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ON THE COMPACTNESS OF A CLASS OF RIEMANNIAN MANIFOLDS

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A class of Riemannian manifolds is studied in this paper. The main conditions are 1) the injectivity is bounded away from 0; 2) a norm of the Riemannian curvature is bounded; 3) volume is bounded above; 4) the Ricci curvature is bounded above by a constant divided by square of the distance from a point. Note the last condition is scaling invariant. It is shown that there exists a sequence of such manifolds whose metric converges to a continuous metric on a manifold.

Introduction. Let $\mathcal{L} = \mathcal{L}(H, K, V, n, i_0)$ be the set of n-dimensional Riemannian manifolds (M, g), *s.t.*,

(0.1) M is diffeomorphic to (B_2, g_0), the standard Euclidean ball of radius 2, center $= 0$;
(0.2) (M, g) has C^∞ curvature tensor in M;
(0.3) for any $x \in M$, the Ricci curvature at x $|Ric(g)(x)| \leq Hr^{-2}$, where $r = dist(x, 0)$;
(0.4) the injectivity of $(M, g) \geq i_0 > 0$;
(0.5) $\int_M |Rm(g)|^{\frac{n}{2}} dg < K$;
(0.6) volume of $(M, g) \leq V$.

In the case when the condition (0.3) is replaced by $|Ric(g)| \leq H$, and (0.6) is replaced by a diameter bound, a compactness property is proved by the first author in a more general setting. The purpose of this paper is to extend some of his results to the present situation where the bound om Ricci curvature of (M, g) blows up like r^{-2} at a point. As an application, we will discuss the compactness of orbifolds with a finite number of singularities.

The main result is:

THEOREM 0.7. *Let* $(M_k, g_k) \in \mathcal{L}$, $k = 1, 2, 3, \ldots$. *Then there exists a subsequence (again denoted by* (M_k, g_k)*), a* C^∞ *manifold* M' *diffeomorphic to* $B_2(0)$*, and a* C^0 *metric* g' *on* M' s.t. $g_k \to g'$ *in* C^0*-norm on* M' *and the convergence is in* $C^{1,\alpha}$*-norm away from* 0.

In Section 1 we study the geodesic balls centered at 0. A compactness estimate of the metric g will be derived. In Section 2, a small geodesic sphere is shown to have a small diameter. In Section 3, some $L^{n/2}$-curvature pinching results are derived, which will be used in Section 4 to show the existence of harmonic coordinates. We will prove in Section 4 the above main result and a slightly different version.

In the definition of $\mathcal{L}$, if (0.3) is replaced by a 1-sided condition

$$Ric(g) \geq -Hr^{-2}g, \tag{0.3$'$}$$

then the above compactness result should be modified as follows. Denote the set of such Riemannian manifolds by $\mathcal{L}'$.

THEOREM 0.8. *Let* $(M_k, g_k) \in \mathcal{L}'$, $k = 1, 2, 3, \ldots$. *Then there exists a subsequence of* (M_k, g_k)*, which converges in* C°*-norm to a* C^∞ *manifold* M' *with a* C° *metric* g'.

1. In this section, we assume that for some $H > 0$, $i_0 > 0$, (M, g) is a Riemannian manifold diffeomorphic to B_2 satisfying

$$Ric(g) \geq -Hr^{-2}g; \tag{1.1}$$

$$inj(g) \geq i_0 > 0. \tag{1.2}$$

Let $B_\rho(0) = \{x \in M | d(0, x) \leq\}$ be the geodesic ball of M centered at 0. Consider a geodesic polar coordinate system $\{r, x^1, \cdots, x^{n-1}\}$ on $B_\rho(0)$, we have

$$ds(g)^2 = dr^2 + \sum_{i=1}^{n-1} g_{ij}(r, x)dx^i dx^j; \tag{1.3}$$

$$R_{irrj} = -\frac{1}{2}\frac{\partial^2}{\partial r^2} g_{ij}(r,x) + \frac{1}{4}\sum g^{kl}\frac{\partial}{\partial r}g_{ik}\frac{\partial}{\partial r}g_{jl}. \tag{1.4}$$

For the Ricci curvature in the radial direction, we have

$$R_{rr} = -\frac{\partial^2}{\partial r^2}\ln\sqrt{g(r)} - \frac{1}{4}\left|\frac{\partial}{\partial r}g(r)\right|^2_{g(r)}, \tag{1.5}$$

where $g(r) = g(r,x)$,

$$\sqrt{g}\, dV_0 = \sqrt{\det(g_(ij)}\, dx^1 \wedge \ldots \wedge\, dx^{n-1}, \tag{1.6}$$

($dV_0 =$ the volume element of the standard Euclidean sphere) and

$$\left|\frac{\partial g}{\partial r}\right|^2_g = \sum g^{ij}g^{kl}\frac{\partial}{\partial r}g_{ij}\frac{\partial}{\partial r}g_{kl}.$$

We start out with the following estimate:

PROPOSITION 1.7. *For $\rho \le \frac{i_0}{2}$, there exists $C_1 = C_1(H,n) > 0$ s.t.* $\int_0^\rho r^2\left|\frac{\partial}{\partial r}g\right|^2 dr \le C_1\rho.$

