, for which Toda has computed the stable orders in [**T**]. When $p > 2$, we obtain upper bounds for the stable orders directly.

The case $k = 1$ is more complicated. It suffices to work with $D_{p^m}(\Omega^2S^3)$, since $D_{p^m}(\Omega^2S^3)$ and $D_{p^m}(\Omega^2S^{2n+1})$ are stably equivalent after a suitable dimension shift.

THEOREM 2.3. *For $p > 2$, The stable order of $D_{p^m}(\Omega^2S^3)$ divides p^m.*

R. Cohen has shown in [**Coh**] that $D_{p^2m+p}(\Omega^2S^3)$ represents the odd-primary Brown-Gitler spectrum $B(m)$. Thus the order of the identity of $B(m)$ is p for all m, since $D_{p^2m+p}(\Omega^2S^3)$ splits off of $D_p(\Omega^2S^3) \wedge D_{p^2m}(\Omega^2S^3)$.

To prove this, we use induction on m and an odd-primary analogue of a cofibration sequence involving the spaces $D_{2^m}(\Omega^2S^3)$ which is originally due to Mahowald [**M**] and reproven by F. Cohen, Mahowald, and Milgram in [**CMM**]. By the same line of proof, this 2-local cofibration gives the following:

THEOREM 2.7. *The stable order of $D_{2^m}(\Omega^2S^3)$ divides 2^{m+1}.*

Brown and Peterson have shown that $D_{2^m}(\Omega^2S^3)$ represents the Brown-Gitler spectrum $B(2^{m-1})$, so we obtain as a corollary,

COROLLARY 2.8. *The order of* $1 \in [B(2^{m-1}), B(2^{m-1})]$ *divides* 2^{m+1}.

This result has already been obtained and shown to be best possible by W.H. Lin in [**L**], using lambda algebra and Adams spectral sequence techniques. Also included in Lin's paper is an alternative proof (for the dual Brown-Gitler spectrum) using results of Goerss.

To proceed with the inductive step, we introduce some new methods. At the prime 2, it is also necessary to distinguish several cases.

THEOREM 4.1. *The stable order of* $D_{2^m}(\Omega^4 S^{4n+1})$ *divides* 2^{m+2}.

THEOREM 4.2. *Let* $m \geqslant 1$ *and* $k \geqslant 1$.
(1) *The stable order of* $D_{2^m}(\Omega^{4k+2}S^{4n+3})$ *divides* $2^{2m+3k-1}$.
(2) *The stable order of* $D_{2^m}(\Omega^{4k+4}S^{4n+1})$ *divides* 2^{2m+3k}.
(3) *The stable order of* $D_{2^m}(\Omega^{4k}S^{4n+3})$ *divides* $2^{2m+3k-2}$.
(4) *The stable order of* $D_{2^m}(\Omega^{4k+2}S^{4n+1})$ *divides* $2^{2m+3k-1}$.

It is a consequence of our methods that the sharper result of Theorem 4.1 is possible for $\Omega^4 S^{4n+1}$. It is hoped that the results of Theorem 4.2 may be improved somewhat.

For odd primes, we have a single result which covers all cases.

THEOREM 4.7. *For* $p > 2$ *and* $m \geqslant 1$, *the stable order of* $D_{p^m}(\Omega^{2k}S^{2n+1})$ *divides* p^{m+k-1}.

We believe that the results of Theorem 4.1 and 4.7 are best possible. But to verify this, it is necessary to compute a stable invariant such as integral K-theory, which is difficult for loops on spheres.

We prove these theorems by stabilizing factorizations through $\Omega^{2k-2}S^{2n-1}$ of power maps on the loop spaces $\Omega^{2k}S^{2n+1}$, and using induction on k. These factorizations are due to James and Selick at the prime 2 and F. Cohen, Moore and Neisendorfer for odd primes. Upon stabilization, these loop spaces decompose into the summands $D_j(\Omega^{2k}S^{2n+1})$ by Snaith's splitting theorem. Furthermore, stabilizing an r-th power map on an H-space gives the degree r map plus a "deviation term".

PROPOSITION 3.2. *Let* X *be an* H*-space with* r*-th power map* r. *Then* $\Sigma^2 r = [r] + \Sigma d_r$, *where* d_r *factors through* $\Sigma(X \wedge X)$.

Thus we are able to bound the order of the map d_r and therefore the order of the degree r map $[r]$. Combining this with a careful analysis of how the summands map to each other in the stabilized factorization using cellular approximation enables us to obtain the upper bounds on each summand.

The paper is organized as follows: In Section 1 we review some properties of the spaces $D_j(\Omega^{2k}S^{2n+1})$, including their description as Thom spaces. We also describe the relationship between $D_{p^m}(\Omega^{2k}S^{2n+1})$ and $D_j(\Omega^{2k}S^{2n+1})$ for j not a power of p. In Section 2 we prove Theorem 2.3, including a proof of the odd-primary analogue of Mahowald's cofibration. We then briefly describe the case $p = 2$, in which Mahowald's cofibration may be used to obtain an alternate proof of Lin's result. Section 3 lays the technical groundwork for the proof of Theorems 4.1, 4.2, and 4.7. We recall the factorizations of power maps on loop spaces of spheres mentioned above, and analyze the stabilization of these factorizations. This includes the description of the deviation and the proof of Proposition 3.2. In Section 4 we use all this to prove Theorems 4.1, 4.2, and 4.7. Finally, in Section 5 we formulate and prove in some generality the relationship between $D_{p^m}(\Omega^n\Sigma^nX)$ and $D_j(\Omega^n\Sigma^nX)$, for j not a power of p. This is well-known, but does not seem to have appeared in the literature, except for the case Ω^2S^3 which is done by R. Cohen and Goerss (with a different proof) in [**CG**].

We use the following notation and conventions: The word "space" will always mean compactly generated space with non-degenerate basepoint, all maps and homotopies are required to preserve the basepoint, and we will denote constant maps by $*$. For a space X, ΣX is the reduced suspension $S^1 \wedge X$ and ΩX is the space of pointed maps from S^1 to X with the compact-open topology. For a space X, we denote the suspension spectrum of X by $\Sigma^\infty X$, and for a finite complex X, we will make no distinction between the stable order of X and the order of the identity of the spectrum $\Sigma^\infty X$. Moreover, where convenient we will also identify X and $\Sigma^\infty X$. For any space X, $[k]$ will denote the degree k map on ΣX, which is k times the identity map in the group $[\Sigma X, \Sigma X]$. We denote the ring of integers by Z, and the field with p elements by Z_p. As usual, $H_*(\ ;R)$ and $H^*(\ ;R)$ denote ordinary singular homology and cohomology with coefficients in the ring R. Finally, a reference such as $a.b$ means the b-th item in section a.

The material in this paper comes from my thesis, written at the University of Rochester. I would like to thank my adviser Joe Neisendorfer for his guidance during this time. I would also like to thank Fred Cohen for his invaluable assistance with some of the material in this work, and the referee for his useful suggestions.

1. The stable decomposition of iterated loops on a sphere. Recall that a theorem of Snaith states that $\Omega^n\Sigma^n X$ has a stable splitting

$$\Sigma^\infty\Omega^n\Sigma^n X \simeq \bigvee_{j\geqslant 1} \Sigma^\infty D_j(\Omega^n\Sigma^n X)\ ,$$

where the pieces $D_j(\Omega^n\Sigma^n X)$ are the subquotients of the canonical filtration in May's combinatorial model of $\Omega^n\Sigma^n X$. Recall further that with coefficients in the field Z_p, the reduced homology of $D_j(\Omega^n\Sigma^n X)$ is spanned by the monomials of weight j in the homology of $\Omega^n\Sigma^n X$.

