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ON FLATNESS OF THE COXETER GRAPH E_8

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We will show flatness of Ocneanu's connections on Coxeter graph E_8. This completes classification of subfactors of the type II_1 AFD factor with indices less than 4, which has been stated by A. Ocneanu.

1. Introduction and main results. Since his celebrated work [**J**], V. Jones theory of index is one of the central topics of the theory of operator algebras, and further deep results have been obtained, for example [**PP**, **K**]. Especially, on classification of subfactors of the approximately finite dimensional (AFD) II_1 factor, A. Ocneanu announced a striking result with the notion of paragroups [**O1**, **O2**]. But the details of his proof have not appeared yet.

Ocneanu's theory has two aspects. One is analytic aspect, which is covered by Popa's deep results [**P1**, **P2**], and the other is combinatorial aspect i.e. the theory of paragroups. Until now, existence and non-existence results of paragroups corresponding to the Coxeter graphs except E_8 have been obtained [**B**, **K**, **SV**, **I**]. The purpose of this paper is to prove the existence of the E_8 paragroup, which shows that Ocneanu's classification list in [**O1**] is correct.

The contents of this paper are as follows. In Section 2 we will show that the study of flat connections on E_8 is reduced to that of other connections on some four graphs, two of which are E_8. In Section 3 we will prove the main result by computing the above-mentioned connections. While we will treat only E_8 case our method is applicable to the other cases of the Coxeter graphs. Throughout this paper we will freely use the contents and the notations in [**K**].

The author would like to thank Y. Kawahigashi. Without his kind explanation the author could not understand the theory of paragroups.

2. Reduction to another embedding of string algebras. We fix the following numbering on vertices of E_8.

$$\begin{array}{ccccccccccccc} & & & & & & & & 7 & & & & \\ & & & & & & & & | & & & & \\ 0 & — & 1 & — & 2 & — & 3 & — & 4 & — & 5 & — & 6 \end{array}$$

In [**O1**, page 159],[**K**, Theorem 3.1] it is shown that there are exactly two connections on the Coxeter graph E_8 up to gauge transformation, which we will explain later, and these two are mutually complex conjugate. We fix one of the connections on E_8 and consider a double complex of string algebras $A_{n,m}(0 \leq n, m \leq \infty)$ as in [**K**, **O1**]. Note that to obtain a flat connection the distinguished point $*$ must be 0 [**I**, Theorem 6.1],[**O1**, page 161]. We do this assumption. Let $\iota : A_{0,\infty} \hookrightarrow A_{5,\infty}$ be the embedding map constructed by the connection and ξ a horizontal path

$$\xi = 0 \to 1 \to 2 \to 3 \to 4 \to 7.$$

Then it is shown in [**K**, Theorem 2.1] that the existence of the subfactors corresponding to E_8 is equivalent to

$$\iota((\xi, \xi)) \in A_{5,0}{}'. \tag{2.1}$$

Let η_1, η_2 be vertical paths

$$\eta_1 = 0 \to 1 \to 2 \to 3 \to 4 \to 5, \quad \eta_2 = 0 \to 1 \to 2 \to 3 \to 4 \to 7,$$

$p = (\eta_1, \eta_1) + (\eta_2, \eta_2)$ and B the $*$subalgebra of $A_{5,0}$ generated by the vertical Jones projections in $A_{5,0}$.. Then the following hold [**K**, Section 1].

$$A_{5,0} = B \oplus C(\eta_1, \eta_1) \oplus C(\eta_2, \eta_2).$$

$$B \oplus Cp \subset \iota(A_{0,\infty})'.$$

So, to show (2.1) it suffices to show

$$p\iota((\xi, \xi))p \in (pA_{5,0}p)'. \tag{2.2}$$

Let $A_n = A_{0,n}$, $B_n = p(A_{5,n})p$ and $\rho(x) \equiv p\iota(x)p$ for $x \in A_\infty$. Then B_n is the string algebra of E_8 with distinguished points $*_1 \equiv$

$5, *_2 \equiv 7$ and ρ is a filtered unital embedding of the string algebra A_∞ into the string algebra B_∞, which preserves the standard Jones projections. By the word "filtered" we mean $\rho(A_n) \subset B_n$ for any non-negative integer n. Summing up the above argument, we have the following lemma.

LEMMA 2.1. *Let A_n, B_n be string algebras of E_8 with distinguished points $* = 0$ and $*_1 = 5, *_2 = 7$, and*

$$\xi = 0 \to 1 \to 2 \to 3 \to 4 \to 7,$$
$$x = (\xi, \xi) \in A_5.$$

If for any unital filtered embedding $\rho : A_\infty \hookrightarrow B_\infty$ which preserves the standard Jones projections, $\rho(x)$ commutes with $(*_1, *_1) \in B_0$ (equivalently $(*_2, *_2)$), then for E_8 there exist two and only two subfactors of the AFD *factor of type* II_1 *up to conjugacy.*

Let us recall Ocneanu's result on embeddings of string algebras [**O3**]. Let $\mathcal{G}_1, \mathcal{G}_2$ be finite bipartite graphs with distinguished points. As in [**K**, **EK**] we admit that $\mathcal{G}_1$ and $\mathcal{G}_2$ have several distinguished points. Let $(A_n), (B_n)$ be the string algebras of $\mathcal{G}_1, \mathcal{G}_2$, and $\rho : A_\infty \hookrightarrow B_\infty$ a filtered unital embedding preserving the standard Jones projections. Then for large $n \in N$ the inclusion matrices of $\rho(A_n) \subset B_n$ and $\rho(A_{n+2}) \subset B_{n+2}$ coincide and we denote by $\mathcal{F}_1, \mathcal{F}_2$ the corresponding graphs of $\rho(A_{2n}) \subset B_{2n}, \rho(A_{2n+1}) \subset B_{2n+1}$. A slight modification of the argument in [**O3**] shows that ρ comes from a connection on the following cells satisfying the renormalization rule and the unitarity.

(2.3)
$$\begin{tikzcd} a \arrow[r, "\xi_2"] \arrow[d, "\xi_1"'] & b \arrow[d, "\xi_3"] \\ c \arrow[r, "\xi_4"'] & d \end{tikzcd}$$

$$\xi_1 \in \mathcal{F}_1, \xi_2 \in \mathcal{G}_1, \xi_3 \in \mathcal{F}_2, \xi_4 \in \mathcal{G}_2 \quad \text{or}$$
$$\xi_1 \in \mathcal{F}_2, \xi_2 \in \mathcal{G}_1, \xi_3 \in \mathcal{F}_1, \xi_4 \in \mathcal{G}_2.$$

The unitarity means that the following matrix is unitary.

$$\begin{array}{ccc} a & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & d \end{array}.$$

The renormalization rule is

$$\begin{array}{ccc} a & \xrightarrow{\xi_2} & b \\ {\scriptstyle \xi_1}\big\downarrow & & \big\downarrow{\scriptstyle \xi_3} \\ c & \xrightarrow{\xi_4} & d \end{array} = \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}}\; \overline{\begin{array}{ccc} a & \xrightarrow{\tilde{\xi}_2} & b \\ {\scriptstyle \xi_3}\big\downarrow & & \big\downarrow{\scriptstyle \xi_1} \\ d & \xrightarrow[\tilde{\xi}_4]{} & c \end{array}},$$

where $\tilde{\xi}$ is the reverse path of ξ. (See Section 3 for the notations.)

We come back to our case, namely we assume $\mathcal{G}_1 = \mathcal{G}_2 = E_8$. To distinguish the vertices of $\mathcal{G}_2$ from those of $\mathcal{G}_1$, we use the preceding numbering for $\mathcal{G}_1$ and the following numbering for $\mathcal{G}_2$.

$$\begin{array}{ccccccccccccc} & & & & & & & & \bar{7} & & & & \\ & & & & & & & & | & & & & \\ \bar{0} & — & \bar{1} & — & \bar{2} & — & \bar{3} & — & \bar{4} & — & \bar{5} & — & \bar{6} \end{array}.$$

The Bratteli diagram of (A_n) and (B_n) are as in Fig.1.