Proof. The function is essentially the same as that given in [**12**], p.5-6. For any piecewise C^∞ function ϕ of r with $\phi(\rho) = 0$, we have

$$\begin{aligned}\left(\frac{1}{4} - \epsilon\right)\int_0^\phi r^2\phi^2\left|\frac{\partial}{\partial r}g\right|^2 dr \\ \le \frac{n-1}{2\epsilon}\int_0^\rho (r^2\phi'^2 + \phi^2)dr - \int_0^\rho r^2\phi^2 R_n dr.\end{aligned} \tag{1.8}$$

Take $\epsilon = \frac{1}{8}$, $\phi = \rho - r$, and use $-R_{rr} \le Hr^{-2}$, we get

$$\begin{aligned}&\int_0^\phi r^2(\phi - r)^2\left|\frac{\partial}{\partial r}g\right|^2 dr \\ &\le 32(n-1)\int_0^\phi (r^2 + (\phi - r)^2)\, dr + H\int_0^\phi (r^2(\phi - r)^2)r^{-2}\, dr \\ &\le C(H,n)\rho^3.\end{aligned}$$

Thus,

$$\int_0^{\frac{\phi}{2}} r^2 \left|\frac{\partial}{\partial r} g\right|^2 dr \leq \frac{1}{\left(\frac{\rho}{2}\right)^2} \int_0^{\phi} r^2(\phi - r)^2 \left|\frac{\partial}{\partial r} g\right|^2 dr \leq \frac{1}{2} C_1(H, n)\rho.$$

□

PROPOSITION 1.9. *There exists* $C_2 = C_2(H, i_0, n) > 0$ s.t. *for any* $r \in \left(0, \frac{i_0}{2}\right)$, *we have*

$$r\left|\frac{\partial}{\partial r} \ln \sqrt{g}\right| \leq C_2.$$

Proof. From (1.5) and integration by parts,

$$\int_0^{\phi} r^2 R_n \, dr = -\frac{1}{2} r^2 \frac{\partial}{\partial r} \ln g + \frac{1}{2} \int_0^{\phi} 2r \frac{\partial}{\partial r} \ln g - \frac{1}{2} \int_0^{\phi} r^2 \left|\frac{\partial}{\partial r} g\right|^2 dr.$$

Thus

$$\frac{1}{2} r^2 \frac{\partial}{\partial r} \ln \sqrt{g} \leq H \int_0^{\phi} r^{-2} r^2 \, dr + \frac{1}{4} C_1 r + \left(\int_0^{\phi} r^2 \left|\frac{\partial}{\partial r} \ln g\right|^2\right)^{\frac{1}{2}} r^{\frac{1}{2}}$$

$$\leq \tfrac{1}{3} H r + \tfrac{1}{4} C_1 r + (n-1)^{\frac{1}{2}} \left(\int_0^{\phi} r^2 \left|\frac{\partial}{\partial r} g\right|^2 dr\right)^{\frac{1}{2}} r^{\frac{1}{2}}$$

$$\leq C_2(H, i_0, n) r.$$

□

Next we study the induced metric $g(r) = \sum g_{ij}(r, x)\, dx^i\, dx^j$ on the geodesic sphere

$$S_r(0) = \{x \in M : d(x, 0) = r\}, \quad r \leq \frac{i_0}{2}.$$

PROPOSITION 1.10. *There exists* $C_3 = C_3(H, n) > 0$ s.t. *for* $0 < r_1 < r_2 \leq \frac{i_0}{2}$, *we have*

$$e^{C_3 r_2 r_1^{-1}} g(r_1) \leq g(r_2) \leq e^{C_3 r_2 r_1^{-1}} g(r_1).$$

Proof. From Proposition 1.7, we have, for any vector $\nu = (\nu^1, \ldots, \nu^n) \in TS_1$,

$$\left|\ln \frac{h(r_2)}{h(r_1)}\right| \leq \int_{r_1}^{r_2} \left|\frac{\partial}{\partial r} \ln h(r)\right| dr \leq \left(\int_{r_1}^{r_2} \left|\frac{\partial}{\partial r} g\right| r dr\right) r_1^{-1}$$
$$\leq \sqrt{r_2}(C_1 r_2)^{\frac{1}{2}} r_1^{-1} = \sqrt{C_1} \frac{r_2}{r_1},$$

where $h(r) = g_{ij}(r)\, d\nu^i d\nu^j$. Hence $e^{C_3 r_2 r_1^{-1}} \leq \frac{h(r_2)}{h(r_1)} \leq e^{C_3 r_2 r_1^{-1}}$, where $c_3 = \sqrt{c_1}$. □

Before we go any further, let us make some remarks regarding conditions (0.3) and (0.5). Let $\tau > 0$ be small. Define a new metric g^τ on M by $g^\tau(x) = \tau^{-2} g(\tau x)$.

REMARK.

(1.11) If g satisfyes $(0.3)'$, so does g^τ.

$$\int_{B_1} |R(g^\tau)|^{\frac{n}{2}}\, dg^\tau = \int_{B_\tau} |R(g)|^{\frac{n}{2}}\, dg. \tag{1.12}$$

Therefore, by a scaling of this type if necessary, we can assume that g satisfies (0.3) and (0.5) with $K \ll 1$.

Once we have Proposition 1.10 we can control the $L^{n/2}$ norm of the Riemannian curvature tensor $Rm(r)$ of $g(r)$, the induced metric on $S(0,r)$.