LEMMA 1.1. *There exist maps $D_r(\Omega^n\Sigma^n X)\wedge D_s(\Omega^n\Sigma^n X) \xrightarrow{\mu_{r,s}} D_{r+s}(\Omega^n\Sigma^n X)$ such that $(\mu_{r,s})_*(m_r\otimes m_s) = m_r m_s$ where m_r and m_s are monomials of weight r and s in $H_*(\Omega^n\Sigma^n X)$.*

Proof. The May filtration F_jC_nX gives maps $F_r\times F_s \xrightarrow{\mu_{r,s}} F_{r+s}$ which correspond to loop space multiplication in $\Omega^n\Sigma^n X$. These maps have the above property on homology.

The following diagram can clearly be completed so as to be commutative, yielding the desired maps on the bottom row.

$$\begin{tikzcd}
F_rC_nX\times F_sC_nX \arrow[r, "\mu_{r,s}"] \arrow[d] & F_{r+s}C_nX \arrow[d] \\
D_r(\Omega^n\Sigma^n X)\wedge D_s(\Omega^n\Sigma^n X) & D_{r+s}(\Omega^n\Sigma^n X)
\end{tikzcd}$$

□

PROPOSITION 1.2.
(1) $\tilde{H}_*D_j(\Omega^{2k}S^{2n+1}) = 0$ *(with Z_p coefficients) unless $j\equiv 0\ (p)$ or $j\equiv 1\ (p)$.*

(2) *For $r \geqslant 1$, $D_{pr+1}(\Omega^{2k}S^{2n+1})$ and $\Sigma^{2n-2k+1}D_{pr}(\Omega^{2k}S^{2n+1})$ are homotopy equivalent when localized at p.*

Proof. The proof given in [**Coh**] for the case $m = 1$ goes through without change. (1) follows from considering all possible monomials of weight j in $H_*(\Omega^{2k}S^{2n+1})$. For (2), note that $D_1(\Omega^{2k}S^{2n+1}) \simeq S^{2n-2k+1}$ and the map $D_1(\Omega^{2k}S^{2n+1}) \wedge D_{pr}(\Omega^{2k}S^{2n+1}) \xrightarrow{\mu_{1,pr}} D_{pr+1}(\Omega^{2k}S^{2n+1})$ is an isomorphism in mod-p homology. $\square$

We see from this proposition that to compute upper bounds for the stable orders of the spaces $D_j(\Omega^{2k}S^{2n+1})$ that we need only consider the cases $j = pr$ where p is the prime at which we are localized. The next proposition combined with basic properties of suspension orders of wedges and smash products (see [**T**]) reduces the computation to yet another special case. We assume that all spaces are localized at the prime under consideration.

PROPOSITION 1.3. *Let $pr = a_1p^{i_1} + \ldots + a_lp^{i_l}$ such that $0 \leqslant i_1 < \ldots < i_l$ and $1 \leqslant a_j \leqslant p-1$. Then D_{pr} is a stable wedge summand of $D_{p^{i_1}}^{[a_1]} \wedge D_{p^{i_2}}^{[a_2]} \wedge \ldots \wedge D_{p^{i_l}}^{[a_l]}$. (Here $X^{[m]}$ is the m-fold smash product, and $D_j(\Omega^{2k}S^{2n+1})$ is abbreviated to D_j.)*

Note that this statement makes sense when $p = 2$. In this case, the a_i's must all be equal to 1. For example, D_{12} splits off of $D_4 \wedge D_8$. The proof of this proposition is postponed until Section 5, where in fact we will prove a slightly more general statement.

Let $\xi_{n,j}$ be the j-dimensional real vector bundle

$$\mathbf{R}^j \to \mathcal{C}_n(j) \times_{\Sigma_j} \mathbf{R}^j \to \mathcal{C}_n(j) \times_{\Sigma_j} * \,.$$

It is shown in [**Mi**] that $D_j(\Omega^nS^{n+m})$ is homeomorphic to the Thom space of $m\xi_{n,j}$, the m-fold Whitney sum of $\xi_{n,j}$.

We will need the following results on the periodicity of the bundles $\xi_{n,j}$.

PROPOSITION 1.4. *The bundle $2\xi_{2,j}$ is trivial.*

This is proved in [**CMM**] by observing that the Whitney sum $2\xi_{2,j}$ is the bundle

$$\mathbf{R}^{2j} \to \mathcal{C}_2(j) \times_{\Sigma_j} \mathbf{R}^{2j} \to \mathcal{C}_2(j) \times_{\Sigma_j} *$$

and constructing an explicit bundle isomorphism between this bundle and a trivial bundle ε_{2j} over $\mathcal{C}_2(j) \times_{\Sigma_j} *$. This has the following important corollary.

COROLLARY 1.5. *$D_j(\Omega^2 S^{q+2}) \simeq \Sigma^{j(q-1)} D_j(\Omega^2 S^3)$ for q odd.*

This is actually a homeomorphism, but a stable homotopy equivalence is all we need. More generally, let $\phi_{n,j}$ be the stable order of the bundle $\xi_{n,j}$. A sharp bound on $\phi_{n,j}$ is obtained in [**CCKN**]. This leads to the following:

COROLLARY 1.6. *There is a stable equivalence $D_j(\Omega^n S^{n+r+\phi_{n,j}}) \simeq \Sigma^{j\phi_{n,j}} D_j(\Omega^n S^{n+r})$.*

2. Suspension orders in the stable decomposition of $\Omega^2 S^3$.

Throughout this section, all spaces are assumed to be localized at a fixed odd prime p, and all homology groups have coefficients in Z_p. For ease of notation $D_j(\Omega^2 S^3)$ will be written as D_j.

THEOREM 2.1. *Let $pk = a_1 p^{i_1} + \ldots + a_l p^{i_l}$ such that $0 \leqslant i_1 < \ldots < i_l$ and $1 \leqslant a_j \leqslant p-1$, and define $\phi(k) = i_1$. Then the stable order of D_{pk} divides $p^{\phi(k)}$.*

Recall from [**Coh**] that if $k = mp + r$ with $1 \leqslant r \leqslant p-1$, then $\Sigma^\infty D_{pk}$ is homotopy equivalent to the odd-primary Brown-Gitler spectrum $\Sigma^{2k(p-1)} B(m)$.

COROLLARY 2.2. *The order of $1 \in [B(m), B(m)]$ is equal to p for $m \geqslant 0$.*

Note that as k varies, there are many D_{pk} that do not correspond to odd-primary Brown-Gitler spectra, for example D_{p^n} for $n > 1$.

By 1.3, it suffices to prove the following:

THEOREM 2.3. *The stable order of D_{p^n} divides p^n.*

The first step in the proof of 2.3 is to obtain an analogue of the cofibration sequence constructed by Mahowald in [**M**].