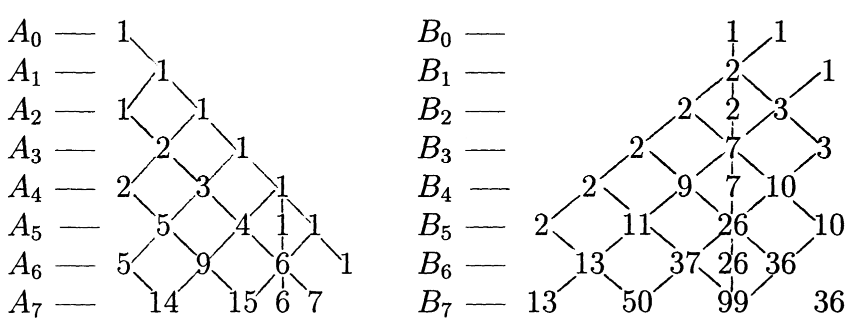

FIGURE 1. The Bratteli diagrams of (A_n) and (B_n).

We have to determine $\mathcal{F}_1, \mathcal{F}_2$, or equivalently the inclusion matrices of $\rho(A_n) \subset B_n$, which we denote by Γ_n. Let G_1, G_2 be the matrices corresponding to $\mathcal{G}_1, \mathcal{G}_2$ i.e.

$$G_1 = \begin{array}{cc} & \begin{array}{cccc} 0 & 2 & 4 & 6 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}, \quad G_2 = \begin{array}{cc} & \begin{array}{cccc} \bar{0} & \bar{2} & \bar{4} & \bar{6} \end{array} \\ \begin{array}{c} \bar{1} \\ \bar{3} \\ \bar{5} \\ \bar{7} \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Then G_1, G_2, Γ_n satisfy the following relations.

$$G_2^t \Gamma_{2n} = \Gamma_{2n+1} G_1,$$

$$G_2 \Gamma_{2n+1} = \Gamma_{2n+2} G_1^t.$$

Note that the graph corresponding to Γ_n is a part of that corresponding to Γ_{n+2}, and the edges in $\mathcal{F}_1$ or $\mathcal{F}_2$, connected to the vertices in $\mathcal{G}_1$ which appear in the former, have already been determined by Γ_n. Taking this fact into account, we can easily see that the possible matrices are as follows.

$$\Gamma_0 = \begin{array}{cc} & \begin{array}{cccc} 0 & 2 & 4 & 6 \end{array} \\ \begin{array}{c} \bar{1} \\ \bar{3} \\ \bar{5} \\ \bar{7} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \Gamma_1 = \begin{array}{cc} & \begin{array}{cccc} 1 & 3 & 5 & 7 \end{array} \\ \begin{array}{c} \bar{0} \\ \bar{2} \\ \bar{4} \\ \bar{6} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\Gamma_2 = \begin{array}{cc} & \begin{array}{cccc} 0 & 2 & 4 & 6 \end{array} \\ \begin{array}{c} \bar{1} \\ \bar{3} \\ \bar{5} \\ \bar{7} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}, \quad \Gamma_3 = \begin{array}{cc} & \begin{array}{cccc} 1 & 3 & 5 & 7 \end{array} \\ \begin{array}{c} \bar{0} \\ \bar{2} \\ \bar{4} \\ \bar{6} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$\Gamma_4 = \begin{array}{cc} & \begin{array}{cccc} 0 & 2 & 4 & 6 \end{array} \\ \begin{array}{c} \bar{1} \\ \bar{3} \\ \bar{5} \\ \bar{7} \end{array} & \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{pmatrix} \end{array}, \quad \Gamma_{2n-1} = \begin{array}{cc} & \begin{array}{cccc} 1 & 3 & 5 & 7 \end{array} \\ \begin{array}{c} \bar{0} \\ \bar{2} \\ \bar{4} \\ \bar{6} \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 2 & 3 & 2 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \quad (n \geq 3)$$

$$
\Gamma_{2n} = \begin{array}{c} \\ \bar{1} \\ \bar{3} \\ \bar{5} \\ \bar{7} \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 2 & 4 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} \end{array} \quad (n \geq 3).
$$

So $\mathcal{F}_1$ and $\mathcal{F}_2$ are as in Fig.2.

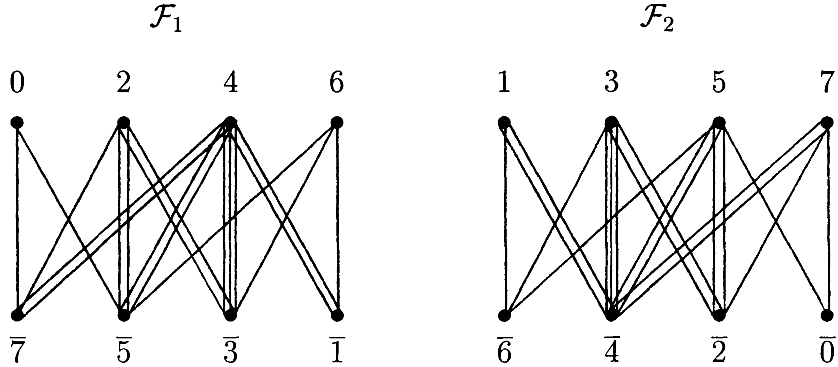

FIGURE 2. $\mathcal{F}_1$ and $\mathcal{F}_2$.

3. Main theorem. In this section we will compute the connections on the cells (2.3). Our aim is not to determine all entries of the connections, but to show that all possible connections satisfy the assumption of Lemma 2.1. For this, it suffices to show that "sufficiently many" entries of the connections take value 0.

Before starting computation we recall the gauge transformations of connections, (in [**O1**, page 154] Ocneanu calls perturbations), which become a key technique later. For a vertices x, y we denote by $Path^{(1)}_{x,y}$ the set of edges between x and y. We consider unitaries

$$
(u(\xi, \eta))_{\xi,\eta} \in \mathrm{End}(l^2(\mathrm{Path}^{(1)}_{x,y}))
$$

for all possible vertices x, y in $\mathcal{G}_1, \mathcal{G}_2, \mathcal{F}_1, \mathcal{F}_2$, between which there exists at least one edge. For given connection

$$
W\left(\begin{array}{ccc} a & \xrightarrow{\xi_2} & b \\ {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_3} \\ c & \xrightarrow[\xi_4]{} & d \end{array}\right),
$$

the transformed connection W' is defined as follows.

$$W' \begin{array}{ccc} a & \xrightarrow{\xi_2'} & b \\ {\scriptstyle \xi_1'}\downarrow & & \downarrow{\scriptstyle \xi_3'} \\ c & \xrightarrow[\xi_4']{} & d \end{array} = \sum_{\xi_1,\xi_2,\xi_3,\xi_4} \alpha W \begin{array}{ccc} a & \xrightarrow{\xi_2} & b \\ {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_3} \\ c & \xrightarrow[\xi_4]{} & d \end{array}$$

$$\text{for} \quad \xi_1' \in \mathcal{F}_1, \xi_2' \in \mathcal{F}_2,$$

$$W' \begin{array}{ccc} a & \xrightarrow{\xi_2'} & b \\ {\scriptstyle \xi_1'}\downarrow & & \downarrow{\scriptstyle \xi_3'} \\ c & \xrightarrow[\xi_4']{} & d \end{array} = \sum_{\xi_1,\xi_2,\xi_3,\xi_4} \overline{\alpha} W \begin{array}{ccc} a & \xrightarrow{\xi_2} & b \\ {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_3} \\ c & \xrightarrow[\xi_4]{} & d \end{array}$$

$$\text{for} \quad \xi_1' \in \mathcal{F}_2, \xi_2' \in \mathcal{F}_1,$$

where $\alpha = u(\xi_1', \xi_1)u(\xi_2', \xi_2)u(\xi_3', \xi_3)u(\xi_4', \xi_4)$. Note that the condition in Lemma 2.1 does not depend on the choice of gauges.