THEOREM 1.13. *If $(M,g) \in \mathcal{L}'$ then for any $\rho \leq \frac{i_0}{4}$, there exist $r_\rho \in (\frac{\rho}{2}, \rho)$, $C_4 = C_4(H, K, i_0, n) > 0$,* s.t.

$$\int_{S(0,r_\rho)} |Rm(r_\rho)|_{g(r_\rho)}^{\frac{n}{2}}\, dg(r_\rho) \leq C_4 r_\rho^{-1}. \tag{1.15}$$

Proof. By Lemma 1.17 in [**12**],$\exists C_5 = C_5(H, i_0, n)$ *s.t.* for $\rho < \frac{i_0}{4}$,

$$\int_{\frac{\rho}{2}}^{\rho} \left|\frac{\partial}{\partial r} g\right|^n dr \leq C_5 \left(\frac{1}{\rho^n} + \int_{\frac{\rho}{2}}^{\rho} |Rm(g)|^{\frac{n}{2}}\, dr\right).$$

From Proposition 1.10, there exists $C = C(H, i_0, n)$ *s.t.*

$$C^{-1}\sqrt{g}(\rho) \leq \sqrt{g}(r) \leq C_3\sqrt{g}(\rho)$$

for $r \in (\frac{\rho}{2}, \rho)$, i.e., $\sqrt{g}(r)$ is equivalent to $\sqrt{g}(\rho)$. Thus for some constant $C_6 = C_6(H, i_0, n) > 0$, we have

$$\int_{\frac{\rho}{2}}^{\rho} \left|\frac{\partial}{\partial r} g\right|^n \sqrt{g}(r)\, dr \leq C_6 \left(\rho^{-n}\sqrt{g}(\rho) + \int_{\frac{\rho}{2}}^{\rho} |Rm(g)|^{\frac{n}{2}} \sqrt{g}(r)\, dr\right).$$

Integrating over $S_\rho(0)$, we get

$$\int_{B_\rho \setminus B_{\frac{\rho}{2}}} \left|\frac{\partial}{\partial r} g\right|^n dg \leq C_6 \rho^{-n} \int_{S_\rho} dg(\rho) + C_6 \int_{B_\rho} |Rm(g)|^{\frac{n}{2}} dg.$$

Taking $\rho = \frac{i_0}{4}$, we get

$$\int_{B_{\frac{i_0}{4}} \setminus B_{\frac{i_0}{8}}} \left|\frac{\partial}{\partial r} g\right|^n dg \leq C_6 \left(\frac{i_0}{4}\right)^{-n} vol\left(S_{\frac{i_0}{4}}\right) + C_6 \int_{B_{\frac{i_0}{4}}} |Rm(g)|^{\frac{n}{2}} dg.$$

By Bishop's volume estimate [**1**], $\exists C_7 = C_7(H, i_0, n)$ *s.t.* $vol\left(S_{\frac{i_0}{4}})\right) \leq C_7$. Thus we get a constant $C_8 = C_8(H, i_0, n) > 0$ *s.t.*

$$\int_{B_{\frac{i_0}{4}} \setminus B_{\frac{i_0}{8}}} \left|\frac{\partial}{\partial r} g\right|^n dg \leq C_8 + C_8 \int_{B_{\frac{i_0}{4}}} |Rm(g)|^{\frac{n}{2}} dg. \tag{1.16}$$

Define $g^\tau = r^{-2} g$ with $r = \frac{4\rho}{i_0}$. Noticing that $Ric(g^\tau) \geq -Hr^{-2}$, $inj(g^\tau) \geq i_0$, we can apply (1.16) to g^τ. By the scaling invariance of (1.16), we get

$$\begin{aligned}
\int_{B_\rho \setminus B_{\frac{\rho}{2}}} \left|\frac{\partial}{\partial r} g\right|^n dg &= \int_{B_{\frac{i_0}{4}} \setminus B_{\frac{i_0}{8}}} \left|\frac{\partial}{\partial r} g^\tau\right|^n dg^\tau \\
&\leq C_8 + C_8 \int_{B_{\frac{i_0}{4}}} |Rm(g^\tau)|^{\frac{n}{2}} dg^\tau \\
&= C_8 + C_8 \int_{B_\rho} |Rm(g^\tau)|^{\frac{n}{2}} dg \\
&\leq C_8 + C_8 K = C_9.
\end{aligned}$$

Hence

$$\int_{\frac{\rho}{2}}^{\rho} \left(\int_{S_r} \left| \frac{\partial}{\partial r} g \right|^n dg(r) \right) dr \le C_9. \tag{1.17}$$

(1.17) and the Gauss formula on S,

$$Rm(g)_{ijkl} = Rm(g(r))_{ijkl} + \frac{1}{4} \left(\frac{\partial}{\partial r} g_{ik} \frac{\partial}{\partial r} g_{jl} - \frac{\partial}{\partial r} g_{jk} \frac{\partial}{\partial r} g_{il} \right)$$

imply that there exists a constant $C = C(H, K, i_0, n) > 0$ *s.t.*

$$\begin{aligned} &\int_{\frac{\rho}{2}}^{\rho} \left(\int_{S_r} |Rm(g(r))|^{\frac{n}{2}} dg(r) \right) dr \\ &\quad \le C + C \int_{\frac{\rho}{2}}^{\rho} \left(\int_{S_r} |Rm(g)|^{\frac{n}{2}} dg(r) \right) dr \\ &\quad \le C + CK. \end{aligned}$$

This implies the existence of $r_\rho \in \left[\frac{\rho}{2}, \rho\right]$ and $C_4 = C_4(H, K, i_0, n) > 0$ *s.t.*

$$\int_{S_{r_\rho}} |Rm(r_\rho)|^{\frac{n}{2}} dg(r_\rho) \le C_4 r_\rho^{-1}.$$

□

We now state and prove the compactness estimate of the induced metric on small geodesic spheres.