THEOREM 2.4. *If r is even and sufficiently larger than k, there is a (p-local) cofibration sequence*

$$\Sigma^{pk(r-2)+2p-2} D_{pk-p} \to \Sigma^{pk(r-2)} D_{pk} \to \Sigma^{k(pr-2)} D_k.$$

Proof. This proof is obtained by modifying the proof at the prime 2 in [**CMM**]. Let $\Omega S^{r+1} \xrightarrow{h} \Omega S^{pr+1}$ be the p-th James-Hopf invariant, and denote May's model for $\Omega^n\Sigma^n X$ by C_nX. Then we have $\Omega h : C_2S^{r-1} \to C_2S^{pr-1}$, which for dimensional reasons, restricts to

$$F_iC_2S^{r-1} \to F_{[i/p]}C_2S^{pr-1} \qquad \text{if } i \leqslant pk.$$

Since $\tilde{H}_*F_pC_2S^{r-1} = \operatorname{span}\{x_1, \beta Q_{p-1}x_1, Q_{p-1}x_1\}$, $F_p = F_pC_2S^{r-1}$ has a cell decomposition $F_p = S^{r-1} \cup e^{pr-2} \cup e^{pr-1}$. Let $F_p{}'$ be the $(pr-2)$-skeleton of F_p. Note that the composite $F_p{}' \hookrightarrow C_2S^{r-1} \xrightarrow{\Omega h} C_2S^{pr-1}$ is null-homotopic, since Ω^2S^{pr+1} is $(pr-2)$-connected. Then the commutative diagram

$$\begin{array}{ccc} F_p{}' \times F_{pk-p} & \xrightarrow{\mu_{p,pk-p}} & F_{pk} \\ \Big\downarrow{\scriptstyle \Omega h \times \Omega h} & & \Big\downarrow{\scriptstyle \Omega h} \\ * \times F_{k-1} & \xrightarrow{\mu_{0,k-1}} & F_{k-1} \hookrightarrow F_k \end{array}$$

shows that the composite

$$F_p{}'C_2S^{r-1} \times F_{pk-p}C_2S^{r-1} \xrightarrow{\mu_{p,pk-p}} F_{pk}C_2S^{r-1} \xrightarrow{\Omega h} F_kC_2S^{pr-1}$$

has its image in $F_{k-1}C_2S^{pr-1}$. By passing to quotients, we obtain the sequence

$$S^{pr-2} \wedge D_{pk-p}(C_2S^{r-1}) \xrightarrow{\mu} D_{pk}(C_2S^{r-1}) \xrightarrow{\bar{h}} D_k(C_2S^{pr-1})$$

with $\bar{h} \circ \mu \simeq *$.

It now suffices to show that the above sequence gives a short exact sequence in mod p homology. $\square$

LEMMA 2.5. *$h_*(x_r^{np}) = \lambda y_{rp}^n$ and $h_*(x_r^m) = 0$ if $m \neq 0$ (p), where $H_*(\Omega S^{r+1}) \cong Z_p[x_r]$ and $H_*(\Omega S^{pr+1}) \cong Z_p[y_{pr}]$ and λ is a unit mod p.*

A proof may be found in [**H**]. In fact, it is not hard to show that $\lambda \equiv 1$ (p). It then follows that $(\Omega h)_*(Q_{p-1}^i x_{r-1}) = Q_{p-1}^{i-1}x_{pr-1}$ and $(\Omega h)_*(\beta Q_{p-1}^i x_{r-1}) = \beta Q_{p-1}^{i-1}x_{pr-1}$ since the Bockstein is natural. This implies that $\bar{h}_*$ is surjective, and it is clear that μ_* is injective. Now if $m \in ker\ \bar{h}_*$, then $m = (\beta Q_{p-1}x_{r-1})m'$ where m'

is a monomial of weight $pk - p$. Then $m = \mu_*(\beta Q_{p-1}x_{r-1} \otimes m')$ so $ker\ \bar{h}_* \subseteq im\ \mu_*$. Since $\bar{h} \circ \mu \simeq *$, it follows that $ker\ \bar{h}_* = im\ \mu_*$. Now apply Corollary 1.5 to obtain the cofibration in the statement of the theorem.

PROPOSITION 2.6. *The Moore space $S^1 \cup_p e^2$ has stable order p.*

This is a well-known fact. A proof may be found in [**N1**].

Proof of 2.3. The proof is by induction on n, the case $n = 1$ is the Moore space $S^1 \cup_p e^2$, since it easy to see by inspecting its homology that D_p is this Moore space. Let μ be the first map in the cofibration of 2.4, and view μ as $\Sigma^{pk(r-2)}\Sigma^{2p-2}D_{pk-p} \to \Sigma^{pk(r-2)}D_{pk}$. Consider the following diagram, in which cumbersome suspensions have been omitted. The square commutes by construction of μ.

$$\begin{tikzcd}
S^{2p-2} \wedge D_{p^n-p} \arrow[d, "{[p]\wedge[1]}"'] & \\
S^{2p-2} \wedge D_{p^n-p} \arrow[r, "\mu"] \arrow[d, "{i\wedge[1]}"'] & D_{p^n} \arrow[d, equal] \\
D_p \wedge D_{p^n-p} \arrow[r, "\mu_{p,p^n-p}"] & D_{p^n}
\end{tikzcd}$$

Since the attaching map of D_p is of degree p, the vertical composite is null-homotopic, and therefore μ has order p.

Now consider the following commutative diagram, where l is the appropriate dimension shift:

$$\begin{tikzcd}
 & D_{p^n} \arrow[r, "\bar{h}"] \arrow[d, "{[p^{n-1}]}"'] & \Sigma^l D_{p^{n-1}} \arrow[d, "{[p^{n-1}]}"'] \\
S^{2p-2} \wedge D_{p^n-p} \arrow[r, "\mu"] & D_{p^n} \arrow[r, "\bar{h}"] \arrow[d, "{[p]}"'] & \Sigma^l D_{p^{n-1}} \arrow[d, equal] \\
 & D_{p^n} & \Sigma^l D_{p^{n-1}}.
\end{tikzcd}$$

Since the middle row is a cofibration and $p\mu$ is null-homotopic, the map $[p]$ extends to the cofibre, thus the bottom row can be completed with a map $\Sigma^l D_{p^{n-1}} \to D_{p^n}$. By the inductive hypothesis,

$[p^{n-1}] \simeq *$ on $D_{p^{n-1}}$. Therefore $[p^n] = [p] \circ [p^{n-1}] \simeq *$ on D_{p^n}. This completes the proof. □

As we noted in the introduction, this proof may be carried out in the case $p = 2$, using the 2-local (stable) cofibration

$$S^2 \wedge D_{2^n-2} \longrightarrow D_{2^n} \longrightarrow \Sigma^l D_{2^{n-1}}$$

constructed in [**M**] and [**CMM**]. The induction begins with the fact that the mod-2 Moore space $S^1 \cup_2 e^2$ has stable order 4. This leads to the following result, which agrees with that obtained by W.H. Lin in [**L**].

THEOREM 2.7. *The stable order of* $D_{2^n}(\Omega^2 S^3)$ *divides* 2^{n+1}.

COROLLARY 2.8. *The order of* $1 \in [B(2^{n-1}), B(2^{n-1})]$ *divides* 2^{n+1}.

3. Factorizations of power maps, deviations, and stabilizations. In this section we introduce the machinery which we will use to obtain upper bounds for the stable orders of the pieces in the stable decomposition of $\Omega^{2k} S^{2n+1}$, for $k > 1$. The main idea is to relate the map $[r]$ on the suspension of an H-space X to the suspension of the r-th power map on X coming from the H-space structure. These two maps are not the same in general, but differ by a deviation which we will describe. In particular, we will apply this to r-th power maps on $\Omega^{2k} S^{2n+1}$. By stabilizing known factorizations of these maps, we will have factorizations of $[r]$ on the stable summands, which allows a double induction to obtain the upper bounds.