We will use the following conventions. As in [**K**] we will omit the sign "W" of connections if no confusion arises. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ have multiedges we need a numbering of edges, and we use that starting from 1. We mean by

$$\begin{array}{ccc} a & \longrightarrow & . \\ \downarrow & & \downarrow \\ . & \longrightarrow & d \end{array}$$

the unitary matrix $(v_{i,j})_{i,j}$ where $i = (\xi_1, \xi_4), j = (\xi_2, \xi_3)$ and

$$v_{i,j} = \xi_1 \begin{array}{ccc} a & \xrightarrow{\xi_2} & b \\ {\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_3} \\ c & \xrightarrow[\xi_4]{} & d \end{array}$$

For example we will write as follows.

$$\begin{array}{ccc} 5 & \longrightarrow & . \\ \downarrow & & \downarrow \\ . & \longrightarrow & \overline{1} \end{array} = \begin{array}{c} \\ \overline{0} \\ \overline{_12} \\ \overline{_22} \end{array} \begin{array}{c} \begin{array}{ccc} 4_1 & 4_2 & 6 \end{array} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \end{array},$$

where the $(\overline{{}_1 2}, 4_2)$ entry means

$$\begin{array}{ccc} 5 & \longrightarrow & 4 \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle 2} \\ \overline{2} & \longrightarrow & \overline{3} \end{array}$$

Let

$$\mu(0) = 1, \mu(1) = \frac{\sin \frac{2\pi}{30}}{\sin \frac{\pi}{30}}, \quad \mu(2) = \frac{\sin \frac{3\pi}{30}}{\sin \frac{\pi}{30}}, \quad \mu(3) = \frac{\sin \frac{4\pi}{30}}{\sin \frac{\pi}{30}},$$

$$\mu(4) = 2\cos\frac{\pi}{5} \cdot \frac{\sin \frac{3\pi}{30}}{\sin \frac{\pi}{30}}, \quad \mu(5) = 2\cos\frac{\pi}{5} \cdot \frac{\sin \frac{2\pi}{30}}{\sin \frac{\pi}{30}}, \quad \mu(6) = 2\cos\frac{\pi}{5},$$

$$\mu(7) = \frac{1}{2\cos\frac{\pi}{5}} \cdot \frac{\sin \frac{4\pi}{30}}{\sin \frac{\pi}{30}}.$$

Then using $(2\cos\pi/5)^2 = 1 + 2\cos\pi/5$, we can easily see that $(\mu(i))_i$ is the Perron-Frobenius eigenvector of E_8, and the following equations hold.

$$\begin{aligned} \mu(3) &= \mu(6)\mu(7), \\ \mu(4) &= \mu(2)\mu(6) = \mu(1)\mu(7), \\ \mu(5) &= \mu(1)\mu(6). \end{aligned} \tag{3.1}$$

Now we start computing the connections. Using the gauge freedom of the following edges in order,

$$\begin{array}{cccccccc} 1 & 2 & 4 & 3 & 2 & 5 & 4 & 7 \\ | & | & | & | & | & | & | & | \\ \overline{4}, & \overline{3}, & \overline{1}, & \overline{2}, & \overline{5}, & \overline{2}, & \overline{5}, & \overline{4}, \end{array}$$

$$\begin{array}{ccccccc} 4 & 5 & 0 & 5 & 6 & 3 & 4 \\ | & | & | & | & | & | & | \\ \overline{7}, & \overline{4}, & \overline{5}, & \overline{0}, & \overline{5}, & \overline{4}, & \overline{3} \end{array}$$

we may assume as follows.

(3.2)

$$\begin{array}{ccc} 1 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \overline{7} \end{array} = \begin{array}{c} \\ \overline{{}_1 4} \\ \overline{{}_2 4} \end{array} \!\!\begin{array}{c} \begin{array}{cc} 0 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \qquad \begin{array}{ccc} 1 & \longrightarrow & 4 \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \overline{3} \end{array} = \begin{array}{c} \\ \overline{{}_1 4} \\ \overline{{}_2 4} \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2_1 & 2_2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array},$$

$$\begin{array}{ccc} 7 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{1} \end{array} = \begin{array}{cc} & \begin{matrix} 4_1 & 4_2 \end{matrix} \\ \begin{matrix} \bar{0} \\ \bar{2} \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \qquad \begin{array}{ccc} 3 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{1} \end{array} = \begin{array}{cc} & \begin{matrix} 4_1 & 4_2 \end{matrix} \\ \begin{matrix} \overline{{}_1 2} \\ \overline{{}_2 2} \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array},$$

$$\begin{array}{ccc} 2 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{6} \end{array} = \begin{array}{cc} & \begin{matrix} 1 & 3 \end{matrix} \\ \begin{matrix} \overline{{}_1 5} \\ \overline{{}_2 5} \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \qquad \begin{array}{ccc} 6 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{2} \end{array} = \begin{array}{cc} & \begin{matrix} 5_1 & 5_2 \end{matrix} \\ \begin{matrix} \bar{1} \\ \bar{3} \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array},$$

$$\begin{array}{ccc} 4 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{6} \end{array} = \begin{array}{cc} & \begin{matrix} 3 & 5 \end{matrix} \\ \begin{matrix} \overline{{}_1 5} \\ \overline{{}_2 5} \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \qquad \begin{array}{ccc} 7 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{5} \end{array} = \begin{array}{cc} & \begin{matrix} 4_1 & 4_2 \end{matrix} \\ \begin{matrix} \overline{{}_1 4} \\ \overline{{}_2 4} \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array},$$

$$\begin{array}{ccc} 7 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{7} \end{array} = \begin{array}{cc} & \begin{matrix} 4_1 & 4_2 \end{matrix} \\ \begin{matrix} \overline{{}_1 4} \\ \overline{{}_2 4} \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \qquad \begin{array}{ccc} 5 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{7} \end{array} = \begin{array}{cc} & \begin{matrix} 4_1 & 4_2 \end{matrix} \\ \begin{matrix} \overline{{}_1 4} \\ \overline{{}_2 4} \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array},$$

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \bar{5} & \longrightarrow & \bar{6} \end{array} = 1, \qquad \begin{array}{ccc} 6 & \longrightarrow & 5 \\ \downarrow & & \downarrow \\ \bar{5} & \longrightarrow & \bar{6} \end{array} = 1, \qquad \begin{array}{ccc} 6 & \longrightarrow & 5 \\ \downarrow & & \downarrow \\ \bar{1} & \longrightarrow & \bar{0} \end{array} = 1,$$

$$\begin{array}{ccc} 3 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{7} \end{array} = \begin{array}{cc} & \begin{matrix} 2 & 4_1 & 4_2 \end{matrix} \\ \begin{matrix} \overline{{}_1 4} \\ \overline{{}_2 4} \\ \overline{{}_3 4} \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \qquad \begin{array}{ccc} 7 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \bar{3} \end{array} = \begin{array}{cc} & \begin{matrix} 4_1 & 4_2 & 4_3 \end{matrix} \\ \begin{matrix} \bar{2} \\ \overline{{}_1 4} \\ \overline{{}_2 4} \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Applying the renormalization rule to

$$\begin{array}{ccc} 1 & \longrightarrow & 0 \\ {\scriptstyle i}\downarrow & & \downarrow \\ \bar{4} & \longrightarrow & \bar{7} \end{array}, \quad \begin{array}{ccc} 7 & \longrightarrow & 4 \\ \downarrow & & \downarrow{\scriptstyle j} \\ \bar{0} & \longrightarrow & \bar{1} \end{array}, \qquad i, j = 1, 2$$

and using (3.1), we have

$$\begin{array}{ccc} 0 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \bar{4} \end{array} = \begin{array}{cc} & \begin{array}{cc} 1_1 & 1_2 \end{array} \\ \begin{array}{c} \bar{5} \\ \bar{7} \end{array} & \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{ccc} 4 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \bar{0} \end{array} = \begin{array}{cc} & \begin{array}{cc} 5 & 7 \end{array} \\ \begin{array}{c} \overline{{}_1 1} \\ \overline{{}_2 1} \end{array} & \begin{pmatrix} * & 1 \\ * & 0 \end{pmatrix} \end{array}.$$

So, due to the unitarity we can put as follows.