Let $(M, g) \in \mathcal{L}'$, $\rho \le \frac{i_0}{4}$, let $r_\rho \in \left[\frac{\rho}{2}, \rho\right]$ as in Theorem 1.13. We have the following

THEOREM 1.18. *There exists $C_{10} = c_{10}(H, K, i_0, n) > 0$ and a C^∞ Riemannian metric $h(r_\rho)$ on the geodesic sphere S_{r_ρ}* s.t.

$$C_{10}^{-1} g(r_\rho) \le {r_\rho}^2 h(r_\rho) \le C_{10} g(r_\rho); \tag{1.19}$$

$$|Rm(h(r_\rho))| \le C_{10}. \tag{1.20}$$

Proof. Proposition 1.10 and Theorem 1.13 are sufficient for carrying through the argument in [**12**]. □

2. In this section, we show that the diameter of a small geodesic sphere is small. More precisely,

THEOREM 2.1. *There exists* $C_{11} = C_{11}(H, K, i_0, V, n)$ s.t. *for any* $(M, g) \in \mathcal{L}'$, *any* $r \in \left(0, \frac{i_0}{2}\right)$, $diam(g(r)) \leq C_{11} r$.

Proof. First observe that there exists a constant $C = C(H, K, i_0, V, n) > 0$ *s.t.*

$$diam\left(S_{\frac{i_0}{4}}\right) \leq C. \tag{2.2}$$

To prove (2.2), we normalize by scaling so that $i_0 = 4$. Let γ be a minimal geodesic on the geodesic sphere $S_1(0)$. We show that there exists $\tilde{C} = \tilde{C}(H, i_0, V)$ *s.t.*

$$length\, \gamma \leq \tilde{C}.$$

Let α be any curve in the annulus $B_{\frac{3}{2}}(0) \backslash B_{\frac{1}{2}}(0)$ *s.t.* for $0 \leq t_1 < t_2 < \cdots \leq 1$, $\alpha|[t_i, t_{i+1}]$ is a minimal geodesic in the annulus. The geodesic balls centered at $\gamma(t_i)$ with radius δ can be made mutually disjoint by choosing $\delta > 0$ sufficiently small. Let N be the number of these balls. By Gromov's relative volume estimate [**6**], the volume of each small bal is bounded from below by a constant $C' = C'(H, i_0, V, n)$. But the total volume of the mannifold M is bounded from above by V (cf. (0.6)). Hence $N \leq V/C'$. Since the induced metric $g(r_1)$ and $g(r_2)$ are equivalent (by Proposition 1.10), we can project $\alpha|[t_i, t_{i+1}]$ into $S_1(0)$, to get (2.2).

Next, apply (2.2) to the metric g^τ defined by $g^\tau(x) = \tau^{-2} g(\tau x)$. By scaling properties, we get

$$diam\,(g(r)) \leq C \frac{4r}{i_0}.$$

□

3. Let (M,g) be in $\mathcal{L}'$. As before we use the geodesic polar coordinates at 0, i.e.,

$$g = dr^2 + \sum_{i,j=1}^{n-1} g_{ij}(x,r)\,dx^i\,dx^j = dr^2 + g(r),$$

where $g(r) = g(x,r)$ is the induced metric on the geodesic sphere $S_r(0)$.

We will begin with the following estimate:

PROPOSITION 3.1. *For $\rho \leq \frac{i_0}{4}$, $\eta \in (0,\rho)$, we have*

$$\int_{T\left(\frac{\eta}{4},\frac{\eta}{2}\right)} \left(\max_{\eta\leq\rho} \int_{S(x,r)} \left| B(x,r) + \frac{1}{r} g(x,r) \right|^{\frac{n}{2}} dg(r) \right) dg(x)$$
$$\leq C(H,n,\eta,\rho) \int_{B(\rho+\eta)} |R_m(g)|^{\frac{n}{2}}\,dg,$$

where $B(x,r)$ is the second fundamental form of $S(x,r)$,

$$T\left(\frac{\eta}{4},\frac{\eta}{2}\right) = \left\{ x \in M | dist(x,0) \in \left(\frac{\eta}{4},\frac{\eta}{2}\right) \right\}.$$

Proof. Let $x \in T\left(\frac{\eta}{4},\frac{\eta}{2}\right)$, $y \in M$ *s.t.* $d(x,y) = \rho \leq \frac{i_0}{2}$. Let γ be the minimal geodesic from x to y with $\gamma(0) = x$, $\gamma(\rho) = y$, $d(x,y) = \rho$. Observe that, as a consequence of Proposition 1.10, there exists a constant $C_{12} = C_{12}(H, i_0, n) > 0$ *s.t.* for any Jacobi field X on γ with $X(\gamma(0)) = 0$, $< X(\gamma(l)), \gamma'(l) >= 0$, we have

$$|X(\gamma(t))| \leq C_{12}|X(\gamma(l))|$$

$\forall\, t \in [0,l]$, where $l =$ the length of γ.