DEFINITION 3.1. Let X be an H-space, and r be a positive integer. The r-th power map on X, $r : X \to X$ is the composite

$$X \xrightarrow{\Delta^r} X^r \xrightarrow{m} X .$$

Here m is the iterated multiplication (performed in some fixed order), and Δ^r is the r-fold diagonal map. In particular, we have r-th power maps on loop spaces, and $\Omega^n r \simeq r$ for any n.

Now if X is any space, ΣX has a degree r map denoted $[r]$ coming from the *co-H* structure in ΣX. We now describe the relationship between Σr and $[r]$ on ΣX, where X is an H-space.

PROPOSITION 3.2. *Let X be an H-space and let r be its r-th power map. Then $\Sigma^2 r = [r] + \Sigma d_r$ where d_r factors through $\Sigma(X \wedge X)$.*

Proof. By induction on r, beginning with $r = 2$. Consider the following diagram:

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma\Delta"] \arrow[d, "\mu"'] & \Sigma(X \times X) \arrow[r, "\Sigma m"] \arrow[d, "\theta"'] & \Sigma X \\
\Sigma X \vee \Sigma X \vee \Sigma X \arrow[r, "1\vee 1\vee\Sigma\bar{\Delta}"] & \Sigma X \vee \Sigma X \vee \Sigma(X \wedge X) \arrow[r, "1\vee 1\vee q"] & \Sigma X \vee \Sigma X \vee \Sigma X. \arrow[u, "\nabla"']
\end{tikzcd}$$

Here θ is the canonical homotopy equivalence and $\bar{\Delta}$ is the reduced diagonal $X \xrightarrow{\Delta} X \times X \xrightarrow{\pi} X \wedge X$ where π is the quotient map. We claim that this diagram (strictly) commutes. It then follows that $\Sigma 2 = [2] + q \circ \Sigma\bar{\Delta}$, and we set $d_2 = q \circ \Sigma\bar{\Delta}$. To show that the diagram commutes, recall that for any product $A \times B$ the equivalence θ is the composite

$$\Sigma(A \times B) \xrightarrow{\mu} \Sigma(A \times B) \vee \Sigma(A \times B) \vee \Sigma(A \times B)$$
$$\xrightarrow{\Sigma\pi_A \vee \Sigma\pi_B \vee \Sigma\pi} \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B).$$

With this description, a check on the point-set level shows that the left square commutes. Since θ is an equivalence, it has an inverse. Thus there is a map $q : \Sigma(X \wedge X) \to \Sigma X$ such that the right square commutes.

Now suppose that $\Sigma^2(r-1) = [r-1] + \Sigma d_{r-1}$, and consider the following diagram:

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma(r-1,1)"] \arrow[d, "\mu"'] & \Sigma(X \times X) \arrow[r, "\Sigma m"] \arrow[d, "\theta"'] & \Sigma X \\
\Sigma X \vee \Sigma X \vee \Sigma X \arrow[r, "\alpha"] & \Sigma X \vee \Sigma X \vee \Sigma(X \wedge X) \arrow[r, "1\vee 1\vee q"] & \Sigma X \vee \Sigma X \vee \Sigma X \arrow[u, "\nabla"']
\end{tikzcd}$$

where $\alpha = \Sigma(r-1) \vee 1 \vee \Sigma\bar{\Delta}_{r-1,1}$ and $\bar{\Delta}_{r-1,1} = \pi \circ (r-1, 1)$. As before, the left square commutes by a direct check. The top row is clearly Σr. It then follows that

$$\Sigma r = \Sigma(r-1) + 1 + q \circ \bar{\Delta}_{r-1,1}.$$

Suspend this equation. Then we are working in the abelian group $[\Sigma^2 X, \Sigma^2 X]$, and we have

$$\Sigma^2 r = \Sigma^2(r-1) + 1 + \Sigma(q \circ \bar{\Delta}_{r-1,1}) = [r] + \Sigma d_{r-1} + \Sigma(q \circ \bar{\Delta}_{r-1,1}) .$$

Now setting $d_r = d_{r-1} + q \circ \bar{\Delta}_{r-1,1}$ completes the proof. $\square$

COROLLARY 3.3. *$\Sigma^\infty r = [r] + \Sigma^\infty d_r$.*

We will apply this result to the following situation:

THEOREM 3.4. *Localized at 2, there exists a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega^3 S^{2n+1} \arrow[r, "4"] \arrow[d, "f"'] & \Omega^3 S^{2n+1} \\
\Omega S^{2n-1} \arrow[r, equal] & \Omega S^{2n-1}. \arrow[u, "\Omega\Sigma^2"']
\end{tikzcd}$$

If we restrict to n even, a factorization of 2 exists.

THEOREM 3.5. *Localized at 2, there exists a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega^4 S^{4n+1} \arrow[r, "2"] \arrow[d, "f"'] & \Omega^4 S^{4n+1} \\
\Omega^2 S^{4n-1} \arrow[r, equal] & \Omega^2 S^{4n-1}. \arrow[u, "g"']
\end{tikzcd}$$

The factorization of 4 in 3.4 is essentially due to James [**J**]. In this form it may also be attributed to Barratt (unpublished) and Moore (unpublished). The factorization of 2 in 3.5 is implicit in work of Selick [**Se**]. Detailed proofs of 3.4 and 3.5 may be found in [**Co**]. At odd primes, a single sharper statement is possible. For $p = 3$ this is due to Neisendorfer and for $p > 3$ to F. Cohen, Moore, and Neisendorfer. Proofs may be found in [**CMN**] and [**N2**].

THEOREM 3.6. *Localized at $p > 2$, there exists a homotopy com-*

mutative diagram

$$\begin{tikzcd} \Omega^2 S^{2n+1} \arrow[r, "p"] \arrow[d, "f"'] & \Omega^2 S^{2n+1} \\ S^{2n-1} \arrow[r, equal] & S^{2n-1}. \arrow[u, "\Sigma^2"'] \end{tikzcd}$$

These diagrams clearly remain commutative when looped down.

Now apply the functor Σ^∞ to any of the three previous diagrams (after looping further), and apply Snaith's stable decomposition. This yields the following commutative diagram, where we omit the symbol Σ^∞.

$$\begin{tikzcd} \bigvee_{j\geqslant 1} D_j(\Omega^{2k}S^{2n+1}) \arrow[r, "{[r]+d_r}"] \arrow[d, "f"'] & \bigvee_{j\geqslant 1} D_j(\Omega^{2k}S^{2n+1}) \\ \bigvee_{j\geqslant 1} D_j(\Omega^{2k-2}S^{2n-1}) \arrow[r, equal] & \bigvee_{j\geqslant 1} D_j(\Omega^{2k-2}S^{2n-1}) \arrow[u] \end{tikzcd}$$

Here $r = 2, 4$ or p and n must be even if $r = 2$. By 3.2, the map d_r factors through $\bigvee_{j\geqslant 1} D_j(\Omega^{2k}S^{2n+1}) \wedge \bigvee_{j\geqslant 1} D_j(\Omega^{2k}S^{2n+1})$. We now consider the behavior of particular summands in this diagram. All spaces are assumed to be localized at the prime under consideration.