(3.3)

$$\begin{array}{ccc} 0 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \bar{4} \end{array} = \begin{array}{cc} & \begin{array}{cc} 1_1 & 1_2 \end{array} \\ \begin{array}{c} \bar{5} \\ \bar{7} \end{array} & \begin{pmatrix} 0 & \varepsilon_1 \\ 1 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{ccc} 4 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \bar{0} \end{array} = \begin{array}{cc} & \begin{array}{cc} 5 & 7 \end{array} \\ \begin{array}{c} \overline{{}_1 1} \\ \overline{{}_2 1} \end{array} & \begin{pmatrix} 0 & 1 \\ \varepsilon_2 & 0 \end{pmatrix} \end{array},$$

$$\begin{array}{ccc} 2 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \bar{2} \end{array} = \begin{array}{cc} & \begin{array}{cc} 3_1 & 3_2 \end{array} \\ \begin{array}{c} \overline{{}_1 3} \\ \overline{{}_2 3} \end{array} & \begin{pmatrix} x & y \\ -\varepsilon_3 \bar{y} & \varepsilon_3 \bar{x} \end{pmatrix} \end{array}, \quad \begin{array}{ccc} 6 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \bar{4} \end{array} = \begin{array}{cc} & \begin{array}{cc} 5_1 & 5_2 \end{array} \\ \begin{array}{c} \bar{3} \\ \bar{5} \end{array} & \begin{pmatrix} u & v \\ -\varepsilon_4 \bar{v} & \varepsilon_4 \bar{u} \end{pmatrix} \end{array},$$

$$\varepsilon_i \ (i = 1, 2, 3, 4), x, y, u, v \in C,$$

$$|\varepsilon_i| = 1, \quad |x|^2 + |y|^2 = |u|^2 + |v|^2 = 1.$$

Applying the renormalization rule to

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ \bar{5} & \longrightarrow & \bar{4} \end{array}, \quad \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow \\ \bar{5} & \longrightarrow & \bar{6} \end{array}, \quad \begin{array}{ccc} 2 & \longrightarrow & 1 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ \bar{5} & \longrightarrow & \bar{6} \end{array}, \quad \begin{array}{ccc} 4 & \longrightarrow & 5 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ \bar{1} & \longrightarrow & \bar{0} \end{array},$$

$$\begin{array}{ccc} 4 & \longrightarrow & 5 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ \bar{5} & \longrightarrow & \bar{6} \end{array}, \quad \begin{array}{ccc} 6 & \longrightarrow & 5 \\ \big\downarrow & & \big\downarrow \\ \bar{1} & \longrightarrow & \bar{0} \end{array}, \quad \begin{array}{ccc} 6 & \longrightarrow & 5 \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ \bar{1} & \longrightarrow & \bar{2} \end{array}, \quad , \begin{array}{ccc} 6 & \longrightarrow & 5 \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ \bar{5} & \longrightarrow & \bar{6} \end{array},$$

$$i = 1, 2,$$

and in the same way as above, we obtain the following.

$$
(3.4)\qquad
\begin{tikzcd}
1 \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \bar{5}
\end{tikzcd}
=
\begin{array}{c}
\\
\overline{{}_14} \\
\overline{{}_24} \\
\bar{6}
\end{array}
\!\!
\begin{array}{c}
\begin{array}{ccc} 0 & 2_1 & 2_2 \end{array} \\
\begin{pmatrix}
0 & 0 & \omega_1 \\
\frac{\bar{\varepsilon}_1\sqrt{\mu(2)}}{\mu(1)} & -\frac{\bar{\varepsilon}_1}{\mu(1)} & 0 \\
\frac{1}{\mu(1)} & \frac{\sqrt{\mu(2)}}{\mu(1)} & 0
\end{pmatrix}
\end{array},
$$

$$
\begin{tikzcd}
5 \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \bar{1}
\end{tikzcd}
=
\begin{array}{c}
\\
\bar{0} \\
\overline{{}_12} \\
\overline{{}_22}
\end{array}
\!\!
\begin{array}{c}
\begin{array}{ccc} 4_1 & 4_2 & 6 \end{array} \\
\begin{pmatrix}
0 & \frac{\bar{\varepsilon}_2\sqrt{\mu(2)}}{\mu(1)} & \frac{1}{\mu(1)} \\
0 & -\frac{\bar{\varepsilon}_2}{\mu(1)} & \frac{\sqrt{\mu(2)}}{\mu(1)} \\
\omega_2 & 0 & 0
\end{pmatrix}
\end{array}
$$

$$
\begin{tikzcd}
5 \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \bar{5}
\end{tikzcd}
=
\begin{array}{c}
\\
\overline{{}_14} \\
\overline{{}_24} \\
\bar{6}
\end{array}
\!\!
\begin{array}{c}
\begin{array}{ccc} 4_1 & 4_2 & 6 \end{array} \\
\begin{pmatrix}
\omega_3\bar{u} & \frac{\bar{\varepsilon}_4 v}{\mu(1)} & -\frac{\bar{\varepsilon}_4 v\sqrt{\mu(2)}}{\mu(1)} \\
\omega_3\bar{v} & -\frac{\bar{\varepsilon}_4 u}{\mu(1)} & \frac{\bar{\varepsilon}_4 u\sqrt{\mu(2)}}{\mu(1)} \\
0 & \frac{\sqrt{\mu(2)}}{\mu(1)} & \frac{1}{\mu(1)}
\end{pmatrix}
\end{array}
$$

$$\omega_i\ (i = 1, 2, 3) \in C, \quad |\omega_i| = 1.$$

Using the same type of argument as above, from (3.1), (3.2), (3.3), (3.4) we have the following.

$$
(3.5)\qquad
\begin{tikzcd}
2 \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \bar{4}
\end{tikzcd}
=
\begin{array}{c}
\\
\overline{{}_13} \\
\overline{{}_23} \\
\overline{{}_15} \\
\overline{{}_25} \\
\bar{7}
\end{array}
\!\!
\begin{array}{c}
\begin{array}{ccccc} 1_1 & 1_2 & 3_1 & 3_2 & 3_3 \end{array} \\
\begin{pmatrix}
\sqrt{\frac{\mu(6)}{\mu(2)}} & 0 & 0 & * & * \\
0 & \sqrt{\frac{\mu(6)}{\mu(2)}} & -\frac{\sqrt{\mu(6)}}{\mu(7)} & * & * \\
0 & -\frac{\varepsilon_1}{\mu(2)} & \frac{\varepsilon_1}{\sqrt{\mu(2)\mu(7)}} & * & * \\
\frac{\overline{\omega_1}\mu(1)}{\mu(2)} & 0 & 0 & * & * \\
0 & \frac{1}{\sqrt{\mu(2)}} & \frac{\mu(7)}{\mu(2)} & 0 & 0
\end{pmatrix}
\end{array},
$$