Let E be the parallel vector field along γ with

$$E(\gamma(l)) = X(\gamma(l)),$$

then the vector field A, defined by $A = X - \frac{t}{l}E$, is again a Jacobi field. Assume $|X(\gamma(l))| = 1$. We have

$$\int_0^l |A'|^2 = \int_0^l < A'', A > \ dt \leq \int_0^l |Rm||X||A|\,dt$$
$$\leq C_{12}(C_{12}+1) \int_\gamma |Rm| = C_{13} \int_\gamma |Rm|,$$

where $C_{13} = C_{13}(H, i_0, n)$.

Next, by a cut-off function argument, one can show that (c.f. [**12**], p.31)

$$|A'|^2(\gamma(l)) \leq C_{14} \int_\gamma |Rm|^2. \tag{3.2}$$

We claim that there exists $C_{15} = C_{15}(H, K, i_0, n)$ *s.t.*

$$\left|B(x,r) + \frac{1}{l} g(\gamma(l))\right|^2 (\gamma(l)) \leq C_{15} \int_\gamma |Rm|^2.$$

To see this, let X, Y be vector fields on $S(x,l)$ *s.t.*

$$|X(\gamma(l))| = |Y(\gamma(l))| = 1,$$

and let $E, \overline{E}$ be parallel vector fields on γ with

$$E(\gamma(l)) = X(\gamma(l)),$$

$$\overline{E}(\gamma(l)) = Y(\gamma(l)).$$

Extended X, Y to the geodesic ball $B(x,l)$ *s.t.* they are Jacobi fields on each radial geodesic. Then, clearly $B(X,Y) = - < \nabla_{\gamma'}, X, Y >= - < X', Y >$. We have, from (3.2), that

$$\begin{aligned}
&|B(X,Y) + \frac{1}{l} < X, Y > |^2(\gamma(l)) \\
&= | < X', Y > - \frac{1}{l} < E, Y > |^2(\gamma(l)) \\
&= | < X' - \frac{1}{l} E, Y > |^2(\gamma(l)) \\
&\leq C_{14} |Y(\gamma(l))|^2 \int_\gamma |Rm|^2 = C_{14} \int_\gamma |Rm|^2.
\end{aligned}$$

To finish the proof, we define $f(x,y)$, for x, y with $d(x,y) = \rho + \frac{\eta}{2} \leq \frac{i_0}{2}$, by

$$f(x,y) = \max_{\eta \leq r \leq \rho} \left|B(x,r) + \frac{1}{r} g(x,r)\right|^{\frac{n}{2}} (\gamma(r)),$$

where γ is the minimal geodesic from x to y, $\quad r =$ distance from x.

Let

$$\Omega = \bigcup_{x \in T\left(\frac{\eta}{4}, \frac{\eta}{2}\right)} S\left(x, \rho + \frac{\eta}{2}\right) \subset M,$$

and

$$\Sigma = \bigcup_{x \in T\left(\frac{\eta}{4}, \frac{\eta}{2}\right)} \left(x, S\left(x, \rho + \frac{\eta}{2}\right)\right) \subset M \times M.$$

Then

$$\begin{aligned}\int_\Sigma \int f(x,y) &= \int_{x \in T\left(\frac{\eta}{4}, \frac{\eta}{2}\right)} \left(\int_{S\left(x, \rho + \frac{\eta}{2}\right)} f(x,y)\, dg_x(y) \right) dg(x) \\ &= \int_\Omega \left(\int_{\Omega_y} f(x,y)\, dg_y(x) \right) dg(y),\end{aligned}$$

where g_x is the induced metric of $S\left(x, \rho + \frac{\eta}{2}\right)$, and $\Omega_y = T\left(\frac{\eta}{4}, \frac{\eta}{2}\right) \cap S\left(y, \rho + \frac{\eta}{2}\right) \subset S\left(y, \rho + \frac{\eta}{2}\right)$. We have

$$\int_\Sigma \int f(x,y) \leq \int_\Omega \left(\int_{\Omega_y} f(x,y)\, dg_y(x) \right) dg(y).$$

Define $\overline{\gamma}(t) = \gamma(t)$ for $t \in [0, \rho]$. From (3.3) we get

$$\begin{aligned}&\int_{\Omega_y} f(x,y)\, dg_y(x) \\ &\leq C(H, \eta, \rho) \int_{\Omega_y} \left(\int_{\overline{\gamma}} |Rm(g)|^{\frac{n}{2}} \right) dg_y \\ &\leq C(H, \eta, \rho) \int_\delta^{\rho+\delta} \left(\int_{\Omega_y} |Rm(g)|^{\frac{n}{2}} \left(\gamma\left(\rho + \frac{\eta}{2} - t\right) \right) dg_y \right) dt.\end{aligned}$$

By Proposition 1.10,

$$dg_y\left(\gamma\left(\rho + \frac{\eta}{2} - t\right)\right) \geq C\left(H, n, \frac{\rho}{\eta}\right) dg_y(x).$$

Therefore

$$\int_{\Omega_y} f(x,y)\, dg_y(x) \leq C\left(H, n, \eta, \frac{\rho}{\eta}\right) \int_{B(\rho+\eta)} |Rm(g)|^{\frac{n}{2}} dg.$$

Finally we have

$$\int_{\Omega_y} f(x,y) \leq C\left(H,n,\eta,\frac{\rho}{\eta}\right) vol\left(T\left(\frac{\eta}{4},\rho+\eta\right)\right)\int_{B(\rho+\eta)} |Rm(g)|^{\frac{n}{2}}dg$$
$$\leq C\left(H,n,\eta,\frac{1}{\eta},\rho,V,i_0\right)\int_{B(\rho+\eta)} |Rm(g)|^{\frac{n}{2}}dg.$$

□

Let $\dot{R}m(r)$ be the scalar curvature free curvature tensor of $g(r)$. We have the following proposition.