PROPOSITION 3.7. *Suppose $pk \leqslant n$, where p is the prime under consideration.*

(1) *There exists an induced commutative diagram*

$$\begin{tikzcd} \bigvee_{j\geqslant 2} D_j(\Omega^{2k}S^{2n+1}) \arrow[r, "{[r]+\bar{d}_r}"] \arrow[d, "\bar{f}"'] & \bigvee_{j\geqslant 2} D_j(\Omega^{2k}S^{2n+1}) \\ \bigvee_{j\geqslant 2} D_j(\Omega^{2k-2}S^{2n-1}) \arrow[r, equal] & \bigvee_{j\geqslant 2} D_j(\Omega^{2k-2}S^{2n-1}) \arrow[u] \end{tikzcd}$$

obtained by pinching D_1 off of each bouquet.

(2) *For $p \geqslant 2$ the map $\bar{f}$ restricted to $D_{p^m}(\Omega^{2k}S^{2n+1})$ is homotopic to a map with image in $\bigvee_{2\leqslant j<p^{m+1}} D_j(\Omega^{2k-2}S^{2n-1})$.*

(3) *Let* $q : \bigvee_{j\geqslant 2} D_j(\Omega^{2k}S^{2n+1}) \to \bigvee_{j\geqslant 3} D_j(\Omega^{2k}S^{2n+1})$ *be the evident pinch map. For* $p = 2$ *the map* $q \circ \bar{d}_r$ *restricted to* $D_{2^m}(\Omega^{2k}S^{2n+1})$ *factors through* $\bigvee_{2<i+j<2^{m+1}} D_i(\Omega^{2k}S^{2n+1}) \wedge D_j(\Omega^{2k}S^{2n+1})$.

(4) *For* $p > 2$ *the map* $\bar{d}_r$ *restricted to* $D_{p^m}(\Omega^{2k}S^{2n+1})$ *factors through* $\bigvee_{2<i+j<2p^m} D_i(\Omega^{2k}S^{2n+1}) \wedge D_j(\Omega^{2k}S^{2n+1})$.

Proof. First note that $D_1(\Omega^{2k}S^{2n+1}) \simeq S^{2n-2k+1}$, and that the functor Σ^∞ does not change the dimension, connectivity, or homology of a finite complex. The proposition then follows by cellular approximation and the next lemma. □

LEMMA 3.8.

(1) *The dimension of* $D_{2^m}(\Omega^{2k}S^{2n+1})$ *is* $2^{m+1}n - 2k + 1$.

(2) *The connectivity of* $D_{2^m}(\Omega^{2k}S^{2n+1})$ *is* $2^{m+1}(n-k) + 2^m - 1$.

(3) *The dimension of* $D_{p^m}(\Omega^{2k}S^{2n+1})$ *is* $2p^m n - 2k + 1$.

(4) *The connectivity of* $D_{p^m}(\Omega^{2k}S^{2n+1})$ *is* $2p^m(n-k) + 2p^{m-1}(p-1) - 1$.

Proof. Recall that $\tilde{H}_*(D_{2^m}(\Omega^{2k}S^{2n+1})$ is spanned by the monomials of weight 2^m in $H_*(\Omega^{2k}S^{2n+1})$. Then it is clear that the bottom homology class is $x_{2n-2k+1}^{2^m}$ which has dimension $2^{m+1}(n-k)+2^m$ and the top class is $Q_{2k-1}^m x_{2n-2k+1}$ which by induction on m has dimension $2^{m+1}n-2k+1$. For p odd, the bottom class is $(\beta Q_1 x_{2n-2k+1})^{p^{m-1}}$ and the top class is also $Q_{2k-1}^m x_{2n-2k+1}$. □

To see why the hypotheses on k and n are needed, the inequalities which need to hold for (2) and (3) in 3.7 are

$$2^{m+1}n - 2k + 1 \leqslant 2^{m+2}(n-k) + 2^{m+1} - 1$$
$$2p^m n - 2k + 1 \leqslant 2p^{m+1}(n-k) + 2p^m(p-1) - 1$$
$$2p^m n - 2k + 1 \leqslant 4p^m(n-k) + 4p^{m-1}(p-1) - 1$$

and these are easily seen to be true if $pk \leqslant n$. We will need a further refinement of 3.7. Let i_{p^m} denote the inclusion

$$D_{p^m}(\Omega^{2k}S^{2n+1}) \hookrightarrow \bigvee_{j\geqslant 2} D_j(\Omega^{2k}S^{2n+1}).$$

PROPOSITION 3.9. *Let $p \geqslant 2$. For fixed m and k, we may choose n sufficiently large such that $\bar{d}_r \circ i_{p^m}$ factors through $\bigvee_{2<i+j<2p^m} D_i(\Omega^{2k}S^{2n+1}) \wedge D_j(\Omega^{2k}S^{2n+1})$ with $i+j \neq p^m+1$. (If $p=2$ the conclusion holds for $q \circ \bar{d}_r \circ i_{2^m}$.)*

Proof. We are trying to exclude the summand $(D_1 \wedge D_{p^m}) \vee (D_{p^m} \wedge D_1)$, which has connectivity $2n-2k+2p^m(n-k)+2p^{m-1}(p-1)$ (for $p>2$). Now the dimension of $D_{p^m}(\Omega^{2k}S^{2n+1})$ is $2p^m-2k+1$, so the inequality which must hold reduces to

$$2p^m k \leqslant 2n + 2p^{m-1}(p-1) - 1 \ ,$$

and similarly for $p=2$. □

By 1.6, we see that to compute stable orders for $D_j(\Omega^{2k}S^{2n+1})$, we may increase n as needed. We now state the corollary to 3.7 and 3.9 two which we will need in the next section. We make the standing assumption that $i+j>2$ and $i+j \neq p^m+1$ when considering the space $\bigvee_{i+j<2p^m} D_i(\Omega^{2k}S^{2n+1}) \wedge D_j(\Omega^{2k}S^{2n+1})$. This is justified by the previous proposition. Furthermore, we will abuse notation somewhat and identify f with $\bar{f}$ and d_r with $\bar{d}_r$ and $q \circ \bar{d}_r$ where it is convenient to do so.

COROLLARY 3.10. *Let $p \geqslant 2$.*

(1) *The map $f \circ i_{p^m}$ has order dividing the stable order of $\bigvee_{2\leqslant j\leqslant p^m} D_j(\Omega^{2k-2}S^{2n-1})$.*

(2) *The map $d_r \circ i_{p^m}$ has order dividing the stable order of $\bigvee_{i+j<2p^m} D_i(\Omega^{2k}S^{2n+1}) \wedge D_j(\Omega^{2k}S^{2n+1})$.*

Proof. The only thing to check is that the stable order of $D_j(\Omega^{2k}S^{2n+1})$ for $p^m<j<p^{m+1}$ is no larger than the stable order of $D_{p^m}(\Omega^{2k}S^{2n+1})$. This follows from 1.3. Furthermore, the stable order of $\bigvee_{2\leqslant j\leqslant p^m} D_j(\Omega^{2k-2}S^{2n-1})$ is equal to the stable order of $D_{p^m}(\Omega^{2k-2}S^{2n-1})$ and the stable order of $\bigvee_{i+j<2p^m} D_i(\Omega^{2k}S^{2n+1}) \wedge$

$D_j(\Omega^{2k}S^{2n+1})$ is equal to the stable order of $D_{p^{m-1}}(\Omega^{2k}S^{2n+1}) \wedge D_{p^{m-1}}(\Omega^{2k}S^{2n+1})$. □