(3.6)

$$
\begin{array}{ccccccc}
4 & \longrightarrow & \cdot \\
\big\downarrow & & \big\downarrow \\
\cdot & \longrightarrow & \overline{2}
\end{array}
=
\begin{array}{cc}
 & \begin{array}{ccccc} 3_1 & 3_2 & 5_1 & 5_2 & 7 \end{array} \\
\begin{array}{c} \overline{{}_1 1} \\ \overline{{}_2 1} \\ \overline{{}_1 3} \\ \overline{{}_2 3} \\ \overline{{}_3 3} \end{array} &
\begin{pmatrix}
\sqrt{\frac{\mu(6)}{\mu(2)}} & 0 & 0 & \frac{\overline{\omega_2}\mu(1)}{\mu(2)} & 0 \\
0 & \sqrt{\frac{\mu(6)}{\mu(2)}} & -\frac{\varepsilon_2}{\mu(2)} & 0 & \frac{1}{\sqrt{\mu(2)}} \\
0 & -\frac{\sqrt{\mu(6)}}{\mu(7)} & \frac{\varepsilon_2}{\sqrt{\mu(2)\mu(7)}} & 0 & \frac{\mu(7)}{\mu(2)} \\
* & * & * & * & 0 \\
* & * & * & * & 0
\end{pmatrix}
\end{array},
$$

(3.7)

$$
\begin{array}{ccc}
4 & \longrightarrow & \cdot \\
\big\downarrow & & \big\downarrow \\
\cdot & \longrightarrow & \overline{4}
\end{array}
=
\begin{array}{cc}
 & \begin{array}{ccccccc} 3_1 & 3_2 & 3_3 & 5_1 & 5_2 & 7_1 & 7_2 \end{array} \\
\begin{array}{c} \overline{{}_1 3} \\ \overline{{}_2 3} \\ \overline{{}_3 3} \\ \overline{{}_1 5} \\ \overline{{}_2 5} \\ \overline{{}_1 7} \\ \overline{{}_2 7} \end{array} &
\begin{pmatrix}
* & * & * & * & * & 0 & 0 \\
* & * & * & a & b & \frac{\sqrt{\mu(6)}}{\mu(1)} & 0 \\
* & * & * & c & d & 0 & \frac{\sqrt{\mu(6)}}{\mu(1)} \\
* & a & b & \frac{\overline{\omega_3}u\mu(1)}{\mu(2)} & \frac{\overline{\omega_3}v\mu(1)}{\mu(2)} & \frac{1}{\sqrt{\mu(2)}} & 0 \\
* & c & d & \frac{\varepsilon_4\overline{v}}{\mu(2)} & -\frac{\varepsilon_4\overline{u}}{\mu(2)} & 0 & \frac{1}{\sqrt{\mu(2)}} \\
0 & \frac{\sqrt{\mu(6)}}{\mu(1)} & 0 & \frac{1}{\sqrt{\mu(2)}} & 0 & \frac{1}{\mu(1)} & 0 \\
0 & 0 & \frac{\sqrt{\mu(6)}}{\mu(1)} & 0 & \frac{1}{\sqrt{\mu(2)}} & 0 & \frac{1}{\mu(1)}
\end{pmatrix}
\end{array},
$$

where

$$a = -\frac{\mu(2) + \overline{\omega_3}u\mu(1)^2}{\sqrt{\mu(2)\mu(6)}\mu(2)}, \quad b = -\frac{\overline{\omega_3}v\mu(1)^2}{\sqrt{\mu(2)\mu(6)}\mu(2)},$$

$$c = -\frac{\varepsilon_4\overline{v}\mu(1)}{\sqrt{\mu(2)\mu(6)}\mu(2)}, \quad d = \frac{-\mu(2) + \varepsilon_4\overline{u}\mu(1)}{\sqrt{\mu(2)\mu(6)}\mu(2)}.$$

$$(3.8)\qquad \begin{tikzcd} 3 \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \bar{3} \end{tikzcd} = \begin{array}{c@{}c} & \begin{array}{ccccc} 2_1 & 2_2 & 4_1 & 4_2 & 4_3 \end{array} \\ \begin{array}{c} \overline{{}_12} \\ \overline{{}_22} \\ \overline{{}_14} \\ \overline{{}_24} \\ \overline{{}_34} \end{array} & \left(\begin{array}{ccccc} \frac{\bar{x}\mu(2)}{\mu(3)} & -\frac{\bar{\varepsilon}_3 y\mu(2)}{\mu(3)} & * & * & * \\ \frac{\bar{y}\mu(2)}{\mu(3)} & -\frac{\bar{\varepsilon}_3 x\mu(2)}{\mu(3)} & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right) \end{array},$$

$$(3.9)\qquad \begin{tikzcd} 3 \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \bar{5} \end{tikzcd} = \begin{array}{c@{}c} & \begin{array}{cccc} 2_1 & 2_2 & 4_1 & 4_2 \end{array} \\ \begin{array}{c} \overline{{}_14} \\ \overline{{}_24} \\ \overline{{}_34} \\ \bar{6} \end{array} & \left(\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & \sqrt{\frac{\mu(2)}{\mu(1)\mu(3)}} & \frac{1}{\sqrt{\mu(6)}} & 0 \end{array}\right) \end{array},$$

$$(3.10)\qquad \begin{tikzcd} 5 \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \bar{3} \end{tikzcd} = \begin{array}{c@{}c} & \begin{array}{cccc} 4_1 & 4_2 & 4_3 & 6 \end{array} \\ \begin{array}{c} \overline{{}_12} \\ \overline{{}_22} \\ \overline{{}_14} \\ \overline{{}_24} \end{array} & \left(\begin{array}{cccc} * & * & * & 0 \\ * & * & * & \sqrt{\frac{\mu(2)}{\mu(1)\mu(3)}} \\ * & * & * & \frac{\bar{u}}{\sqrt{\mu(6)}} \\ * & * & * & \frac{\bar{v}}{\sqrt{\mu(6)}} \end{array}\right) \end{array}.$$

Applying the renormalization rule to

$$\begin{tikzcd} 2 \arrow[r] \arrow[d, "1"'] & 3 \arrow[d, "1"] \\ \bar{3} \arrow[r] & \bar{4} \end{tikzcd}, \quad \begin{tikzcd} 2 \arrow[r] \arrow[d, "2"'] & 3 \arrow[d, "1"] \\ \bar{3} \arrow[r] & \bar{2} \end{tikzcd}, \quad \begin{tikzcd} 2 \arrow[r] \arrow[d, "1"'] & 3 \arrow[d, "1"] \\ \bar{5} \arrow[r] & \bar{4} \end{tikzcd}, \quad \begin{tikzcd} 2 \arrow[r] \arrow[d, "2"'] & 3 \arrow[d, "1"] \\ \bar{5} \arrow[r] & \bar{4} \end{tikzcd},$$

$$\begin{tikzcd} 4 \arrow[r] \arrow[d, "1"'] & 3 \arrow[d, "1"] \\ \bar{3} \arrow[r] & \bar{2} \end{tikzcd}, \quad \begin{tikzcd} 4 \arrow[r] \arrow[d, "1"'] & 3 \arrow[d, "2"] \\ \bar{3} \arrow[r] & \bar{2} \end{tikzcd}, \quad \begin{tikzcd} 4 \arrow[r] \arrow[d, "1"'] & 5 \arrow[d, "1"] \\ \bar{3} \arrow[r] & \bar{2} \end{tikzcd}, \quad \begin{tikzcd} 4 \arrow[r] \arrow[d, "1"'] & 5 \arrow[d, "2"] \\ \bar{3} \arrow[r] & \bar{2} \end{tikzcd},$$