Proposition 3.4. *For any $x \in T\left(\frac{\eta}{4},\frac{\eta}{2}\right)$, where $\eta \in (0,\rho)$ with $\rho \leq \frac{i_0}{4}$, we have*

$$\int_\eta^\rho \left(\int_{S(x,r)} |\dot{R}m(r)|^{\frac{n}{4}}\, dg_x(r)\right) dr$$
$$\leq C(H,n,\eta,\rho,i_0)\left(\left(\int_{B_x(\rho)} |Rm(g)|^{\frac{n}{2}}\, dg\right)^{\frac{1}{2}}\right.$$
$$+\left(\max_{\eta\leq\rho}\int_{S(x,r)}\left|A(r)+\frac{1}{r}g_x(r)\right|^{\frac{n}{2}}\, dg_x(r)\right)^{\frac{1}{2}}$$
$$\left.+\max_{\eta\leq\rho}\int_{S(x,r)}\left|A(r)+\frac{1}{r}g_x(r)\right|^{\frac{n}{2}}\, dg_x(r)\right).$$

Proof. $\dot{R}m(r)$ can be expressed as

$$(\dot{R}m(r))_{ijkl}$$
$$= (Rm(r))_{ijkl} - \frac{R(r)}{(n-1)(n-2)}(g_{ik}(r)g_{jl}(r) - g_{il}(r)g_{jk}(r)),$$

where $R(r)$ is the scalar curvature of $g(r)$. We have

$$\int_{S(x,r)}\left|B_{ik}(r)B_{jl}(r) - \frac{1}{r^2}g_{ik}(r)g_{jl}(r)\right|^{\frac{n}{4}}\, dg(r)$$

$$= \int_{S(x,r)} B_{ik}(r)\left(B_{jl}(r) + \frac{1}{r}g_{jl}(r)\right)$$
$$-\frac{1}{r}g_{jl}(r)\left(B_{ik}(r) + \frac{1}{r}g_{ik}(r)\right)^{\frac{n}{4}} dg(r)$$

$$\leq C\int_{S(x,r)} |B|^{\frac{n}{4}}\left|B(r) + \frac{1}{r}g(r)\right|^{\frac{n}{4}} dg(r)$$
$$+C\int_{S(x,r)}\left|B(r) + \frac{1}{r}g(r)\right|^{\frac{n}{4}} dg(r)$$

$$\leq C\left(\int_{S(x,r)}\left|B(r) + \frac{1}{r}g(r)\right|^{\frac{n}{2}} dg(r)\right)^{\frac{1}{2}}$$
$$+C\int_{S(x,r)}\left|B(r) + \frac{1}{r}g(r)\right|^{\frac{n}{2}} dg(r).$$

This implies that

$$\int_{S(x,r)}\left|(B_{ik}B_{jl} - B_{il}B_{jk}) - \frac{1}{r^2}(g_{ik}g_{jl} - g_{il}g_{jk})\right|^{\frac{n}{4}} dg(r)$$

$$\leq C(H,K,i_0,n)\left(\int_{S(x,r)}\left|B(r) + \tfrac{1}{r}g(r)\right|^{\frac{n}{2}} dg(r)\right)^{\frac{1}{2}}$$
$$+C(H,K,i_0,n)\int_{S(x,r)}\left|B(r) + \tfrac{1}{r}g(r)\right|^{\frac{n}{2}} dg(r).$$

By Gauss formula,

$$(Rm(g))_{ijkl} = (Rm(g(r)))_{ijkl} + B_{ik}(r)B_{jl}(r) - B_{il}(r)B_{jk}(r).$$

Therefore

$$\int_\eta^\rho\left(\int_{S(x,r)}\left|R_{ijkl}(g(r)) - \frac{1}{r^2}(g_{ik}(r)g_{jl}(r)\right.\right.$$
$$\left.\left.-g_{il}(r)g_{jk}(r))\right|^{fracn4} dg(r)\right) dr$$

$$\leq C(H,n,\eta,\rho)\left(\int_{B(x,\rho)}|Rm(g)|^{\frac{n}{2}} dg\right)^{\frac{1}{2}}$$
$$+C(H,n,\eta,\rho)\left(\max_{\eta\leq r\leq\rho}\int_{S(x,r)}\left|B(r) + \frac{1}{r}g(r)\right|^{\frac{n}{2}} dg(r)\right)^{\frac{1}{2}}$$
$$+C(H,n,\eta,\rho)\left(\max_{\eta\leq r\leq\rho}\int_{S(x,r)}\left|B(r) + \frac{1}{r}g(r)\right|^{\frac{n}{2}} dg(r)\right).$$

Observe that

$$\int_{T_x(\eta,\rho)} \left| R(r) - \frac{(n-1)(n-2)}{r^2} \right|^{\frac{n}{4}} dg$$
$$\leq C(H, K, i_0, n, \eta, \rho) \left(\int_{B(x,\rho)} |Rm(g)|^{\frac{n}{2}} dg \right)^{\frac{1}{2}}.$$