4. Suspension orders in the stable decomposition of $\Omega^{2k}S^{2n+1}$.

In this section we obtain upper bounds for the stable orders of each piece in the stable decomposition of $\Omega^{2k}S^{2n+1}$ localized at each prime and for $2k < 2n+1$. This generalizes the results of sections 3 and 4 on the stable decomposition of $\Omega^2 S^{2n+1}$. There are enough differences between the cases $p = 2$ and $p > 2$ to warrant treating them separately, though the method used for $p = 2$ applies directly to (and is actually simpler for) odd primes. We make the standing assumption that all spaces are localized at the prime under consideration, and that $pk \leqslant n$ for any space $\Omega^{2k}S^{2n+1}$, where again p is the prime under consideration. This restriction will later be reduced to $2k < 2n+1$. By 1.3, it is enough to consider $D_{p^m}(\Omega^{2k}S^{2n+1})$

THEOREM 4.1. *The stable order of $D_{2^m}(\Omega^4 S^{4n+1})$ divides 2^{m+2}.*

THEOREM 4.2. *Let $m \geqslant 1$ and $k \geqslant 1$.*
(1) *The stable order of $D_{2^m}(\Omega^{4k+2}S^{4n+3})$ divides $2^{2m+3k-1}$.*
(2) *The stable order of $D_{2^m}(\Omega^{4k+4}S^{4n+1})$ divides 2^{2m+3k}.*
(3) *The stable order of $D_{2^m}(\Omega^{4k}S^{4n+3})$ divides $2^{2m+3k-2}$.*
(4) *The stable order of $D_{2^m}(\Omega^{4k+2}S^{4n+1})$ divides $2^{2m+3k-1}$.*

We will prove Theorem 4.2 by double induction on m and k, with the case $k = 1$ having been done already. For $m = 1$, the spaces D_2 are well known to be suspensions of stunted real projective spaces, for which the stable orders have been computed by Toda. Adopt the standard notation P_k^n for stunted real projective space RP^n/RP^{k-1}.

PROPOSITION 4.3. *$D_2(\Omega^{2k}S^{2n+1})$ is stably equivalent to $P^{2n}_{2n-2k+1}$.*

Here we mean stably equivalent by analogy with vector bundles. We write $X \cong_s Y$ if $\Sigma^p X$ is homeomorphic to $\Sigma^q Y$ for some integers p and q.

Proof. Recall from section 2 that $D_2(\Omega^{2k}S^{2n+1})$ is homeomorphic to the Thom space $T(m\xi_{2k,2})$ where $m = 2n-2k+1$. Let γ_{2k-1} be the canonical line bundle over RP^{2k-1}. Then $\xi_{2k,2}$ is stably equivalent to γ_{2k-1} (as vector bundles), so we have

$$T(m\xi_{2k,2}) \cong_s T(m\gamma_{2k-1}) \ .$$

It is shown in [**H**] that $T(m\gamma_{2k-1})$ is homeomorphic to P_m^{m+2k+1}. This completes the proof. □

Next, we recall Toda's results on the stable order of stunted real projective spaces.

THEOREM 4.4 [T].

(1) *The stable order of* $P^{4n}_{4n-4k+1}$ *is* $2^{2k+\epsilon}$ *where* $\epsilon = 0$ *if* k *is even and* $\epsilon = 1$ *if* k *is odd.*

(2) *The stable order of* $P^{4n}_{4n-4k-1}$ *is* $2^{2k+1+\epsilon}$ *where* $\epsilon = 0$ *if* k *is even and* $\epsilon = 1$ *if* k *is odd.*

(3) *The stable order of* $P^{4n+2}_{4n-4k+3}$ *is* $2^{2k+\epsilon}$ *where* $\epsilon = 0$ *if* k *is even and* $\epsilon = 1$ *if* k *is odd.*

(4) *The stable order of* $P^{4n+2}_{4n-4k+1}$ *is* $2^{2k+1+\epsilon}$ *where* $\epsilon = 0$ *if* k *is odd and* $\epsilon = 1$ *if* k *is even.*

A slightly weaker statement results by always taking $\epsilon = 1$. This will suffice for our results, which are independent of parity of k.

LEMMA 4.5. *Let* $X = \bigvee_{\alpha \in \mathcal{A}} X_\alpha$ *be an arbitrary wedge of co-H-spaces. Then* X *is a co-H-space and the degree* k *map on* X *is homotopic to the wedge of the degree* k *maps on each summand.*

The proof is trivial. In particular, this holds for suspensions and suspension spectra. Now let i_α be the inclusion $X_\alpha \hookrightarrow X$ and let π_α be the pinch map $X \to X_\alpha$.

LEMMA 4.6. *With the above notation, if* $[k] \circ i_\alpha$ *is null homotopic in* X, *then it is null-homotopic in* X_α. *That is,* X_α *has suspension order dividing* k.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
X_\alpha \arrow[r, "i_\alpha"] \arrow[d, "{[k]}"'] & X \arrow[r, "\pi_\alpha"] \arrow[d, "{[k]}"'] & X_\alpha \arrow[d, "{[k]}"'] \\
X_\alpha \arrow[r, "i_\alpha"] & X \arrow[r, "\pi_\alpha"] & X_\alpha
\end{tikzcd}$$

By hypothesis, each composite in the left square is null-homotopic. Thus k times the bottom row is null-homotopic. But the bottom row is the identity on X_α. $\square$

We are now ready to prove the main results of this section.

Proof of 4.1. By induction on m. Consider the following diagram, which is obtained by stabilizing the factorization of 2 in 3.5.

$$\begin{tikzcd} D_{2^m}(\Omega^4 S^{4n+1}) \arrow[r, "i_{2^m}"] & \bigvee_{j\geqslant 2} D_j(\Omega^4 S^{4n+1}) \arrow[r, "{[2]+d_2}"] \arrow[d, "f"'] & \bigvee_{j\geqslant 2} D_j(\Omega^4 S^{4n+1}) \\ & \bigvee_{j\geqslant 2} D_j(\Omega^2 S^{4n-1}) \arrow[r, equal] & \bigvee_{j\geqslant 2} D_j(\Omega^2 S^{4n-1}) \arrow[u] \end{tikzcd}$$

By 3.7, the map $f \circ i_{2^m}$ has its image contained in

$$\bigvee_{2\leqslant j<2^{m+1}} D_j(\Omega^2 S^{4n-1}).$$

Thus it has order dividing the stable order of $D_{2^m}\Omega^2 S^{4n-1}$, which is at most 2^{m+1} by 2.7. Then we have

$$2^{m+1}([2] + d_2 \circ i_{2^m}) = [2^{m+2}] + (2^{m+1}d_2) \circ i_{2^m} \simeq * \, .$$

But d_2 factors through $\bigvee_{i+j<2^{m+1}} D_i(\Omega^4 S^{4n+1}) \wedge D_j(\Omega^4 S^{4n+1})$, so it has order dividing the stable order of $D_{2^{m-1}}(\Omega^4 S^{4n+1}) \wedge D_{2^{m-1}}(\Omega^4 S^{4n+1})$, which is at most 2^{m+1} by induction. Thus $[2^{m+2}] \simeq *$ on $D_{2^m}(\Omega^4 S^{4n+1})$. $\square$