and using (3.1) and the unitarity, we obtain

$$
(3.8')\quad
\begin{array}{ccc}
3 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{3}
\end{array}
=
\begin{array}{c|c}
 & \begin{array}{ccccc} 2_1 & 2_2 & 4_1 & 4_2 & 4_3 \end{array} \\
\begin{array}{c} \overline{{}_12} \\ \overline{{}_22} \\ \overline{{}_14} \\ \overline{{}_24} \\ \overline{{}_34} \end{array} &
\begin{pmatrix}
\frac{\overline{x}\mu(2)}{\mu(3)} & -\frac{\overline{\varepsilon_3}y\mu(2)}{\mu(3)} & 0 & * & * \\
\frac{\overline{y}\mu(2)}{\mu(3)} & -\frac{\overline{\varepsilon_3}x\mu(2)}{\mu(3)} & -\frac{\mu(2)}{\mu(7)^2} & * & * \\
0 & -\frac{\mu(2)}{\mu(7)^2} & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{pmatrix}
\end{array},
$$

$$
(3.9')\quad
\begin{array}{ccc}
3 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{5}
\end{array}
=
\begin{array}{c|c}
 & \begin{array}{cccc} 2_1 & 2_2 & 4_1 & 4_2 \end{array} \\
\begin{array}{c} \overline{{}_14} \\ \overline{{}_24} \\ \overline{{}_34} \\ \overline{6} \end{array} &
\begin{pmatrix}
\frac{\overline{\varepsilon_1}}{\mu(6)\mu(7)} & 0 & 0 & * \\
* & * & * & * \\
* & * & * & * \\
0 & \sqrt{\frac{\mu(2)}{\mu(1)\mu(3)}} & \frac{1}{\sqrt{\mu(6)}} & 0
\end{pmatrix}
\end{array},
$$

$$
(3.10')\quad
\begin{array}{ccc}
5 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{3}
\end{array}
=
\begin{array}{c|c}
 & \begin{array}{cccc} 4_1 & 4_2 & 4_3 & 6 \end{array} \\
\begin{array}{c} \overline{{}_12} \\ \overline{{}_22} \\ \overline{{}_14} \\ \overline{{}_24} \end{array} &
\begin{pmatrix}
\frac{\overline{\varepsilon_2}}{\mu(6)\mu(7)} & * & * & 0 \\
0 & * & * & \sqrt{\frac{\mu(2)}{\mu(1)\mu(3)}} \\
* & * & * & \frac{\overline{u}}{\sqrt{\mu(6)}} \\
* & * & * & \frac{\overline{v}}{\sqrt{\mu(6)}}
\end{pmatrix}
\end{array}.
$$

Thanks to

$$
\begin{array}{ccc}
3 & \longrightarrow & 4 \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\
\overline{4} & \longrightarrow & \overline{5}
\end{array}
= 0,
$$

we have

$$
\begin{array}{ccc}
4 & \longrightarrow & 3 \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\
\overline{5} & \longrightarrow & \overline{4}
\end{array}
= 0.
$$

Hence the unitarity of

$$\begin{array}{ccc} 4 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \overline{4} \end{array}$$

implies that the following two vectors are mutually orthogonal.

$$\begin{pmatrix} 0\,, a\,, b\,, \frac{\overline{\omega_3}u\mu(1)}{\mu(2)}\,, \frac{\overline{\omega_3}v\mu(1)}{\mu(2)}\,, \frac{1}{\sqrt{\mu(2)}}\,, & 0 \end{pmatrix},$$
$$\begin{pmatrix} *\,, c\,, d\,, \frac{\varepsilon_4\overline{v}}{\mu(2)}\,, -\frac{\varepsilon_4\overline{u}}{\mu(2)}\,, 0\,, & \frac{1}{\sqrt{\mu(2)}} \end{pmatrix}.$$

So we have the following.

$$\begin{aligned} 0 &= \overline{a}c + \overline{b}d + \frac{\varepsilon_4\omega_3\overline{uv}\mu(1)}{\mu(2)^2} - \frac{\varepsilon_4\omega_3\overline{uv}\mu(1)}{\mu(2)^2} \\ &= \frac{\varepsilon_4\overline{v}\mu(1)(\mu(2)+\omega_3\overline{u}\mu(1)^2)}{\mu(2)^3\mu(6)} + \frac{\omega_3\overline{v}\mu(1)^2(\mu(2)-\varepsilon_4\overline{u}\mu(1))}{\mu(2)^3\mu(6)} \\ &= \frac{\overline{v}\mu(1)}{\mu(2)^3\mu(6)}\{\varepsilon_4(\mu(2)+\omega_3\overline{u}\mu(1)^2)+\omega_3\mu(1)(\mu(2)-\varepsilon_4\overline{u}\mu(1))\} \\ &= \frac{\overline{v}\mu(1)}{\mu(2)^2\mu(6)}(\varepsilon_4+\omega_3\mu(1)). \end{aligned}$$

From $\varepsilon_4 + \omega_3\mu(1) \neq 0$, we obtain $v = 0$ and consequently

$$\begin{array}{ccc} 6 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \overline{4} \end{array} = \begin{array}{c c} & \begin{array}{cc} 5_1 & 5_2 \end{array} \\ \begin{array}{c} \overline{3} \\ \overline{4} \end{array} & \begin{pmatrix} u & 0 \\ 0 & \varepsilon\overline{u} \end{pmatrix} \end{array}, \quad |u| = 1. \tag{3.11}$$

$$\begin{array}{ccc} 5 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \overline{5} \end{array} = \begin{array}{c c} & \begin{array}{ccc} 4_1 & 4_2 & 6 \end{array} \\ \begin{array}{c} \overline{{}_14} \\ \overline{{}_24} \\ \overline{6} \end{array} & \begin{pmatrix} \omega_3\overline{u} & 0 & 0 \\ 0 & -\frac{\overline{\varepsilon_4}u}{\mu(1)} & \frac{\overline{\varepsilon_4}u\sqrt{\mu(2)}}{\mu(1)} \\ 0 & \frac{\sqrt{\mu(2)}}{\mu(2)} & \frac{1}{\mu(1)} \end{pmatrix} \end{array}.$$

The unitarity of

$$\begin{array}{ccc} 5 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \overline{3} \end{array}$$

implies

$$
(3.10'') \qquad \begin{array}{ccc} 5 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \bar{3} \end{array} = \begin{array}{c} \\ \overline{{}_12} \\ \overline{{}_22} \\ \overline{{}_14} \\ \overline{{}_24} \end{array} \begin{array}{c} \begin{array}{cccc} 4_1 & 4_2 & 4_3 & 6 \end{array} \\ \begin{pmatrix} \frac{\overline{\varepsilon_2}}{\mu(6)\mu(7)} & * & * & 0 \\ 0 & * & * & \sqrt{\frac{\mu(2)}{\mu(1)\mu(3)}} \\ 0 & * & * & \frac{\overline{u}}{\sqrt{\mu(6)}} \\ * & * & * & 0 \end{pmatrix} \end{array},
$$

and hence we have

$$
\begin{array}{ccc} 4 & \longrightarrow & 5 \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\ \bar{3} & \longrightarrow & \bar{4} \end{array} = 0.
$$