Hence (3.4) follows immediately. □

PROPOSITION 3.5. *For* $0 < \eta < \rho \leq \frac{i_0}{4}$, *let* $(M_k, g_k) \in \mathcal{L}'$, $x_k \in M_k$ *with* $dist(x_k, 0) \in \left(\frac{\eta}{4}, \frac{\eta}{2}\right)$. *Assume*

$$\eta_k = \max_{\eta \leq r \leq \rho} \int_{S(x,r)} \left| B(x_k, r) + \frac{1}{r} g_k(r) \right|^{\frac{n}{2}} dg_k(r) \to 0$$

and

$$\mu_k = \int_{B(x_k,\rho)} |Rm(g_k)|^{\frac{n}{2}} dg_k \to 0 \quad as \quad k \to \infty.$$

Then there exists a diffeomorphism $\phi_k : S(1) \to S(x_k, \rho)$ *for each* $k = 1, 2, 3, \cdots$, s.t.

$$\int_{S(1)} |\phi_k^* g_k(r) - r^2\, d\theta^2|^{\frac{n}{2}}\, d\theta \to 0$$

uniformly for $\eta \leq r \leq \rho$, *where* $S(1)$ *is the Euclidean unit sphere, and*

$$|\phi_k^* g_k(\rho) - \rho^2\, d\theta^2|_{C^0} \to 0 \quad as \quad k \to \infty.$$

Proof. Proposition 1.10 and Theorem 1.13 enable us to carry out the arguments in [**12**] (cf. 5.18, 5.21, and 5.25). □

4. In this section we prove the existence of a controllable harmonic coordinate system under the smallness condition of the $L^{n/2}$-norm of curvature tensor.

PROPOSITION 4.1. *For any* $\eta \in (0,1)$, *there exists* $\epsilon = \epsilon(H, n, i_0, \eta) > 0$ s.t. *if* $(M, g) \in \mathcal{L}$ *satisfies* $\int_M |Rm(g)|^{\frac{n}{2}} dg \leq$

ϵ, *then there exists a diffeomorphism*

$$F = (h^1, h^2, \cdots, h^n): \quad T\left(1+\frac{\eta}{2}, \frac{3\eta}{2}\right) \to T\left(1+\frac{\eta}{2}, \frac{3\eta}{2}\right) \subset \mathbb{R}^n$$

having the following properties:

(a) $\Delta = 0$;

(b) $F^{-1}\left(T\left(1+\frac{\eta}{4}, \frac{\eta}{4}+\eta\right)\right) \supset T(1-\eta, 2\eta)$ *and the image of* $F \supset T\left(1+\frac{\eta}{4}, \frac{5\eta}{4}\right)$;

(c) $|h^{ij} - \delta^{ij}|_{C^0} < \frac{\eta^2}{100n}$ *on* $T\left(1+\frac{\eta}{4}, \frac{5\eta}{4}\right)$; *where* $h^{ij} = < \nabla h^i, \nabla h^j >$;

(d) $|dh^{ij}|_{C^0} \leq C(H, n, \eta)$ *for some* $\alpha \in (0,1)$ *on* $T\left(1+\frac{\eta}{4}, \frac{5\eta}{4}\right)$;

(e) $||F|^2 - r^2| \leq \frac{\eta}{100n}$, *where* $|F|^2 = \sum_i (h^i)^2$, $r = dist(x, 0)$;

(f) $\|d^2 h^{ij}\|_{L^q} \leq C(H, n, \eta)$ *on* $T\left(1+\frac{\eta}{4}, \frac{5\eta}{4}\right)$ *for some* $q > n$.

Proof. Suppose for $k = 1, 2, \cdots$, $(M_g, g_k) \in \mathcal{L}$ with $\int_{M_k} |Rm(g_k)|^{\frac{n}{2}} \leq \frac{1}{k}$.

Proposition 3.1 implies that $\exists y_k \in T\left(\frac{\eta}{2}, \frac{\eta}{4}\right)$ *s.t.*

$$\begin{aligned}\eta_k &= \max_{\eta \leq r \leq 1} \int_{S_k(y_k, r)} \left| B_k(y_k, r) + \frac{1}{r} g_k(y_k, r) \right|^{\frac{n}{2}} dg_k(y_k, r) \\ &\leq C\left(H, n, i_0, \eta, \frac{1}{\eta}\right) \int_{B_2} |Rm(g_k)|^{\frac{n}{2}} dg_k \\ &\leq Ck^{-1}.\end{aligned}$$

Proposition 3.5 implies that there exists $\phi_k : S_1 \to S_k(y_k) \approx S_1$ *s.t.*

$$\int_{T(1,\eta)} |\phi_k^* g_k - g_0|^{\frac{n}{2}} dg_0 < Ck^{-1},$$

where ϕ_k has been extended trivially to $T(1, \eta)$, g_0 is the flat metric on B_1. In the Euclidean coordinates $x = (x^1, \cdots, x^n)$, $g_0 = \delta_{ij}$.