Proof of 4.2. We will first show that (1) implies (2), and then prove (1) by a double induction on m and k. Suppose that (1) holds for all m and k. Consider the following diagram:

$$\begin{tikzcd} D_{2^m}(\Omega^{4k+4} S^{4n+1}) \arrow[r, "i_{2^m}"] & \bigvee_{j\geqslant 2} D_j(\Omega^{4k+4} S^{4n+1}) \arrow[r, "{[2]+d_2}"] \arrow[d, "f"'] & \bigvee_{j\geqslant 2} D_j(\Omega^{4k+4} S^{4n+1}) \\ & \bigvee_{j\geqslant 2} D_j(\Omega^{4k+2} S^{4n-1}) \arrow[r, equal] & \bigvee_{j\geqslant 2} D_j(\Omega^{4k+2} S^{4n-1}). \arrow[u] \end{tikzcd}$$

For $m = 1$, we have that the stable order of $D_2(\Omega^{4k+4} S^{4n+1})$ divides 2^{3k+2} by Toda's result. By (1) and 3.10, $f \circ i_{2^m}$ has order dividing

$2^{2m+3k-1}$. This gives $2^{2m+3k-1}([2]+d_2\circ i_{2^m})\simeq *$. But d_2 has order dividing $2^{2m+3k-2}$, by 3.10 and induction on m. Therefore the term with d_2 vanishes and $2^{2m+3k}\simeq *$ on $D_{2^m}(\Omega^{4k+4}S^{4n+1})$.

We now turn to the proof of (1). For $k=1$, triple-loop the factorization of 4 in 3.4 and stabilize:

$$\begin{tikzcd} D_{2^m}(\Omega^6 S^{4n+3}) \arrow[r, "i_{2^m}"] & \bigvee_{j\geqslant 2} D_j(\Omega^6 S^{4n+3}) \arrow[r, "{[4]+d_4}"] \arrow[d, "f"'] & \bigvee_{j\geqslant 2} D_j(\Omega^6 S^{4n+3}) \\ & \bigvee_{j\geqslant 2} D_j(\Omega^4 S^{4n+1}) \arrow[r, equal] & \bigvee_{j\geqslant 2} D_j(\Omega^4 S^{4n+1}). \arrow[u] \end{tikzcd}$$

By 4.1 and 3.10, the map $f\circ i_{2^m}$ has order dividing 2^{m+2}. This gives $2^{m+1}([4]+d_4\circ i_{2^m})\simeq *$. But d_4 has order dividing 2^{2m} by 3.10 and induction on m. Therefore, if we multiply the last equation by 2^{m-1}, then the term with d_4 vanishes and $[2^{2m+2}]\simeq *$ on $D_{2^m}(\Omega^6 S^{4n+3})$.

To prove (1) for all $m\geqslant 1$ and $k\geqslant 1$, first note that (1) is true for $m=1$ and $k\geqslant 1$ by 4.4. We will proceed by using the following form of double induction: Let $P(k,m)$ represent the statement that (1) is true for k and m. We will show that $P(\leqslant k,\leqslant m-1)$ and $P(\leqslant k-1,m)$ implies $P(k,m)$.

$$\begin{tikzcd} D_{2^m}(\Omega^{4k+2} S^{4n+3}) \arrow[r, "i_{2^m}"] & \bigvee_{j\geqslant 2} D_j(\Omega^{4k+2} S^{4n+3}) \arrow[r, "{[4]+d_4}"] \arrow[d, "f"'] & \bigvee_{j\geqslant 2} D_j(\Omega^{4k+2} S^{4n+3}) \\ & \bigvee_{j\geqslant 2} D_j(\Omega^{4k} S^{4n+1}) \arrow[r, equal] & \bigvee_{j\geqslant 2} D_j(\Omega^{4k} S^{4n+1}). \arrow[u] \end{tikzcd}$$

Since (1) is assumed true for $k-1$ and m and (1) implies (2), the stable orders are known for $j\leqslant 2^m$ in the lower left corner. Thus $f\circ i_{2^m}$ has order dividing $2^{2m+3k-3}$. Now $d_4\circ i_{2^m}$ has order dividing $2^{2m+3k-3}$, by induction. Therefore $[2^{2m+3k-1}]\simeq *$ on $D_{2^m}(\Omega^{4k+2}S^{4n+3})$. To prove (3) and (4), proceed exactly as for (1) and (2), only start with the diagram of 3.4 and the result of 2.7 to verify (3) for $k=1$. This completes the proof. □

We now state and prove the analogous results for odd primes. Because of the factorization of p from 3.6 for all odd spheres, it is not necessary to distinguish cases as before and we have one simpler result.

THEOREM 4.7. *For $p > 2$ and $m \geqslant 1$, the stable order of $D_{p^m}(\Omega^{2k}S^{2n+1})$ divides p^{m+k-1}.*

As before, this suffices to compute the stable order of all $D_j(\Omega^{2k}S^{2n+1})$.

For the case $m = 1$, we compute upper bounds for the stable orders of the p-adic pieces directly rather than proceed by analogy with Proposition 4.3.

PROPOSITION 4.8. *The stable order of $D_p(\Omega^{2k}S^{2n+1})$ divides p^k.*

Proof. For $k = 1$, $D_p(\Omega^2 S^{2n+1})$ is a Moore space which has stable order p by 2.6. Now stabilize the factorization of p from 3.6. It then suffices to show that d_p is null-homotopic when restricted to $D_p(\Omega^{2k}S^{2n+1})$. But d_p factors through $\bigvee_{i+j<2p} D_i(\Omega^{2k}S^{2n+1}) \wedge D_j(\Omega^{2k}S^{2n+1})$ and this is a contractible space, by 1.2. □

The proof of 4.7 now proceeds by induction on m and k, as in the proof of 4.2, beginning with the factorization of p on $\Omega^2 S^{2n+1}$ from 3.6.

Finally, we show that the restriction $pk \leqslant n$ for the spaces $\Omega^{2k}S^{2n+1}$ may be weakened.

PROPOSITION 4.9. *If $2k < 2n + 1$, then the results of 4.1, 4.2, and 4.7 hold for $D_j(\Omega^{2k}S^{2n+1})$.*

Proof. By 1.6, $D_j(\Omega^{2k}S^{2n+1})$ has the same stable homotopy type as $D_j(\Omega^{2k}S^{2n+1+\phi_{2k,j}})$, where $\phi_{2k,j}$ is a multiple of the stable order of the bundle $\xi_{2k,j}$. Now choose a multiple r of $\phi_{2k,j}$ such that $pk \leqslant \left(n + \frac{r\phi_{2k,j}}{2}\right)$, so that $\Omega^{2k}S^{2n+1+r\phi_{2k,j}}$ satisfies our previous condition. □

5. Proof of proposition 1.3. We now give the proof of 1.3. In fact, we will prove something slightly stronger. To simplify the statement and proof slightly we will assume that $p = 2$. The case $p > 2$ is analogous.

Recall that for a space X, $QX = \Omega^\infty\Sigma^\infty X = \varinjlim \Omega^n\Sigma^n X$. For X path-connected, May's model (with $n = \infty$) is equivalent to QX, and the quotients $D_j(QX)$ may be defined as in the case $n < \infty$. Moreover, there is a canonical map $D_j(\Omega^n\Sigma^n X) \xrightarrow{\theta} D_j(QX)$ induced by the inclusion $\mathcal{C}_n(j) \hookrightarrow \mathcal{C}_\infty(j) = \varinjlim \mathcal{C}_n(j)$.

PROPOSITION 5.1. *Let* $2r = 2^{i_1} + ... + 2^{i_l}$ *with* $0 < i_1 \leqslant ... \leqslant i_l$. *Then* $D_{2r}(QX)$ *is a stable wedge summand of* $D_{2^{i_1}}(QX) \wedge ... \wedge D_{2^{i_l}}(QX)$.