So the unitarity of

$$
\begin{array}{ccc} 4 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \bar{4} \end{array}
$$

shows

(3.7′)

$$
\begin{array}{ccc} 4 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \bar{4} \end{array} = \begin{array}{c} \\ \overline{{}_13} \\ \overline{{}_23} \\ \overline{{}_33} \\ \overline{{}_15} \\ \overline{{}_25} \\ \overline{{}_17} \\ \overline{{}_27} \end{array} \begin{array}{c} \begin{array}{ccccccc} 3_1 & 3_2 & 3_3 & 5_1 & 5_2 & 7_1 & 7_2 \end{array} \\ \begin{pmatrix} * & 0 & * & 0 & * & 0 & 0 \\ 0 & * & 0 & a & 0 & \frac{\sqrt{\mu(6)}}{\mu(1)} & 0 \\ * & 0 & * & 0 & d & 0 & \frac{\sqrt{\mu(6)}}{\mu(1)} \\ 0 & a & 0 & \frac{\overline{\omega_3}u\mu(1)}{\mu(2)} & 0 & \frac{1}{\sqrt{\mu(2)}} & 0 \\ * & 0 & d & 0 & -\frac{\varepsilon_4\overline{u}}{\mu(2)} & 0 & \frac{1}{\sqrt{\mu(2)}} \\ 0 & \frac{\sqrt{\mu(6)}}{\mu(1)} & 0 & \frac{1}{\sqrt{\mu(2)}} & 0 & \frac{1}{\mu(1)} & 0 \\ 0 & 0 & \frac{\sqrt{\mu(6)}}{\mu(1)} & 0 & \frac{1}{\sqrt{\mu(2)}} & 0 & \frac{1}{\mu(1)} \end{pmatrix} \end{array}.
$$

Applying the renormalization rule to

$$\begin{array}{ccc} 4 & \longrightarrow & 3 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ \overline{3} & \longrightarrow & \overline{4} \end{array} \quad (i, j, = 1, 2, 3), \qquad \begin{array}{ccc} 4 & \longrightarrow & 5 \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle l} \\ \overline{3} & \longrightarrow & \overline{4} \end{array} \quad (k = 1, 2, 3,\ l = 1, 2),$$

$$\begin{array}{ccc} 4 & \longrightarrow & 3 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle n} \\ \overline{5} & \longrightarrow & \overline{4} \end{array} \quad (m = 1, 2, \quad n = 1, 2, 3),$$

and using (3.1), we have

$$(3.8'') \quad \begin{array}{ccc} 3 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \overline{3} \end{array} = \begin{array}{c} \\ \overline{{}_12} \\ \overline{{}_22} \\ \overline{{}_14} \\ \overline{{}_24} \\ \overline{{}_34} \end{array} \begin{array}{c} \begin{array}{ccccc} 2_1 & 2_2 & 4_1 & 4_2 & 4_3 \end{array} \\ \begin{pmatrix} \frac{\overline{x}\mu(2)}{\mu(3)} & -\frac{\overline{\varepsilon_3}y\mu(2)}{\mu(3)} & 0 & * & * \\ \frac{\overline{y}\mu(2)}{\mu(3)} & -\frac{\overline{\varepsilon_3}x\mu(2)}{\mu(3)} & -\frac{\mu(2)}{\mu(7)^2} & * & * \\ 0 & -\frac{\mu(2)}{\mu(7)^2} & * & 0 & * \\ * & * & 0 & * & 0 \\ * & * & * & 0 & * \end{pmatrix} \end{array},$$

$$(3.9'') \quad \begin{array}{ccc} 3 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \overline{5} \end{array} = \begin{array}{c} \\ \overline{{}_14} \\ \overline{{}_24} \\ \overline{{}_34} \\ \overline{6} \end{array} \begin{array}{c} \begin{array}{cccc} 2_1 & 2_2 & 4_1 & 4_2 \end{array} \\ \begin{pmatrix} \frac{\overline{\varepsilon_1}}{\mu(6)\mu(7)} & 0 & 0 & * \\ * & * & \overline{a}\sqrt{\frac{\mu(2)}{\mu(6)}} & 0 \\ * & * & 0 & \overline{d}\sqrt{\frac{\mu(2)}{\mu(6)}} \\ 0 & \sqrt{\frac{\mu(2)}{\mu(1)\mu(3)}} & \frac{1}{\sqrt{\mu(6)}} & 0 \end{pmatrix} \end{array},$$

(3.10‴)

$$\begin{array}{ccc} 5 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \overline{3} \end{array} = \begin{array}{cc} & \begin{array}{cccc} 4_1 & 4_2 & 4_3 & 6 \end{array} \\ \begin{array}{c} \overline{{}_12} \\ \overline{{}_22} \\ \overline{{}_14} \\ \overline{{}_24} \end{array} & \begin{pmatrix} \frac{\overline{\varepsilon_2}}{\mu(6)\mu(7)} & * & * & 0 \\ 0 & * & * & \sqrt{\frac{\mu(2)}{\mu(1)\mu(3)}} \\ 0 & \overline{a}\sqrt{\frac{\mu(2)}{\mu(6)}} & 0 & \frac{\overline{u}}{\sqrt{\mu(6)}} \\ * & 0 & \overline{d}\sqrt{\frac{\mu(2)}{\mu(6)}} & 0 \end{pmatrix} \end{array}.$$

So the unitarity of

$$\begin{array}{ccc} 3 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \overline{5} \end{array}, \quad \begin{array}{ccc} 5 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \overline{3} \end{array}$$

implies the following.

(3.9‴)

$$\begin{array}{ccc} 3 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \overline{5} \end{array} = \begin{array}{cc} & \begin{array}{cccc} 2_1 & 2_2 & 4_1 & 4_2 \end{array} \\ \begin{array}{c} \overline{{}_14} \\ \overline{{}_24} \\ \overline{{}_34} \\ \overline{6} \end{array} & \begin{pmatrix} \frac{\overline{\varepsilon_1}}{\mu(6)\mu(7)} & 0 & 0 & * \\ 0 & * & \overline{a}\sqrt{\frac{\mu(2)}{\mu(6)}} & 0 \\ * & 0 & 0 & \overline{d}\sqrt{\frac{\mu(2)}{\mu(6)}} \\ 0 & \sqrt{\frac{\mu(2)}{\mu(1)\mu(3)}} & \frac{1}{\sqrt{\mu(6)}} & 0 \end{pmatrix} \end{array},$$

(3.10⁗)

$$\begin{array}{ccc} 5 & \longrightarrow & \cdot \\ \big\downarrow & & \big\downarrow \\ \cdot & \longrightarrow & \overline{3} \end{array} = \begin{array}{cc} & \begin{array}{cccc} 4_1 & 4_2 & 4_3 & 6 \end{array} \\ \begin{array}{c} \overline{{}_12} \\ \overline{{}_22} \\ \overline{{}_14} \\ \overline{{}_24} \end{array} & \begin{pmatrix} \frac{\overline{\varepsilon_2}}{\mu(6)\mu(7)} & 0 & * & 0 \\ 0 & * & 0 & \sqrt{\frac{\mu(2)}{\mu(1)\mu(3)}} \\ 0 & \overline{a}\sqrt{\frac{\mu(2)}{\mu(6)}} & 0 & \frac{\overline{u}}{\sqrt{\mu(6)}} \\ * & 0 & \overline{d}\sqrt{\frac{\mu(2)}{\mu(6)}} & 0 \end{pmatrix} \end{array}.$$

In the same way we obtain the following from $(3.9''')$ and $(3.10'''')$.

$$
(3.5')\quad
\begin{tikzcd}
2 \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \bar{4}
\end{tikzcd}
=
\begin{array}{c}
\\ \overline{{}_1 3} \\ \overline{{}_2 3} \\ \overline{{}_1 5} \\ \overline{{}_2 5} \\ \overline{7}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 1_1 & 1_2 & 3_1 & 3_2 & 3_3 \end{array} \\
\begin{pmatrix}
\sqrt{\frac{\mu(6)}{\mu(2)}} & 0 & 0 & * & 0 \\
0 & \sqrt{\frac{\mu(6)}{\mu(2)}} & -\frac{\sqrt{\mu(6)}}{\mu(7)} & 0 & * \\
0 & -\frac{\varepsilon_1}{\mu(2)} & \frac{\varepsilon_1}{\sqrt{\mu(2)\mu(7)}} & 0 & * \\
\frac{\bar{\omega}_1 \mu(1)}{\mu(2)} & 0 & 0 & * & 0 \\
0 & \frac{1}{\sqrt{\mu(2)}} & \frac{\mu(7)}{\mu(2)} & 0 & 0
\end{pmatrix}
\end{array},
$$