Next we solve the Dirichlet problem

$$\begin{cases} \Delta F = 0 & \text{in } T(1, \eta) \\ F = x & \text{on } \partial T(1, \eta). \end{cases}$$

By Proposition 1.10, we can show (as in [**14**])

$$\int_{T(1,\eta)} |\nabla F - \nabla x|_g^2 \, dg \le \frac{1}{k} C\left(H, n, \frac{1}{\eta}, \eta, i_0\right).$$

By a standard argument involving DeGiorgi-Nash-Moser iteration, it follows that F is the desired diffeomorphism. □

THEOREM 4.2. *For each $M_k, g_k \in \mathcal{L}$, there exists, for $l = 1, 2, \cdots$, open sets $F_k(l) \subset M_k$ s.t. $F_k(l+1) \supset F_k(l)$ and $F_k(l) \cup B(l^{-1}) = M_k$. There also exists a diffeomorphism $\phi_k(l)$ for each pair of k and l: $\phi_k(l) : T(1, l^{-1}) \subset \mathbb{R}^n \to F_k(l)$ such that $\phi_k(l)^* g_k$ converges in $C^{1,\alpha}$ norm to some $C^{1,\alpha}$ metric g_l' on $T(1, l^{-1}) \subset \mathbb{R}^n$.*

Proof. By rescaling, we can assume that g_k satisfies

$$\int_{M_k} |Rm(g_k)|^{\frac{n}{2}} dg_k \le \epsilon$$

where $\epsilon > 0$ is given by Proposition 4.1. Therefore we have harmonic coordinates

$$h^k : T_k\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right) \subset M_k \to D(\eta) = T\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right) \subset \mathbb{R}^n,$$

satisfying (a)-(f) of 4.1. Taking $\eta = l^{-1}$, by the Hölder estimate (d), we have, for each $l = 1, 2, \cdots$, a subsequence of (M_k, g_k), denoted by $g_k(l)$, *s.t.* $g_k(l)$ converges in the C^2-norm on $T_k\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right) \subset M$ to a $C^{1,\alpha}$ metric g_l' on $D(l)$. We can then take

$$F_k(l) = T_k\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right), \quad \eta = \frac{1}{l}.$$

By passing to a subsequence if necessary, we can make $F_k(l+1) \supset F_k(l)$. □

THEOREM 4.3. *Let g' be a metric on $M' \cong B_1 | \{0\}$ defined by $g'(x) = g_l'(x)$ if $x \in F_k(l)$. Then g' can be extended as a C^0 metric on B_1.*

Proof. Theorem 2.1 says that the diameter of a small geodesic sphere around 0 is small. Hence 0 is the only possible singularity. To show that 0 is a removable singular point, let, for fixed $N = 1, 2, \cdots$,

$$C(\rho, N) = \left\{ x \in M' | \frac{\rho}{N} < d(x, 0) < 2\rho \right\}.$$

By Theorem 4.2, a subsequence (M_k, g_k) converges to M' away from 0. Thus for each $\rho, \exists k = k(\rho), \exists$ a submanifold $C_k(\rho, N) \subset (M_k, g_k), \exists y_\rho \in C_k(\rho, N)$ *s.t.* $y_\rho \to x_\rho \in C(\rho, N)$ (with $dist(x_\rho, 0) = \rho$), and such that

$$\left| \int_{C_k(\rho,N)} |Rm(g_k)|^{\frac{n}{2}} \, dg_k - \int_{C(\rho,N)} |RM(g')|^{\frac{n}{2}} \, dg' \right| \le \rho^2,$$

and

$$\left\| \left(\frac{1}{\rho} C(\rho, N), x_\rho \right) - \left(\frac{1}{\rho} C_k(\rho, N), y_k \right) \right\|_{C^{1,\alpha}} < \rho.$$

By (0.5),

$$\int_{C(\rho,N)} |RM(g')|^{\frac{n}{2}} \, dg' \to 0 \quad \text{as} \quad \rho \to 0.$$

Consequently,

$$\int_{C_k(\rho,N)} |RM(g_k)|^{\frac{n}{2}} \, dg_k \to 0 \quad \text{as} \quad \rho \to 0.$$

Therefore, from the zero pinching theorem of [**12**], it follows that $\left(\frac{1}{\rho} C_k(\rho, N), y_\rho\right)$ converges to a flat manifold D_N in $C^{1,\alpha}$-norm as $\rho \to 0$. Thus $\left(\frac{1}{\rho} C(\rho, N), x_\rho\right)$ converges to (D_N, e_N) in $C^{1,\alpha}$-norm. The direct union of (D_N, e_N) has to be $(U(0), e)$ where 0 is the isolated singular point, e is a unit vector in $|BbbR^n$, and $U(0)$ is a simply connected flat manifold since $\frac{1}{\rho} C(\rho, N)$ is the $C^{1,\alpha}$ limit of simply connected manifolds $\frac{1}{\rho} C_k(\rho, N)$. Hence $U(0) \cong B(2) - \{0\}$. Letting $N \to \infty$ have that $\left(\frac{1}{\rho} C(\rho, 0), x_\rho\right)$ converges to $\{B(2) - \{0\}, e\}$ in $C^{1,\alpha}$-norm. It follows that g' can extend to a C^0 metric on M', diffeomorphic to $B_1 \subset \mathbb{R}^n$. □

REMARK. In the case $(M_k, g_k) \in \mathcal{L}'$, we use Proposition 3.5 directly in place of Proposition 4.1 and Theorem 4.2. This, combined with Theorem 4.3, proves Theorem (0.8).

REMARK. Let O be the set of compact orbifolds with finitely many singular points, satisfying (0.3)-(0.6). Let Γ be the group acting on these orbifolds. We can lift a neighbourhood of each singular point via Γ to B^n. It then follows from Theorem (0.7) that O has the same compactness property.

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