It clearly suffices to show that $D_{2j}(QX)$ is a stable wedge summand of $D_{2^{i_1}}(QX) \wedge D_{2r-2^{i_1}}(QX)$.

COROLLARY 5.2. *If* θ_* *is injective in mod-2 homology, then the same statement holds with* $\Omega^n\Sigma^n X$ *replacing* QX.

In particular, if $X = S^{2m+1}$ then this condition holds since $H_*(QX; Z_2) \cong Z_2[Q_I x_{2m+1}]$ where I is an admissable and the operations Q_i are defined for $i \geqslant 0$. See [**CLM**] or [**Co**] for further details.

Proof. To begin the proof, we recall the stable homotopy version of the transfer. Let G be a finite group and H a subgroup of index N, and suppose that G (and hence H) acts freely on a space Y. Then there is an N-sheeted covering $Y/H \xrightarrow{\pi} Y/G$. In [**KP**], Kahn and Priddy show that there exists a stable map $Y/G \xrightarrow{\tau} Y/H$ such that $(\pi \circ \tau)_*$ is multiplication by N. Thus the classical transfer for covering spaces may be realized by a (stable) map of spaces. See [**A**] for a thorough treatment of the transfer. □

Let $F(\mathbf{R}^n, j)$ be the space of j distinct ordered points in $\mathbf{R}^n$. It is shown in [**M**] that $F(\mathbf{R}^n, j)$ and $\mathcal{C}_n(j)$ have the same homotopy type, and that the connectivity of $F(\mathbf{R}^n, j)$ increases with n. Thus $F(\mathbf{R}^\infty, j) = \varinjlim F(\mathbf{R}^n, j)$ is a contractible space. Furthermore, the spaces $F(\mathbf{R}^n, j)$ may be used in May's model for $\Omega^n\Sigma^n X$, and then the defintions of section 2 show that $D_j(\Omega^n\Sigma^n X)$ is a quotient space of $F(\mathbf{R}^n, j) \times_{\Sigma_j} X^j$ and similarly for $D_j(QX)$. Now there is a covering space of degree $\binom{j+k}{j}$

$$F(\mathbf{R}^n, j+k) \times_{\Sigma_j \times \Sigma_k} X^{j+k} \to F(\mathbf{R}^n, j+k) \times_{\Sigma_{j+k}} X^{j+k}$$

induced by the inclusion $\Sigma_j \times \Sigma_k \hookrightarrow \Sigma_{j+k}$, and similarly for $n = \infty$. Consider the following commutative diagram (of stable maps):

$$\begin{array}{ccc}
F(\mathbf{R}^n, j+k) \times_{\Sigma_{j+k}} X^{j+k} & \longrightarrow & F(\mathbf{R}^\infty, j+k) \times_{\Sigma_{j+k}} X^{j+k} \\
{\scriptstyle \tau}\big\downarrow & & {\scriptstyle \tau}\big\downarrow \\
F(\mathbf{R}^n, j+k) \times_{\Sigma_j \times \Sigma_k} X^{j+k} & \longrightarrow & F(\mathbf{R}^\infty, j+k) \times_{\Sigma_j \times \Sigma_k} X^{j+k} \\
{\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\
F(\mathbf{R}^n, j) \times_{\Sigma_j} X^j & & F(\mathbf{R}^\infty, j) \times_{\Sigma_j} X^j \\
\times F(\mathbf{R}^n, k) \times_{\Sigma_k} X^k & \longrightarrow & \times F(\mathbf{R}^\infty, k) \times_{\Sigma_k} X^k \\
{\scriptstyle \mu}\big\downarrow & & {\scriptstyle \mu}\big\downarrow \\
F(\mathbf{R}^n, j+k) \times_{\Sigma_{j+k}} X^{j+k} & \longrightarrow & F(\mathbf{R}^\infty, j+k) \times_{\Sigma_{j+k}} X^{j+k}.
\end{array}$$

Here the second vertical maps are given by projections and the third vertical maps are given by the multiplication in the May filtration. The diagram commutes by naturality of the constructions under the inclusion $\mathbf{R}^n \hookrightarrow \mathbf{R}^\infty$. Notice that the second vertical arrow in the right column is a homotopy equivalence since the spaces $F(\mathbf{R}^\infty, j)$ are contractible. It follows that in homology, the right column is multiplication by $\binom{j+k}{j}$.

Now consider the following commutative diagram which is obtained from the above diagram by passing to quotients.

$$\begin{array}{ccccc}
D_{j+k}(\Omega^n\Sigma^n X) & \xrightarrow{\tau} & D_j(\Omega^n\Sigma^n X) \wedge D_k(\Omega^n\Sigma^n X) & \xrightarrow{\mu\circ\pi} & D_{j+k}(\Omega^n\Sigma^n X) \\
{\scriptstyle \theta}\big\downarrow & & {\scriptstyle \theta\wedge\theta}\big\downarrow & & {\scriptstyle \theta}\big\downarrow \\
D_{j+k}(QX) & \xrightarrow{\tau} & D_j(QX) \wedge D_k(QX) & \xrightarrow{\mu\circ\pi} & D_{j+k}(QX).
\end{array}$$

It is not hard to show that the quotient map $F(\mathbf{R}^\infty, j+k) \times_{\Sigma_{j+k}} X^{j+k} \to D_{j+k}(QX)$ is onto in homology. Thus in homology the bottom row of this diagram is multiplication by $\binom{j+k}{j}$. Now if $j+k = 2r$ and $j = 2^{i_1}$ then this index is odd and $\mu \circ \pi \circ \tau$ is a 2-local equivalence. This proves Proposition 5.1. To prove Corollary 5.2, note that if θ_* is injective, then the top row is injective in homology, hence an isomorphism since the domain and range are equal vector spaces of finite type.

REMARK. For $p = 2$ and X an even sphere, the Corollary holds since in this case θ_* is injective. But for $p > 2$ and X an even sphere,

θ_* fails to be injective due to the presence of nontrivial Browder operations which become 0 in $H_*(QS^{2m}; Z_p)$. For $X = S^{2m+1}$ and $p > 2$, the proof goes through without change since the index is a unit mod p and θ_* is injective. We are indebted to Fred Cohen for discussions on this point and for his specific suggestions for the previous proof.

Finally, we would also like to point out that a slightly easier proof of 1.3 is possible for $D_{pr}(\Omega^2 S^3)$. By stabilizing the diagonal $\Omega^2 S^3 \to \Omega^2 S^3 \times \Omega^2 S^3$ and pinching off onto a factor, it is easy to obtain maps $D_{pr} \to D_{p^{i_1}} \wedge D_{pr-p^{i_1}} \to D_{pr}$ where p^{i_1} is from the p-adic representation of pr in 1.3. Then since $\bar{H}^*(D_{pr})$ is a cyclic $\mathcal{A}$-module (for $\Omega^2 S^3$), it follows by checking on the bottom class that the above map is an isomorphism in mod-p homology. This is carried out in **[CG]** for odd primes and the proof given there easily modifies for the prime 2. We would like to acknowledge this paper as being most helpful in understanding these ideas and for motivating Propositions 1.3 and 5.1.

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UNIVERSITY OF ROCHESTER AND WABASH COLLEGE
Current address: QUEENS COLLEGE
FLUSHING, NY 11367
E-mail address:pslq@qcunix.acc.qc.edu