(3.6′)

$$
\begin{tikzcd}
4 \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \bar{2}
\end{tikzcd}
=
\begin{array}{c}
\\ \overline{{}_1 1} \\ \overline{{}_2 1} \\ \overline{{}_1 3} \\ \overline{{}_2 3} \\ \overline{{}_3 3}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 3_1 & 3_2 & 5_1 & 5_2 & 7 \end{array} \\
\begin{pmatrix}
\sqrt{\frac{\mu(6)}{\mu(2)}} & 0 & 0 & \frac{\bar{\omega}_2 \mu(1)}{\mu(2)} & 0 \\
0 & \sqrt{\frac{\mu(6)}{\mu(2)}} & -\frac{\varepsilon_2}{\mu(2)} & 0 & \frac{1}{\sqrt{\mu(2)}} \\
0 & -\frac{\sqrt{\mu(6)}}{\mu(7)} & \frac{\varepsilon_2}{\sqrt{\mu(2)\mu(7)}} & 0 & \frac{\mu(7)}{\mu(2)} \\
* & 0 & 0 & * & 0 \\
0 & * & * & 0 & *
\end{pmatrix}
\end{array}.
$$

Due to $(3.5')$ and $(3.6')$ we have

$$
\begin{tikzcd}
3 \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \bar{3}
\end{tikzcd}
=
\begin{array}{c}
\\ \overline{{}_1 2} \\ \overline{{}_2 2} \\ \overline{{}_1 4} \\ \overline{{}_2 4} \\ \overline{{}_3 4}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 2_1 & 2_2 & 4_1 & 4_2 & 4_3 \end{array} \\
\begin{pmatrix}
\frac{\bar{x}\mu(2)}{\mu(3)} & -\frac{\overline{\varepsilon_3 y}\mu(2)}{\mu(3)} & 0 & * & 0 \\
\frac{\bar{y}\mu(2)}{\mu(3)} & -\frac{\overline{\varepsilon_3 x}\mu(2)}{\mu(3)} & -\frac{\mu(2)}{\mu(7)^2} & 0 & * \\
0 & -\frac{\mu(2)}{\mu(7)^2} & * & 0 & * \\
* & 0 & 0 & * & 0 \\
0 & * & * & 0 & *
\end{pmatrix}
\end{array}.
$$

The unitarity of this matrix shows $y = 0$, $|x| = 1$, and

$$
(3.8''') \quad
\begin{array}{ccc}
3 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{3}
\end{array}
=
\begin{array}{c}
\\
\overline{{}_12} \\ \overline{{}_22} \\ \overline{{}_14} \\ \overline{{}_24} \\ \overline{{}_34}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 2_1 & 2_2 & 4_1 & 4_2 & 4_3 \end{array} \\
\begin{pmatrix}
\frac{\overline{x}\mu(2)}{\mu(3)} & 0 & 0 & * & 0 \\
0 & -\frac{\overline{\varepsilon_3}x\mu(2)}{\mu(3)} & -\frac{\mu(2)}{\mu(7)^2} & 0 & * \\
0 & -\frac{\mu(2)}{\mu(7)^2} & * & 0 & * \\
* & 0 & 0 & * & 0 \\
0 & * & * & 0 & *
\end{pmatrix}
\end{array},
$$

$$
(3.12) \quad
\begin{array}{ccc}
2 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{2}
\end{array}
=
\begin{array}{c}
\\
\overline{{}_13} \\ \overline{{}_23}
\end{array}
\begin{array}{c}
\begin{array}{cc} 3_1 & 3_2 \end{array} \\
\begin{pmatrix}
x & 0 \\
0 & \varepsilon_3\overline{x}
\end{pmatrix}
\end{array}.
$$

Finally we have come to the position to prove our main theorem.

THEOREM 3.1. *For the Coxeter graph E_8, there exist two and only two subfactors of the* AFD *type* II_1 *factor up to conjugacy.*

Proof. Let ξ be the horizontal path defined in Section 2, and ρ the embedding map defined by one of the connections computed above. By definition $\rho((\xi,\xi))$ is as follows.

$$
\rho((\xi,\xi)) = \sum_{\sigma_+^1,\sigma_-^1}
\begin{array}{ccccc}
0 & \xrightarrow{\epsilon} & & \xleftarrow{\epsilon} & 0 \\
\downarrow & & & & \downarrow \\
\overline{5} & \xrightarrow[\sigma_+^1]{} & & \xleftarrow[\sigma_-^1]{} & \overline{5}
\end{array}
(\sigma_+^1, \sigma_-^1)
$$

$$
+ \sum_{\sigma_+^2,\sigma_-^2}
\begin{array}{ccccc}
0 & \xrightarrow{\epsilon} & & \xleftarrow{\epsilon} & 0 \\
\downarrow & & & & \downarrow \\
\overline{5} & \xrightarrow[\sigma_+^2]{} & & \xleftarrow[\sigma_-^2]{} & \overline{7}
\end{array}
(\sigma_+^2, \sigma_-^2)
$$

$$+\sum_{\sigma_+^3,\sigma_-^3} \begin{array}{ccccc} 0 & \xrightarrow{\epsilon} & & \xleftarrow{\epsilon} & 0 \\ \Big\downarrow & & & & \Big\downarrow \\ \overline{7} & \xrightarrow[\sigma_+^3]{} & & \xleftarrow[\sigma_-^3]{} & \overline{5} \end{array} (\sigma_+^3,\sigma_-^3)$$

$$+\sum_{\sigma_+^4,\sigma_-^4} \begin{array}{ccccc} 0 & \xrightarrow{\epsilon} & & \xleftarrow{\epsilon} & 0 \\ \Big\downarrow & & & & \Big\downarrow \\ \overline{7} & \xrightarrow[\sigma_+^4]{} & & \xleftarrow[\sigma_-^4]{} & \overline{7} \end{array} (\sigma_+^4,\sigma_-^4).$$

So to show the assumption of Lemma 2.1 it suffices to prove

$$\begin{array}{ccccc} 0 & \xrightarrow{\epsilon} & & \xleftarrow{\epsilon} & 0 \\ \Big\downarrow & & & & \Big\downarrow \\ \overline{5} & \xrightarrow[\sigma_+]{} & & \xleftarrow[\sigma_-]{} & \overline{7} \end{array} = 0, \tag{3.13}$$

for any pair of paths σ_+, σ_- in $\mathcal{G}_2$ satisfying

$$|\sigma_+| = |\sigma_-| = |\xi|, \quad s(\sigma_+) = \overline{5}, \quad s(\sigma_-) = \overline{7}, \quad r(\sigma_+) = r(\sigma_-).$$

Fix an edge $\zeta \in \mathcal{F}_2$ connected to the vertex 7

$$\begin{array}{c} 7 \\ \Big\downarrow{\scriptstyle \zeta} \\ w \end{array},$$

and assume that the following large connection is non-zero.

$$\begin{array}{ccc} 0 & \xrightarrow{\epsilon} & 7 \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \zeta} \\ z & \xrightarrow[\sigma]{} & w \end{array}, \quad z = \overline{5} \text{ or } \overline{7}.$$

Thanks to (3.2)— (3.12) we can see by direct computation that z is uniquely determined. (Of course σ is not unique.) That is, if w

is $\overline{4_1}$, $\overline{4_2}$, $\overline{2}$, $\overline{0}$, then z must be $\overline{7}$, $\overline{5}$, $\overline{5}$, $\overline{7}$, respectively. This means that (3.13) holds and we finish the proof. □

REMARK. In the same way we can prove the existence of other paragroups. For example, it is much easier to show the existence of E_6 paragroup. In the case of D_{even}, we can also show the existence using induction.

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